Proposition 6 (Upper bound 2) $\exists K_2 < \infty$ s.t. for $\varepsilon > 0$, for t sufficiently large,

$$u(t,x) \le \varepsilon \quad \forall x \ge \sqrt{2}\,t - \frac{3}{2\sqrt{2}} \log t + K_2 \log\log t.$$

Proof: Take t large and $x \in [\sqrt{2}\,t - \frac{3}{2\sqrt{2}} \log t, \sqrt{2}\,t + \log t]$.
For $j \in [0, t-1] \cap \mathbb{Z}$, let

$$E_j = \{ \exists s \in [j, j+1] : B_s < \tfrac{t-s}{t} x - \min(s^{1/4}, (t-s)^{1/4}) \}$$

and $D_j = \mathbb{E}_x [\mathbb{1}_{E_j} e^{\int_0^t (1 - u(t-s, B_s))\, ds} u_0(B_t)]$.

Then by FK formula,

$$u(t,x) \le \sum_{j = \lfloor (\log t)^5 \rfloor}^{\lfloor t - (\log t)^5 \rfloor} D_j + e^t P_x \Big(B_s \ge \tfrac{t-s}{t} x - \min(s^{1/4}, (t-s)^{1/4}) \ \forall s \in [(\log t)^5, t - (\log t)^5], \ B_t \le 0 \Big). \tag{3}$$

For $y \in [0, 10 \log t]$,

$$P_x \Big(B_s \ge \tfrac{t-s}{t} x - \min(s^{1/4}, (t-s)^{1/4}) \ \forall s \in [(\log t)^5, t - (\log t)^5] \ \Big| \ B_t = -y \Big) \le \frac{(\log t)^{10}}{t} \quad \text{for } t \text{ suff large.}$$

Proof ctd:

For $y \in [0, 10\log t]$,

$$P_x\left(B_s \geq \frac{t-s}{t}x - \min(s^{1/4}, (t-s)^{1/4})\ \forall s \in [(\log t)^5, t-(\log t)^5] \,\middle|\, B_t = -y\right)$$

$$\leq \frac{(\log t)^{10}}{t} \quad \text{for } t \text{ suff large.}$$

So $$e^t P_x\left(B_s \geq \frac{t-s}{t}x - \min(s^{1/4}, (t-s)^{1/4})\ \forall s \in [(\log t)^5, t-(\log t)^5], B_t \leq 0\right)$$

$$\leq e^t \frac{(\log t)^{10}}{t} P_x(B_t \in [-10\log t, 0]) + e^t P_x(B_t \leq -10\log t)$$

$$\leq e^t \frac{(\log t)^{10}}{t} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} e^{-\frac{1}{2} x^2/t} + e^t \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} e^{-\frac{1}{2}\frac{(x+10\log t)^2}{t}}$$

$\leq (\log t)^{-1}$ for t suff large if $x \geq \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + K_2 \log\log t$ (for K_2 a suff large const).

Proposition 6 (Upper bound 2) $\exists K_2 < \infty$ s.t. for $\varepsilon > 0$, for t sufficiently large,

$$u(t,x) \le \varepsilon \quad \forall x \ge \sqrt{2}\,t - \frac{3}{2\sqrt{2}}\log t + K_2 \log\log t.$$

Proof ctd: Recall $E_j = \{\exists s \in [j, j+1] : B_s < \frac{t-s}{t}x - \min(s^{1/4}, (t-s)^{1/4})\}$

$$\text{and } D_j = E_x\left[\mathbb{1}_{E_j} e^{\int_0^t (1-u(t-s,B_s))\,ds} u_0(B_t)\right].$$

Take K a large const. For $\lfloor (\log t)^5 \rfloor \le j \le \lfloor t-(\log t)^5 \rfloor$,

$$D_j \le E_x\left[\mathbb{1}_{E_j} \mathbb{1}_{B_s < \sqrt{2}(t-s) - K_1 \log t \ \forall s \in [j, j+K\log t]} e^{t-cK\log t} u_0(B_t)\right]$$

$$+ E_x\left[\mathbb{1}_{E_j} \mathbb{1}_{B_{s^*} \ge \sqrt{2}(t-s^*) - K_1\log t, \text{ some } s^* \in [j, j+K\log t]} e^t u_0(B_t)\right]$$

for t suff large, by Prop$^{\underline{n}}$ 4

$$\le e^t t^{-cK} P_x(B_t \le 0) + e^t P_0\Big(\sup_{s_1, s_2 \in [0, K\log t]} |B_{s_1} - B_{s_2}| \ge \tfrac{1}{2}(\log t)^{5/4}\Big)$$

$$\le e^t t^{-cK} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t)^2} + e^t \cdot 8 \exp\Big(-\frac{1}{2K\log t}\big(\tfrac{1}{4}(\log t)^{5/4}\big)^2\Big)$$

by the reflection principle

$\le t^{-2}$ for t suff large, by choosing K suff large.

So by (3), if $x \ge \sqrt{2}\,t - \frac{3}{2\sqrt{2}}\log t + K_2\log\log t$, $u(t,x) \le t \cdot t^{-2} + (\log t)^{-1}$. $\square$

Corollary 7 (Exponential decay) For t sufficiently large, for $s \geq 0$, for $x \geq \sqrt{2}(t+s) - \frac{3}{2\sqrt{2}}\log(t+s) + K_2 \log\log(t+s)$,

$$u(t,x) \leq e^{1-C^{-1}s}.$$

Proof: Suppose $x \geq \sqrt{2}(t+s) - \frac{3}{2\sqrt{2}}\log(t+s) + K_2\log\log(t+s)$ and $u(t,x) \geq e^{1-C^{-1}s}$.

By Lemma 2 (exponential growth), $u(t+s,x) \geq c$, which is a # for t suff large by Prop$^{\underline{n}}$ 6. $\square$

Proposition 8 (Lower bound 2) $\exists K_3 < \infty$ s.t. for t sufficiently large,

$$u(t,x) \geq c \quad \forall x \leq \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t - K_3(\log\log t)^3.$$

Proof: Let $x = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$. By FK formula,

$$u(t,x) \geq E_x\Big[\mathbb{1}_{B_s \geq \frac{t-s}{t}(x-1) - 2\frac{s}{t} + (2+K_2)\min(s^{1/3},(t-s)^{1/3})\ \forall s\in[(\log\log t)^3,\, t-(\log\log t)^3]} \cdot e^t e^{-\int_0^t u(t-s,B_s)ds} u_0(B_t)\Big].$$

For $s \in [(\log\log t)^3, t-(\log\log t)^3]$, $y \geq \frac{t-s}{t}(x-1) - \frac{s}{t}2 + (2+K_2)\min(s^{1/3},(t-s)^{1/3})$, we have

$$y \geq \sqrt{2}(t-s) - \frac{t-s}{t}\frac{3}{2\sqrt{2}}\log t + (2+K_2)\min(s^{1/3},(t-s)^{1/3}) - 3$$

$$\geq \sqrt{2}\big((t-s) + \min(s^{1/3},(t-s)^{1/3})\big) - \frac{3}{2\sqrt{2}}\log(t-s) + K_2\log\log t,$$

so by Corollary 7, $u(t-s,y) \leq e^{1-C^{-1}\min(s^{1/3},(t-s)^{1/3})}$.

<u>Proposition 8</u> (Lower bound 2) $\exists K_3<\infty$ s.t. for t sufficiently large,

$$u(t,x)\geq c \quad \forall x\leq \sqrt{2}\,t-\frac{3}{2\sqrt{2}}\log t - K_3(\log\log t)^3.$$

Proof: Let $x=\sqrt{2}\,t-\frac{3}{2\sqrt{2}}\log t$. By FK formula,

$$u(t,x)\geq E_x\Big[\mathbb{1}_{B_s\geq \frac{t-s}{t}(x-1)-2\frac{s}{t}+(2+K_2)\min(s^{1/3},(t-s)^{1/3})\ \forall s\in[(\log\log t)^3,\,t-(\log\log t)^3]} \cdot e^t e^{-\int_0^t u(t-s,B_s)ds}\,u_0(B_t)\Big].$$

For $s\in[(\log\log t)^3, t-(\log\log t)^3]$, $y\geq \frac{t-s}{t}(x-1)-\frac{s}{t}2+(2+K_2)\min(s^{1/3},(t-s)^{1/3})$, we have

$$u(t-s,y)\leq e^{1-C^{-1}\min(s^{1/3},(t-s)^{1/3})}$$

Hence

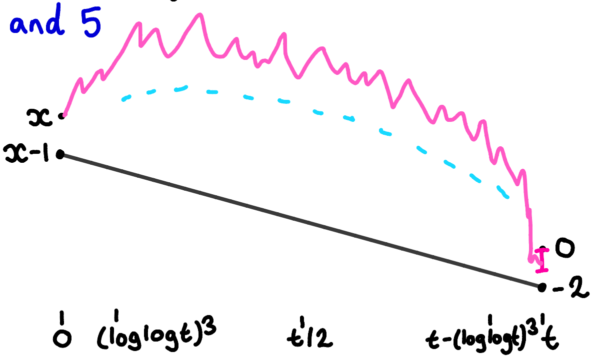
$$u(t,x)\geq e^{t-2(\log\log t)^3}\, e^{-\int_{(\log\log t)^3}^{t-(\log\log t)^3} e^{1-C^{-1}\min(s^{1/3},(t-s)^{1/3})}ds}$$

$$\cdot P_x\big(B_t\leq 0,\ B_s\geq \tfrac{t-s}{t}(x-1)-\tfrac{s}{t}2+(2+K_2)\min(s^{1/3},(t-s)^{1/3})\ \forall s\in[(\log\log t)^3,\ t-(\log\log t)^3]\big)$$

$$\geq e^{t-2(\log\log t)^3}\, e^{-2e\int_0^\infty e^{-C^{-1}s^{1/3}}ds}\ P_x(B_t\in[-1,0])\,\frac{1}{t}$$

for t suff large, by Lemmas 3 and 5

$$\geq e^{-3(\log\log t)^3} \quad \text{for } t \text{ suff large.}$$



The result follows by Lemma 2 (exponential growth). $\square$.

Non-local Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1-\phi * u), \text{ where } \phi \in L^1(\mathbb{R}) \text{ with } \phi \geq 0,\ \phi(x) \geq \sigma \ \forall |x| \leq \sigma, \text{ some } \sigma > 0$$

$$\|\phi\|_1 = 1 \text{ and } \phi * u(t,x) = \int_{-\infty}^{\infty} \phi(y)\, u(t,x-y)\, dy.$$

- Models non-local interaction and competition in a population.

c.f. $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1-u)$ local Fisher-KPP equation

$u(t,x)$ "=" population density at location x at time t.

Non-local Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1-\phi * u), \text{ where } \phi \in L^1(\mathbb{R}) \text{ with } \phi \geq 0,\ \phi(x) \geq \sigma\ \forall |x| \leq \sigma, \text{ some } \sigma > 0$$

$$\|\phi\|_1 = 1 \text{ and } \phi * u(t,x) = \int_{-\infty}^{\infty} \phi(y)\, u(t, x-y)\, dy.$$

- Models non-local interaction and competition in a population.

c.f. $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1-u)$ local Fisher-KPP equation

$u(t,x)$ "=" population density with trait x at time t.

(Wikipedia)

Non-local Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1-\phi * u), \text{ where } \phi \in L^1(\mathbb{R}) \text{ with } \phi \geq 0, \ \phi(x) \geq \sigma \ \forall |x| \leq \sigma, \text{ some } \sigma > 0$$

$$\|\phi\|_1 = 1 \text{ and } \phi * u(t,x) = \int_{-\infty}^{\infty} \phi(y)\, u(t,x-y)\, dy.$$

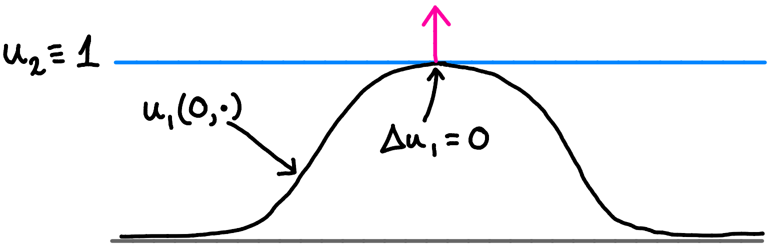
- Models non-local interaction and competition in a population.

- Comparison principle doesn't hold.

If u_1, u_2 solve Fisher-KPP equation with $u_1(0,x) \leq u_2(0,x)$ $\forall x \in \mathbb{R}$, then

$$u_1(t,x) \leq u_2(t,x) \quad \forall t \geq 0, x \in \mathbb{R}.$$

Not true for non-local Fisher-KPP equation.



Non-local Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1-\phi * u), \text{ where } \phi \in L^1(\mathbb{R}) \text{ with } \phi \geq 0,\ \phi(x) \geq \sigma\ \forall |x| \leq \sigma, \text{ some } \sigma > 0$$

$$\|\phi\|_1 = 1 \text{ and } \phi * u(t,x) = \int_{-\infty}^{\infty} \phi(y) u(t, x-y)\, dy.$$

- Models non-local interaction and competition in a population.
- Comparison principle doesn't hold.

Theorem (P. (2018)) Suppose $u_0(x) = 0\ \forall x \geq L$, $\liminf_{x \to -\infty} u_0(x) > 0$ and $\|u_0\|_\infty < \infty$.

1. Suppose $\exists \alpha > 2$ s.t. for r suff large, $\int_r^\infty \phi(x)\, dx \leq r^{-\alpha}$.

   Then for t sufficiently large,

   - for $x \leq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t - A(\log\log t)^3$, $u(t,x) \geq c > 0$.
   - for $x \geq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + A \log\log t$, $u(t,x) = o(1)$.

2. Suppose $\exists \alpha \in (0,2), C_1, C_2$ s.t. for r suff large, $C_1 r^{-\alpha} \leq \int_r^\infty \phi(x)\, dx \leq C_2 r^{-\alpha}$.

   Let $\beta = \frac{2-\alpha}{2+\alpha}$. For $\varepsilon > 0$, for t sufficiently large,

   - for $x \leq \sqrt{2}t - t^{\beta+\varepsilon}$, $u(t,x) \geq c > 0$.
   - for $x \geq \sqrt{2}t - t^{\beta-\varepsilon}$, $u(t,x) = o(1)$.

See also Bouin, Henderson, Ryzhik (2019).

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1-\phi * u)$$

Feynman-Kac formula: for $t \geq 0, x \in \mathbb{R}$, $u(t,x) = E_x\left[u_0(B_t)\, e^{\int_0^t (1-\phi * u(t-s,B_s))ds}\right]$.

Lemma (Global bound) $\exists M < \infty$ s.t. $0 \leq u(t,x) \leq M \quad \forall t \geq 0, x \in \mathbb{R}$.

Lemma (Upper bound 1) For t suff large, for $x \geq \sqrt{2}t + \frac{1}{2\sqrt{2}}\log t$, $\quad u(t,x) \leq t^{-1}$.

Proof: Same as for Lemma 1. By FK formula, $u(t,x) \leq e^t E_x[u_0(B_t)]$. □.

Lemma (Exponential growth) $\exists C < \infty, c > 0$ s.t. for $z > 0$, if $u(t,x) \geq z$ then $u(t + C(\log(1/z)+1)+s, x) \geq c \quad \forall s \geq 0$.

Suppose $\exists \alpha > 2$ s.t. $\int_r^\infty \phi(x)\,dx \leq r^{-\alpha}$ for r suff large.

Proposition (Lower bound 1) $\exists K < \infty$ s.t. for t suff large, $u(t,x) \geq c \quad \forall x \leq \sqrt{2}t - K_1 \log t$.

Proof: Take $\delta < 1/2$ s.t. $\alpha\delta > 1$. Fix t large. Let $f(s) = \sqrt{2}s + \frac{1}{2\sqrt{2}}\log t + \min((s-\log t)^\delta, (t-s)^\delta)$.

For $y \in [0,2]$, by FK formula, $\quad u(\log t, f(\log t)+y) \geq e^{(1-M)\log t} E_{f(\log t)+y}[u_0(B_{\log t})]$

$$\geq t^{-1-M} \quad \text{for } t \text{ suff large.}$$

Suppose $\exists \alpha > 2$ s.t. $\int_r^\infty \phi(x)\,dx \le r^{-\alpha}$ for r suff large.

Proposition (Lower bound 1) $\exists K < \infty$ s.t. for t suff large, $u(t,x) \ge c \quad \forall x \le \sqrt{2}\,t - K_1 \log t$.

Proof: Take $\delta < 1/2$ s.t. $\alpha\delta > 1$. Fix t large. Let $f(s) = \sqrt{2}\,s + \frac{1}{2\sqrt{2}} \log t + \min\left((s - \log t)^\delta, (t-s)^\delta\right)$.

For $y \in [0,2]$, by FK formula, $u(\log t, f(\log t) + y) \ge t^{-1-M}$ for t suff large.

For $x \ge f(t-s)$,

$$\phi * u(t-s,x) = \int_{\sqrt{2}(t-s)+\frac{1}{2\sqrt{2}}\log t}^{\infty} u(t-s,y)\,\phi(x-y)\,dy + \int_{-\infty}^{\sqrt{2}(t-s)+\frac{1}{2\sqrt{2}}\log t} u(t-s,y)\,\phi(x-y)\,dy$$

$$\le (t-s)^{-1} + M\left(\min\left(s^\delta, (t - \log t - s)^\delta\right)\right)^{-\alpha}.$$

Take r large. By FK formula,

$$u(t, f(t)+1) = E_{f(t)+1}\left[e^{\int_0^{t-\log t} (1 - \phi * u(t-s, B_s))\,ds}\, u(\log t, B_{t-\log t}) \right]$$

$$\ge E_{f(t)+1}\Big[\mathbb{1}_{B_s \ge f(t-s)\ \forall s \in [r, t-\log t - r]}\, \mathbb{1}_{B_{t-\log t} - f(\log t) \in [0,2]} \cdot t^{-M-1} e^{t-\log t} e^{-2rM} e^{-\int_r^{t-\log t - r} \left((M+1)(t-\log t - s)^{-1} + M s^{-1}\right) ds} \Big]$$

$$\ge e^t\, t^{-1}\, t^{-M-1}\, t^{-(2M+1)}\, e^{-2rM} \cdot P_{f(t)+1}\left(B_s \ge f(t-s)\ \forall s \in [r, t-\log t - r],\ B_{t-\log t} - f(\log t) \in [0,2]\right)$$

$$\ge e^t\, t^{-3M-3}\, e^{-2rM} \cdot t^{-1} \frac{1}{\sqrt{2\pi t}} e^{-(t-\log t)}$$ by entropic repulsion

$$\ge t^{-3M-4}$$ for t suff large. The result follows by exp growth. $\square$

Suppose for some $\alpha \in (0,2)$, $C_1 r^{-\alpha} \leq \int_r^\infty \phi(x)\,dx \leq C_2 r^{-\alpha}$ for r suff large.

Let $\beta = \frac{2-\alpha}{2+\alpha}$ and take $\varepsilon > 0$. Let $\gamma = \frac{2}{2+\alpha} > 1/2$.

Suppose (for a $\#$) $u(t,x) \geq m > 0 \ \forall t \geq T$, $x \leq \sqrt{2}t - t^{\beta-\varepsilon}$.

Then by FK formula, for $x = \sqrt{2}t - t^{\beta-\varepsilon}$,

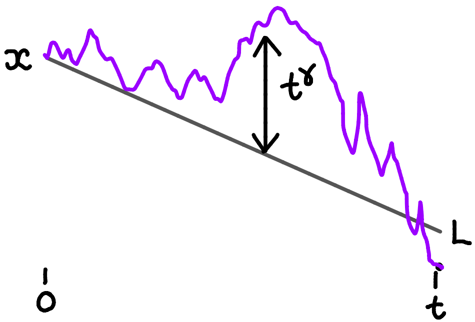
$$u(t,x) = E_x\left[e^{\int_0^t (1-\phi * u(t-s,B_s))\,ds} u_0(B_t)\right]$$

$$\leq E_x\left[e^t \mathbb{1}_{\exists s^* \in [1/4 t, 3/4 t] : B_{s^*} \geq \sqrt{2}(t-s^*) + t^\gamma} \mathbb{1}_{B_t \leq L}\right]$$

$$+ E_x\left[e^t e^{-\int_{1/4 t}^{3/4 t} \phi * u(t-s,B_s)\,ds} \mathbb{1}_{B_s \leq \sqrt{2}(t-s) + t^\gamma \ \forall s \in [1/4 t, 3/4 t]} \mathbb{1}_{B_t \leq L}\right]$$

$$\leq e^t e^{-at^{2\gamma-1}} e^{-t+\sqrt{2}t^{\beta-\varepsilon}}$$

$$+ e^t e^{-\frac{1}{2}t \cdot m(2t^\gamma)^{-\alpha}} e^{-t+\sqrt{2}t^{\beta-\varepsilon}}$$



$= o(1)$ as $t \to \infty$

since $2\gamma - 1 = 1 - \alpha\gamma = \beta$.