The Fisher-KPP equation and related topics

References: Bramson. Convergence of solutions of the Kolmogorov equation to travelling waves.
Etheridge. An introduction to superprocesses.
Durrett. Stochastic calculus: a practical introduction.

Branching Brownian motion

Defn (BBM) Branching Brownian motion has three ingredients:

(i) The spatial motion During its lifetime, each particle moves in $\mathbb{R}$ according to a Brownian motion.

(ii) The branching rate, V Each particle has a lifetime with distribution $\mathrm{Exp}(V)$.

(iii) The branching mechanism, Φ When it dies, a particle leaves behind (at the location where it died) a random number of offspring particles with probability generating function Φ. Conditional on their time and place of birth, offspring particles behave independently of each other.

Notation

- $N(t)$ is the number of particles alive at time t.
- $X_1(t), \dots, X_{N(t)}(t)$ are the locations of the particles alive at time t (e.g. in Ulam–Harris ordering)
- $\mathbb{P}_x$ is law of BBM starting with a single particle at x. $\mathbb{P} := \mathbb{P}_0$.
- $\mathbb{E}_x$ is corresponding expectation. $\mathbb{E} := \mathbb{E}_0$.

Important properties

1. Markov property. BBM is a Markov process (since Exp r.v.s are memoryless and BMs are Markov processes.
2. Branching property. Conditional on $(X_1(t), X_2(t), \dots, X_{N(t)}(t))$, the descendants of each particle alive at time t evolve according to independent BBMs.

The rightmost particle

Let $\Phi(u) = u^2$, $V = 1$ (i.e. branch into two particles at rate 1).

Let $M(t) = \max_{i \in \{1, \dots, N(t)\}} X_i(t)$.

Let $m(t) = \sup\{x \in \mathbb{R} : \mathbb{P}(M(t) \leq x) \leq 1/2\}$, the median of $M(t)$.

Theorem (Bramson, 1978) $m(t) = \sqrt{2}\,t - \frac{3}{2\sqrt{2}} \log t + \mathcal{O}(1)$ as $t \to \infty$.

See Roberts (2013) for a simpler proof.

Notation

- For $s \in [0,t]$, $X_{i,t}(s) =$ location of ancestor of $X_i(t)$ at time s.
- For $x \in \mathbb{R}$, P_x is probability measure under which $(B_t)_{t \geq 0}$ is a BM started at x, and E_x is corresponding expectation.

Lemma (Many-to-one lemma) Suppose $F : C[0,t] \to \mathbb{R}$ is measurable. Then

$$\mathbb{E}_x\Big[\sum_{i=1}^{N(t)} F((X_{i,t}(s), 0 \leq s \leq t))\Big] = e^t E_x\big[F((B_s)_{0 \leq s \leq t})\big].$$

Lemma (Many-to-one lemma) Suppose $F: C[0,t] \to \mathbb{R}$ is measurable. Then

$$\mathbb{E}_x\left[\sum_{i=1}^{N(t)} F((X_{i,t}(s), 0 \leq s \leq t))\right] = e^t \, \mathbb{E}_x\left[F((B_s)_{0 \leq s \leq t})\right].$$

Proof: Conditional on the set of branching events, each path $(X_{i,t}(s), 0 \leq s \leq t)$ is a BM.
So LHS $= \mathbb{E}_x[N(t)] \, \mathbb{E}_x[F((B_s)_{0 \leq s \leq t})]$.

Since each particle branches into two particles at rate 1,

$$\frac{d}{dt}\mathbb{E}[N(t)] = \mathbb{E}[N(t)] \quad \text{and so} \quad \mathbb{E}[N(t)] = e^t. \quad \square.$$

First moment calculation

Lemma (Gaussian tail estimates) If $Z \sim N(0,1)$, then for $x > 0$,

$$\mathbb{P}(Z > x) \leq e^{-\frac{1}{2}x^2} \quad \text{and} \quad \mathbb{P}(Z > x) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{1}{2}x^2}.$$

By the many-to-one lemma, for $y \geq 0$ with $y \ll \sqrt{t}$,

$$\begin{aligned}
\mathbb{E}[\#\{i \leq N(t) : X_i(t) \geq \sqrt{2}t - y\}] &= e^t \, \mathbb{E}_0[\mathbb{1}_{B_t \geq \sqrt{2}t - y}] \\
&= e^t \, \mathbb{P}\left(Z \geq \sqrt{2t} - \frac{y}{\sqrt{t}}\right) \quad \text{where } Z \sim N(0,1) \\
&\sim e^t \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2t}} e^{-\frac{1}{2}\left(\sqrt{2t} - \frac{y}{\sqrt{t}}\right)^2} \\
&= e^t \frac{1}{2\sqrt{\pi t}} e^{-\frac{1}{2}(2t - 2\sqrt{2}y + o(1))} \\
&\sim \frac{1}{2\sqrt{\pi t}} e^{\sqrt{2}y} \quad \text{as } t \to \infty.
\end{aligned}$$

$$\mathbb{E}[\#\{i \le N(t) : X_i(t) \ge \sqrt{2}t - y\}] \sim \frac{1}{2\sqrt{\pi t}} e^{\sqrt{2}y} \quad \text{as } t \to \infty.$$

So $\mathbb{E}[\#\{i \le N(t) : X_i(t) \ge \sqrt{2}t - y\}] = \Theta(1)$ as $t \to \infty$ iff $y = \frac{1}{2\sqrt{2}} \log t + O(1)$.

(Roberts: $\#\{i \le N(t) : X_i(t) \ge \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t,\ X_{i,t}(s) \le \xi(s)\ \forall s \le t\}$).

McKean representation

Back to general V, Φ.

Theorem (McKean (1975), Skorohod (1964))

Suppose $\Phi'(1) < \infty$. Suppose $\psi \in C^2(\mathbb{R})$ with $0 \le \psi(x) \le 1$ $\forall x \in \mathbb{R}$.

For $t \ge 0$, $x \in \mathbb{R}$, let

$$u(t,x) = \mathbb{E}_x\Big[\prod_{i=1}^{N(t)} \psi(X_i(t))\Big].$$

Then u solves the PDE

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + V(\Phi(u) - u), \quad u(0,x) = \psi(x). \qquad (1)$$

Proof: Assume that $u(t,x)$ is twice continuously differentiable in x at time t.
(Once we have shown that u solves (1) under this assumption, regularity theory for the heat semigroup implies that u is smooth.)

Let S = time of first branching event.
B_S = location of original particle at time S.
K = # offspring particles at time S.

Proof ctd: $\qquad u(t,x) = \mathbb{E}_x\left[\prod_{i=1}^{N(t)} \psi(X_i(t))\right].$

Let S = time of first branching event.

$\quad B_S$ = location of original particle at time S.

$\quad K$ = # offspring particles at time S.

Then for $\delta > 0$,

$$\begin{aligned}
u(t+\delta, x) &= \mathbb{E}_x\left[\prod_{j=1}^{N(t+\delta)} \psi(X_j(t+\delta)) \,\middle|\, S \le \delta\right] \mathbb{P}(S \le \delta) \\
&\quad + \mathbb{E}_x\left[\prod_{j=1}^{N(t+\delta)} \psi(X_j(t+\delta)) \,\middle|\, S > \delta\right] \mathbb{P}(S > \delta) \\
&= \sum_{k=0}^{\infty} \mathbb{E}_x\left[\prod_{j=1}^{N(t+\delta)} \psi(X_j(t+\delta)) \,\middle|\, S \le \delta, K=k\right] \mathbb{P}(S \le \delta, K=k) \\
&\quad + E_x\left[\mathbb{E}_{B_\delta}\left[\prod_{j=1}^{N(t)} \psi(X_j(t))\right]\right] \mathbb{P}(S > \delta).
\end{aligned} \tag{2}$$

For the first term on the RHS, for $1 \le i \le K$, let

$\quad N_i$ = set of particles at time $t+\delta$ which are descended from particle i at time S.

For $s \le \delta$ and $i \le k$,

$$\mathbb{E}_x\left[\prod_{j \in N_i} \psi(X_j(t+\delta)) \,\middle|\, (S, B_S, K) = (s, y, k)\right] = E_y\left[u(t, B_{\delta - s})\right] + \mathcal{O}(\delta)$$

since particle i branches before time δ w.p. $\mathcal{O}(\delta)$

Hence by our assumption that $u(t,\cdot)$ is C^2,

$$\mathbb{E}_x\left[\prod_{j \in N_i} \psi(X_j(t+\delta)) \,\middle|\, S \le \delta, K=k\right] = u(t,x) + \mathcal{O}(\delta).$$

Proof ctd: $$u(t,x) = \mathbb{E}_x\Big[\prod_{i=1}^{N(t)} \psi(X_i(t))\Big].$$

Hence by our assumption that $u(t,\cdot)$ is C^2,
$$\mathbb{E}_x\Big[\prod_{j\in N_i} \psi(X_j(t+\delta)) \,\Big|\, S\leq\delta,\ K=k\Big] = u(t,x) + \mathcal{O}(\delta).$$

By the branching property,
$$\mathbb{E}_x\Big[\prod_{i=1}^{N(t+\delta)} \psi(X_j(t+\delta)) \,\Big|\, S\leq\delta,\ K=k\Big] = (u(t,x)+\mathcal{O}(\delta))^k.$$

Since $S \sim \mathrm{Exp}(V)$,
$$\sum_{k=0}^{\infty} \mathbb{E}_x\Big[\prod_{j=1}^{N(t+\delta)} \psi(X_j(t+\delta)) \,\Big|\, S\leq\delta,\ K=k\Big]\ \mathbb{P}(S\leq\delta,\ K=k)$$
$$= \sum_{k=0}^{\infty} \big(u(t,x)^k + k\,\mathcal{O}(\delta)\big)\big(V\delta + \mathcal{O}(\delta^2)\big)\,\mathbb{P}(K=k)$$
$$= V\delta\,\Phi(u(t,x)) + \mathcal{O}(\delta^2) \qquad \text{since } \Phi'(1)<\infty.$$

Since $\frac{1}{2}\Delta$ is the infinitesimal generator of BM,
$$\mathbb{E}_x\Big[\mathbb{E}_{B_\delta}\Big[\prod_{j=1}^{N(t)} \psi(X_j(t))\Big]\Big]\,\mathbb{P}(S>\delta) = \mathbb{E}_x[u(t,B_\delta)]\,\mathbb{P}(S>\delta)$$
$$= \big(u(t,x) + \delta\tfrac{1}{2}\Delta u(t,x) + o(\delta)\big)\big(1 - V\delta + \mathcal{O}(\delta^2)\big).$$

Substituting into (2),
$$u(t+\delta,x) = V\delta\,\Phi(u) + u + \delta\tfrac{1}{2}\Delta u - V\delta u + o(\delta).$$

Hence $$\frac{u(t+\delta,x)-u(t,x)}{\delta} = \tfrac{1}{2}\Delta u + V(\Phi(u)-u) + o(1) \text{ as } \delta\to 0. \ \square.$$

Aside – voting schemes What if we want to solve $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + V(\Phi(u) - u)$, $u(0,x) = \psi(x)$, where Φ is not a pgf of a $\mathbb{N}_0$-valued r.v.?

e.g. $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1-u)(2u-1)$. (3)

Take a BBM with $\Phi(u) = u^3$, $V = 1$ (branch into three particles at rate 1).
Write $(X(t), t \geq 0)$ for the historical process of the BBM
(traces out the trees which record the locations of all particles alive at time s for all $s \in [0,t]$).

Majority voting:

- Each particle $i \in \{1, \dots, N(t)\}$ independently votes 1 w.p. $\psi(X_i(t))$ and otherwise votes 0.
- At each branch point in the tree traced out by the BBM, the vote of the parent is the majority vote of the three offspring.

Let $\mathbb{V}_\psi(X(t))$ = vote of original (root) particle.
Then $u(t,x) := \mathbb{P}_x(\mathbb{V}_\psi(X(t)) = 1)$ solves the PDE (3).
(depends on genealogy, not just locations of particles).

Back to the rightmost particle $\Phi(u) = u^2$, $V = 1$.

$$\mathbb{P}(M(t) \geq x) = 1 - \mathbb{P}\Big(\prod_{i=1}^{N(t)} \mathbb{1}_{X_i(t) \leq x} = 1\Big)$$

$$= 1 - \mathbb{E}_{-x}\Big[\prod_{i=1}^{N(t)} \mathbb{1}_{X_i(t) \leq 0}\Big]$$

$$= 1 - v(t,-x), \quad \text{where } v \text{ solves } \begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\Delta v + v^2 - v \\ v(0,y) = \mathbb{1}_{y \leq 0}. \end{cases}$$

Back to the rightmost particle $\Phi(u)=u^2,\ V=1.$

$$\mathbb{P}(M(t)\geq x) = 1-\mathbb{P}\Big(\prod_{i=1}^{N(t)} \mathbb{1}_{X_i(t)\leq x}=1\Big)$$

$$= 1-\mathbb{E}_{-x}\Big[\prod_{i=1}^{N(t)} \mathbb{1}_{X_i(t)\leq 0}\Big]$$

$$= 1-v(t,-x), \quad \text{where } v \text{ solves } \begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\Delta v + v^2 - v \\ v(0,y) = \mathbb{1}_{y\leq 0}. \end{cases}$$

Hence $u(t,x) := 1-v(t,-x)$ solves $\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u(1-u) \\ u(0,y) = \mathbb{1}_{y\leq 0} \end{cases}$ Fisher-KPP equation

So $\mathbb{P}(M(t)\geq x) = u(t,x).$

Feynman-Kac formula

Proposition Suppose $k:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous and bounded, that $u:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous and bounded, and smooth on $(0,\infty)\times\mathbb{R}$, and that

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + k(t,x)u \quad \forall t>0,\ x\in\mathbb{R}.$$

Then for $0\leq t'\leq t$, $x\in\mathbb{R}$,

$$u(t,x) = \mathbb{E}_x\Big[e^{\int_0^{t'} k(t-s,B_s)\,ds}\, u(t-t',B_{t'})\Big].$$