Back to the rightmost particle $\Phi(u)=u^2$, $V=1$.

$$\mathbb{P}(M(t)\geq x) = 1-\mathbb{P}\Big(\prod_{i=1}^{N(t)} \mathbb{1}_{X_i(t)\leq x}=1\Big)$$

$$= 1-\mathbb{E}_{-x}\Big[\prod_{i=1}^{N(t)} \mathbb{1}_{X_i(t)\leq 0}\Big]$$

$$= 1-v(t,-x), \quad \text{where } v \text{ solves } \begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\Delta v + v^2 - v \\ v(0,y) = \mathbb{1}_{y\leq 0}. \end{cases}$$

Hence $u(t,x) := 1-v(t,-x)$ solves $\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1-u) \\ u(0,y) = \mathbb{1}_{y\leq 0} \end{cases}$ Fisher-KPP equation

So $\mathbb{P}(M(t)\geq x) = u(t,x)$.

Feynman-Kac formula

Proposition Suppose $k:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous and bounded, that $u:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous and bounded, and smooth on $(0,\infty)\times\mathbb{R}$, and that

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + k(t,x)u \quad \forall t>0, x\in\mathbb{R}.$$

Then for $0\leq t'\leq t$, $x\in\mathbb{R}$,

$$u(t,x) = \mathbb{E}_x\Big[e^{\int_0^{t'} k(t-s,B_s)\,ds}\,u(t-t',B_{t'})\Big].$$

Proof: For $s\in[0,t]$, let $M_s = u(t-s,B_s)\,e^{I_s}$,
where $I_s = \int_0^s k(t-s',B_{s'})\,ds'$.

By Itô's formula,

$$dM_s = \frac{\partial u}{\partial x}(t-s,B_s)e^{I_s}\,dB_s + \frac{1}{2}\Delta u(t-s,B_s)\,e^{I_s}\,ds$$

$$-\frac{\partial u}{\partial t}(t-s,B_s)e^{I_s}\,ds + u(t-s,B_s)\,e^{I_s}\,dI_s$$

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$$-\frac{\partial u}{\partial t}(t-s,B_s)e^{I_s}\,ds+u(t-s,B_s)\,e^{I_s}\,\underbrace{dI_s}_{=k(t-s,B_s)ds}$$

$$=\frac{\partial u}{\partial x}(t-s,B_s)\,e^{I_s}\,dB_s+\left(\frac{1}{2}\Delta u+uk-\frac{\partial u}{\partial t}\right)\Big|_{(t-s,B_s)}e^{I_s}\,ds$$

$$=\frac{\partial u}{\partial x}(t-s,B_s)\,e^{I_s}\,dB_s.$$

So $(M_s)_{0\le s\le t}$ is a local martingale, and since it is bounded it is a true martingale.
Therefore for $t'\le t$, $u(t,x)=\mathbb{E}_x[M_0]=\mathbb{E}_x[M_{t'}]$. $\square$.

If u solves $\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+u(1-u)$, $u(0,y)=u_0(y)$, then

$$u(t,x)=\mathbb{E}_x\left[e^{\int_0^t(1-u(t-s,B_s))\,ds}\,u_0(B_t)\right].$$

Note that $u\in[0,1]$ if $u_0\in[0,1]$.

Sketch proof of Bramson's result

Suppose $u_0(x) = \mathbb{1}_{x \le 0}$.

We will show $u(t,x) = o(1)$ for $x \geq \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + o(\log t)$

and $u(t,x) \geq c > 0$ for $x \le \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t - o(\log t)$.

Lemma 1 (Upper bound 1) For $t \geq 1$ and $x \geq \sqrt{2}t + \frac{1}{2\sqrt{2}}\log t$, $u(t,x) \le t^{-1}$.

Proof: By FK formula,

$$u(t,x) \le e^t E_x[u_0(B_t)] \le e^t P_0\left(B_t \geq \sqrt{2}t + \frac{1}{2\sqrt{2}}\log t\right)$$

$$\le e^t \,\mathbb{P}\left(Z \geq \sqrt{2t} + \frac{1}{2\sqrt{2t}}\log t\right) \quad \text{where } Z \sim N(0,1)$$

$$\le e^t \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} e^{-1/2\left(\sqrt{2t} + \frac{1}{2\sqrt{2t}}\log t\right)^2}$$

$$\le \frac{1}{2\sqrt{\pi}} e^t t^{-1/2} e^{-t} t^{-1/2} \le t^{-1}. \quad \square.$$

Lemma 2 (Exponential growth)

$\exists C < \infty, c > 0$ s.t. for $z > 0$, if $u(t,x) \geq z$ then $u(t + C(\log(1/z) + 1) + s, x) \geq c \quad \forall s \geq 0$.

Proof: Can use FK formula.

Defn (Brownian bridge from 0 to 0 in time t) "Brownian motion conditioned on $B_t = 0$".

Let $\xi^t(s) = B_s - \frac{s}{t}B_t$ for $0 \le s \le t$, where $(B_s)_{s\geq 0}$ is a Brownian motion started at 0.

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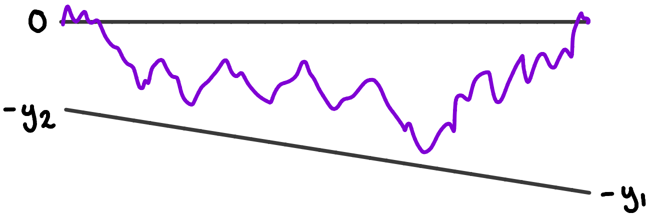
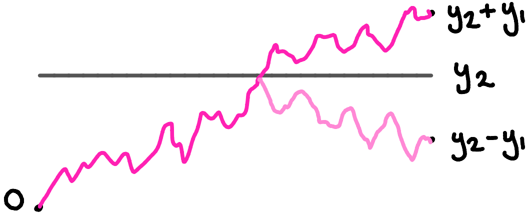
Lemma 3 (Brownian Bridge 1) For $y_1, y_2 \ge 0$,

$$\mathbb{P}\left(\xi^t(s) \ge -\frac{s}{t} y_1 - \frac{t-s}{t} y_2 \ \forall s \in [0,t]\right) = 1 - e^{-\frac{2y_1y_2}{t}} \sim \frac{2y_1y_2}{t} \text{ as } t \to \infty.$$



Proof: $p := \mathbb{P}\left(\exists s \in [0,t] : \xi^t(s) \ge \frac{s}{t} y_1 + \frac{t-s}{t} y_2\right) = \mathbb{P}\left(\exists s \in [0,t] : \xi^t(s) + \frac{s}{t}(y_2 - y_1) \ge y_2\right)$

BB from 0 to $y_2 - y_1$ in time t



By the reflection principle,
$\mathbb{P}(\text{cross } y_2 \text{ and end at } y_2 - y_1) = \mathbb{P}(\text{end at } y_2 + y_1)$.

Let $G_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}x^2}$.

Then $p = \dfrac{G_t(y_2+y_1)}{G_t(y_2-y_1)} = e^{-\frac{1}{2t}(2y_1y_2 + 2y_1y_2)} = e^{-\frac{2y_1y_2}{t}}$. $\square$.

Proposition 4 (Lower bound 1) $\exists K_1 < \infty$ s.t. for t sufficiently large,

$$u(t,x) \geq c \quad \forall x \leq \sqrt{2}\,t - K_1 \log t.$$

Proof: Fix t large. Let $f(s) = \sqrt{2}\,s + \frac{1}{2\sqrt{2}} \log t$.
For $y \in [0,2]$, by FK formula,

$$\begin{aligned} u(\log t, f(\log t) + y) &\geq E_{f(\log t)+y}\left[u_0(B_{\log t})\right] \\ &\geq P_{f(\log t)+2}\left(B_{\log t} \leq 0\right) \\ &\geq \frac{1}{\sqrt{2\pi \log t}}\, e^{-\frac{1}{2\log t}\left(\sqrt{2}\log t + \frac{1}{2\sqrt{2}}\log t + 3\right)^2} \\ &\geq t^{-2} \quad \text{for } t \text{ suff large.} \end{aligned}$$

By FK formula,

$$\begin{aligned} u(t, f(t)+1) &= E_{f(t)+1}\left[e^{\int_0^{t-\log t}(1-u(t-s,B_s))\,ds}\, u(\log t, B_{t-\log t})\right] \\ &\geq E_{f(t)+1}\Big[\mathbb{1}_{B_s \geq f(t-s)\ \forall s \in [0, t-\log t]}\, \mathbb{1}_{B_{t-\log t} \leq f(\log t)+2} \\ &\qquad \cdot e^{\int_0^{t-\log t}(1-(t-s)^{-1})\,ds}\, t^{-2}\Big] \end{aligned}$$

since $u(t-s,x) \leq (t-s)^{-1}$ for $x \geq f(t-s)$ by Lemma 1

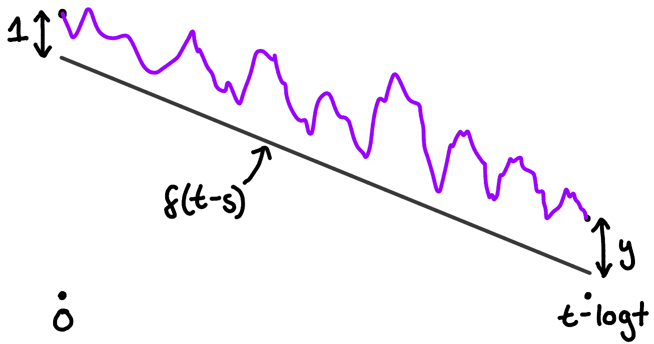
$$\begin{aligned} &\geq e^{t-\log t}\, e^{-\int_1^t s^{-1}\,ds}\, t^{-2} \\ &\qquad \cdot P_{f(t)+1}\left(B_s \geq f(t-s)\ \forall s \in [0, t-\log t],\ B_{t-\log t} \leq f(\log t)+2\right). \end{aligned}$$

Proof ctd:

$$u(t, f(t)+1) \geqslant e^{t-\log t} e^{-\int_0^t s^{-1} ds} t^{-2}$$
$$\cdot \, P_{f(t)+1}\left(B_s \geqslant f(t-s) \ \forall s \in [0, t-\log t], \ B_{t-\log t} \leqslant f(\log t)+2\right).$$

For $y \in [1,2]$,

$$P_{f(t)+1}\left(B_s \geqslant f(t-s) \ \forall s \in [0, t-\log t] \ \middle| \ B_{t-\log t} = f(\log t)+y\right)$$
$$= \mathbb{P}\left(\xi^{t-\log t}(s) + \frac{t-\log t - s}{t-\log t}(f(t)+1) + \frac{s}{t-\log t}(f(\log t)+y) \geqslant f(t-s) \ \forall s \in [0, t-\log t]\right)$$
$$= \mathbb{P}\left(\xi^{t-\log t}(s) \geqslant -\frac{t-\log t - s}{t-\log t} - y\frac{s}{t-\log t} \quad \forall s \in [0, t-\log t]\right)$$
$$= 1 - e^{-\frac{2y}{t-\log t}} \quad \text{by Lemma 3}$$
$$\geqslant t^{-1} \quad \text{for } t \text{ suff large.}$$



So

$$u(t, f(t)+1) \geqslant e^t t^{-4} \cdot t^{-1} P_{f(t)+1}\left(B_{t-\log t} \in [f(\log t)+1, f(\log t)+2]\right)$$
$$\geqslant e^t t^{-5} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2(t-\log t)}\left(\sqrt{2}(t-\log t)\right)^2\right) \quad \text{since } f(x)-f(y) = \sqrt{2}(x-y)$$
$$\geqslant t^{-5} \quad \text{for } t \text{ suff large.}$$

Proposition 4 (Lower bound 1) $\exists K_1 < \infty$ s.t. for t sufficiently large,

$$u(t,x) \geq c \quad \forall x \leq \sqrt{2}t - K_1 \log t.$$

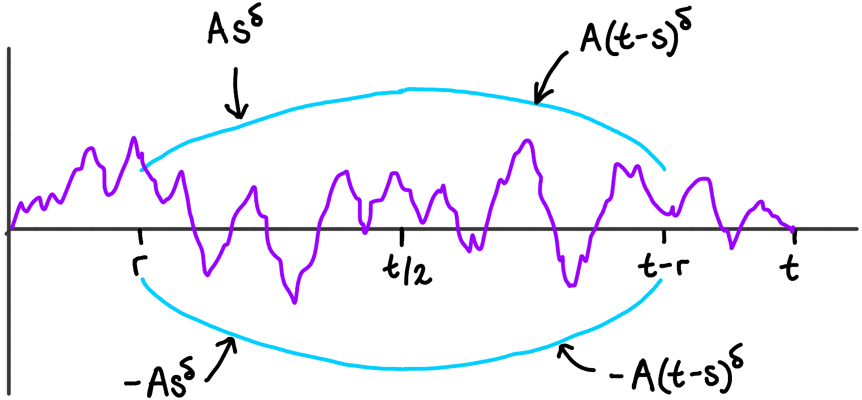
Proof ctd: $g(s) = \sqrt{2}s + \frac{1}{2\sqrt{2}} \log t$, $u(t, g(t)+1) \geq t^{-5}$ for t suff large.

By Lemma 2 (exponential growth),

$$u(t + C(5\log t + 1) + s,\ g(t)+1) \geq c \quad \forall s \geq 0.$$ The result follows. $\square$.

Lemma 5 (Entropic repulsion) For $z \in \mathbb{R}$, $\delta \in (0, 1/2)$ and $A > 0$, for $\varepsilon > 0$, $\exists r_\varepsilon < \infty$ s.t. $\forall r > r_\varepsilon$ and $t > 3r$,

$$\left| \frac{\mathbb{P}(\xi^t(s) > z + \min(As^\delta, A(t-s)^\delta)\ \forall s \in [r, t-r])}{\mathbb{P}(\xi^t(s) > z - \min(As^\delta, A(t-s)^\delta)\ \forall s \in [r, t-r])} - 1 \right| < \varepsilon.$$



Proposition 6 (Upper bound 2) $\exists K_2<\infty$ s.t. for $\varepsilon>0$, for t sufficiently large,

$$u(t,x)\leq \varepsilon \quad \forall x\geq \sqrt{2}t-\frac{3}{2\sqrt{2}}\log t-K_2\log\log t.$$

Proof: Take t large and $x\in[\sqrt{2}t-\frac{3}{2\sqrt{2}}\log t,\ \sqrt{2}t+\log t]$.
For $j\in[0,t-1]\cap\mathbb{Z}$, let

$$E_j=\{\exists s\in[j,j+1]: B_s<\frac{t-s}{t}x-\min(s^{1/4},(t-s)^{1/4})\}$$

and $$D_j=\mathbb{E}_x[\mathbb{1}_{E_j}e^{\int_0^t(1-u(t-s,B_s))ds}u_0(B_t)].$$

Then by FK formula,

$$u(t,x)\leq\sum_{j=\lfloor(\log t)^5\rfloor}^{\lfloor t-(\log t)^5\rfloor}D_j+\mathbb{E}_x[\mathbb{1}_{\bigcap_{j=\lfloor(\log t)^5\rfloor}^{\lfloor t-(\log t)^5\rfloor}E_j^c}\cdot e^t u_0(B_t)]$$

$$\text{(3)}\qquad \leq\sum_{j=\lfloor(\log t)^5\rfloor}^{\lfloor t-(\log t)^5\rfloor}D_j+e^t P_x\Big(B_s\geq\frac{t-s}{t}x-\min(s^{1/4},(t-s)^{1/4})\ \forall s\in[(\log t)^5,t-(\log t)^5],\ B_t\leq 0\Big).$$

For $y\in[0,10\log t]$,

$$P_x\Big(B_s\geq\frac{t-s}{t}x-\min(s^{1/4},(t-s)^{1/4})\ \forall s\in[(\log t)^5,t-(\log t)^5]\ \Big|\ B_t=-y\Big)$$

$$\leq\ \ldots$$

Proof ctd:

For $y \in [0, 10\log t]$,

$$P_x\left(B_s \geq \frac{t-s}{t}x - \min(s^{1/4}, (t-s)^{1/4})\ \forall s \in [(\log t)^5, t-(\log t)^5] \,\middle|\, B_t = -y\right)$$

$$\leq \frac{(\log t)^{10}}{t}$$ for t suff large, by Lemmas 3 and 5 and since whp $|B_{(\log t)^5} - x| \lesssim (\log t)^5$ and $|B_{t-(\log t)^5}| \lesssim (\log t)^5$