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Stable Lévy processes in a cone

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Abstract. Bañuelos and Bogdan (*Potential Anal.* **21** (3) (2004) 263–288) and Bogdan et al. (*Electron. J. Probab.* **23** (2018) 11) analyse the asymptotic tail distribution of the first time a stable (Lévy) process in dimension $d \geq 2$ exits a cone. We use these results to develop the notion of a stable process conditioned to remain in a cone as well as the the notion of a stable process conditioned to absorb continuously at the apex of a cone (without leaving the cone). As self-similar Markov processes, we examine some of their fundamental properties through the lens of its Lamperti–Kiu decomposition. In particular we are interested to understand the underlying structure of the Markov additive process (MAP) that drives such processes. Through the interrogation of the underlying MAP, we are able to provide an answer by example to the open question: If the modulator of a MAP has a stationary distribution, under what conditions does its ascending ladder MAP have a stationary distribution?

With the help of an analogue of the Riesz–Bogdan–Żak transform (cf. Bogdan and Żak (*J. Theoret. Probab.* **19** (1) (2006) 89–120), Kyprianou (*Electron. J. Probab.* **21** (2016) 23), Alili et al. (*Electron. J. Probab.* **22** (2017) 20)) as well as Hunt–Nagasawa duality theory, we show how the two forms of conditioning are dual to one another. Moreover, in the sense of Rivero (*Bernoulli* **11** (3) (2005) 471–509; *Bernoulli* **13** (4) (2007) 1053–1070) and Fitzsimmons (*Electron. Commun. Probab.* **11** (2006) 230–241), we construct the null-recurrent extension of the stable process killed on exiting a cone, showing that it again remains in the class of self-similar Markov processes. Aside from the Riesz–Bogdan–Żak transform and Hunt–Nagasawa duality, an unusual combination of the Markov additive renewal theory of e.g. Alsmeyer (*Stochastic Process. Appl.* **50** (1) (1994) 37–56) as well as the boundary Harnack principle (see e.g. *Electron. J. Probab.* **23** (2018) 11) play a central role to the analysis.

In the spirit of several very recent works (see *Stochastic Process. Appl.* **129** (3) (2019) 954–977; *Electron. J. Probab.* **21** (2016) 23; *Ann. Inst. Henri Poincaré Probab. Stat.* **54** (1) (2018) 343–362; *Potential Anal.* **53** (2020) 1347–1375; *ALEA Lat. Am. J. Probab. Math. Stat.* **15** (1) (2018) 617–690; *Ann. Probab.* **48** (3) (2020) 1220–1265), the results presented here show that previously unknown results of stable processes, which have long since been understood for Brownian motion, or are easily proved for Brownian motion, become accessible by appealing to the properties of the stable process as a self-similar Markov process, combined with its special status as a Lévy processes having semi-tractable potential analysis.

Résumé. Bañuelos et Bogdan (*Potential Anal.* **21** (3) (2004) 263–288) et Bogdan et al. (*Electron. J. Probab.* **23** (2018) 11) ont etudié la queue de distribution du temps pour qu'un processus stable (Lévy) de dimension $d \geq 2$ sorte d'un cône. Nous utilisons ces résultats pour développer le concept de processus stable conditionné à rester dans un cône ainsi que celui du processus stable conditionné à être absorbé continûment au sommet d'un cône (sans sortir de celui-ci). En tant que processus de Markov auto-similaires, nous examinons certaines de leurs propriétés fondamentales en utilisant la décomposition de Lamperti–Kiu. En particulier, nous sommes intéressés à comprendre la structure sous-jacente du processus de Markov additif (Markov Additive Process – MAP) qui dirige des tels processus. En étudiant le MAP sous-jacent, nous sommes en mesure de répondre aux questions ouvertes, telles que : si le modulateur d'un MAP a une distribution stationnaire, dans quelles conditions le processus d'échelle ascendant associé, qui est aussi un MAP, a-t-il une distribution stationnaire ?

À l'aide d'une transformation Riesz–Bogdan–Zak (cf. Bogdan and Żak (*J. Theoret. Probab.* **19** (1) (2006) 89–120), Kyprianou (*Electron. J. Probab.* **21** (2016) 23), Alili et al. (*Electron. J. Probab.* **22** (2017) 20)) ainsi que la théorie de la dualité de Hunt–Nagasawa, nous montrons comment les deux formes de conditionnement sont duales l'une de l'autre. De plus, au sens de Rivero (*Bernoulli* **11** (3) (2005) 471–509 ; *Bernoulli* **13** (4) (2007) 1053–1070) et Fitzsimmons (*Electron. Commun. Probab.* **11** (2006) 230–241), nous construisons l'extension nulle-récurrente du processus stable tué à la sortie d'un cône, montrant que celui-ci appartient encore à la classe des processus de Markov auto-similaire. Outre la transformation de Riesz–Bogdan–Zak et la dualité de Hunt–Nagasawa, une combinaison inhabituelle de la théorie du renouvellement Markovien additif, par ex. Alsmeyer (*Stochastic Process. Appl.* **50** (1) (1994) 37–56), ainsi que le principe de frontière de Harnack (voir par exemple *Electron. J. Probab.* **23** (2018) 11) jouent un rôle central dans notre analyse.

Dans l'esprit de plusieurs travaux très récents (voir *Stochastic Process. Appl.* **129** (3) (2019) 954–977 ; *Electron. J. Probab.* **21** (2016) 23 ; *Ann. Inst. Henri Poincaré Probab. Stat.* **54** (1) (2018) 343–362 ; *Potential Anal.* **53** (2020) 1347–1375 ; *ALEA Lat. Am. J. Probab. Math. Stat.* **15** (1) (2018) 617–690 ; *Ann. Probab.* **48** (3) (2020) 1220–1265), les résultats présentés ici montrent que des résultats jusqu'à maintenant inconnus pour les processus stables, deviennent accessibles en faisant appel aux propriétés du processus stable, en tant que processus de Markov auto-similaire, en conjugaison avec le fait qu'il s'agit des processus de Lévy avec une théorie du potentiel accessible. Ces résultats sont peut être bien compris depuis longtemps pour le mouvement Brownien ou bien peuvent être facilement prouvés pour celui-ci, notre approche permets donc de combler cette lacune dans la théorie des processus stables.

1. Introduction

For $d \geq 2$, let $X := (X_t : t \geq 0)$, with probabilities $\mathbb{P} = (\mathbb{P}_x, x \in \mathbb{R}^d)$, be a d-dimensional isotropic stable process of index $\alpha \in (0, 2)$. That is to say, $(X, \mathbb{P})$ is a $\mathbb{R}^d$-valued Lévy process having characteristic triplet $(0, 0, \Pi)$, where

$$\Pi(B) = \frac{2^{\alpha}\Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} \, \mathrm{d}y, \quad B \in \mathcal{B}(\mathbb{R}). \tag{1.1}$$

Equivalently, this means $(X, \mathbb{P})$ is a d-dimensional Lévy process with characteristic exponent $\Psi(\theta) = -\log \mathbb{E}_0(\mathrm{e}^{\mathrm{i}\langle\theta, X_1\rangle})$ which satisfies

$$\Psi(\theta) = |\theta|^{\alpha}, \quad \theta \in \mathbb{R}^d.$$

Stable processes are also self-similar in the sense that they satisfy a scaling property. More precisely, for $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,

under $\mathbb{P}_x$, the law of $(cX_{c^{-\alpha}t}, t \geq 0)$ is equal to $\mathbb{P}_{cx}$. \tag{1.2}

As such, stable processes are useful prototypes for the study of the class of Lévy processes and, more recently, for the study of the class of self-similar Markov processes. The latter class of processes are regular strong Markov processes which respect the scaling relation (1.2), and accordingly are identified as having self-similarity (Hurst) index $1/\alpha$.

In this article, we are interested in understanding the notion of conditioning such stable processes to remain in a Lipschitz cone,

$$\Gamma = \left\{x \in \mathbb{R}^d : x \neq 0, \arg(x) \in \Omega\right\}, \tag{1.3}$$

where Ω is open on $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$. Note that Γ is an open set which does not include its apex $\{0\}$, moreover, Ω need not be a connected domain. See [21] for the notion of Lipschitz cone and some related facts.

Our motivation comes principally from the desire to show how the rapidly evolving theory of self-similar Markov processes presents a number of new opportunities to contextualise existing theory and methodology in a completely new way to attack problems, which may have otherwise been seen as beyond reach. We note in this respect that the key tool, the Lamperti–Kiu transform for self-similar Markov processes, was formalised only recently in [1,26,42]. It identifies self-similar Markov processes as in one-to-one correspondence with Markov additive processes through a generalised polar decomposition with additional time change, and is the principal tool which, in the last five years or so, has unlocked a number of ways forward for classical problems such as the one we consider here; see [26,32–34,43,45–49]. Moreover, this new perspective opens up an entire new set of challenges both in the setting of the underlying class of Markov additive processes (which have seldom received attention in the general setting since the foundational work of e.g. Çinlar [27–29] and Kaspi [39]) as well as the general class of self-similar Markov processes. Many of these challenges also emerge naturally in the setting of other stochastic processes and random structures where self-similarity plays an inherently fundamental role; see for example [61] in the setting of multi-type fragmentation processes, [12] in the setting of growth fragmentation processes and [8,10] in the setting of planar maps. In this respect interrogating fundamental questions in the stable setting lays the foundations to springboard to problems of significantly greater generality. We mention in this respect, an outstanding problem in the setting of stable Lévy processes, which relates to the extremely deep work of e.g. [14,15,35] which showed how to condition a Brownian motion to stay in a Weyl Chamber and the important relationship this has to random matrix theory. This also inspired similar conditionings of other processes, such as those that appear in queueing theory; see [55].

Our journey towards conditioning stable processes to remain in the cone Γ will take us through a number of striking relations between stable processes killed on exiting Γ and stable processes conditioned either enter or to to absorb continuously at the apex of Γ, which are captured by space-time transformations. Our analysis will necessitate examining new families of Markov additive processes that underly the conditioned stable processes through the Lamperti–Kiu transform. As an example we will exhibit a rare and difficult result which identifies semi-explicitly the existence of a stationary distribution for the at radially extreme points of the conditioned process (showing how the harmonic functions that drive our conditionings influence the strong mixing of the angular process). Moreover, our work will complement a number of other works which have examined the notion entrance laws of self-similar Markov processes, as well as the subsequent notion of recurrent extension. Specifically, in the setting of Brownian motion and stable processes in a Lipschitz cone, this pertains to [7,21], and more generally for self-similar Markov processes, see [13,23,32,36,56,57]. Our approach will consist of an unusual mixture of techniques, coming from potential analysis and Harnack inequalities, Markov additive renewal theory, last exit decompositions *à la* Maisonneuve and Itô synthesis.

2. Harmonic functions in a cone

For simplicity, we denote κ_Γ be the exit time from the cone i.e.

$$\kappa_\Gamma = \inf\{s > 0 : X_s \notin \Gamma\}.$$

Bañuelos and Bogdan [6] and Bogdan et al. [21] analyse the tail behaviour of the stopping time κ_Γ. Let us spend a moment reviewing their findings.

Suppose that we write

$$U_\Gamma(x, \mathrm{d}y) = \int_0^\infty \mathbb{P}_x(X_t \in \mathrm{d}y, t < \kappa_\Gamma) \, \mathrm{d}t, \quad x, y \in \Gamma$$

for the potential of the stable process killed on exiting Γ.

Then it is known e.g. from Section 2 of [21] that $U_\Gamma(x, \mathrm{d}y)$ has a density, denoted by $u_\Gamma(x, y)$ and that

$$M(y) := \lim_{|x| \to \infty} \frac{u_\Gamma(x, y)}{u_\Gamma(x, y_0)}, \quad y \in \Gamma,$$

exists and depends on $y_0 \in \Gamma$ only through a normalising constant. Note that it is a consequence of this definition that $M(x) = 0$ for all $x \notin \Gamma$. Moreover, M is locally bounded on $\mathbb{R}^d$ and homogeneous of degree $\beta = \beta(\Gamma, \alpha) \in (0, \alpha)$, meaning,

$$M(x) = |x|^\beta M\big(x/|x|\big) = |x|^\beta M\big(\arg(x)\big), \quad x \neq 0. \tag{2.1}$$

It is also known that, up to a multiplicative constant, M is the unique function which is harmonic in the sense that

$$M(x) = \mathbb{E}_x\big[M(X_{\tau_B})\mathbf{1}_{(\tau_B < \kappa_\Gamma)}\big], \quad x \in \mathbb{R}^d, \tag{2.2}$$

where B is any open bounded domain and $\tau_B = \inf\{t > 0 : X_t \notin B\}$.

The function M plays a prominent role in the following asymptotic result in Corollary 3.2 of Bogdan et al. [21], which strengthens e.g. Lemma 4.2 of Bañuelos and Bogdan [6].

Proposition 2.1 (Bogdan et al. [21]). *We have*

$$\lim_{a \to 0} \sup_{x \in \Gamma, |t^{-1/\alpha}x| \leq a} \frac{\mathbb{P}_x(\kappa_\Gamma > t)}{M(x)t^{-\beta/\alpha}} = C,$$

where $C > 0$ is a constant.

Before moving to the next section we also mention the earlier works of DeBlaissie [30] and Méndez-Hernández [53], who considered moment properties of the exit time from the cone.

3. Results for stable processes conditioned to stay in a cone

The above summary of the results in [6,21] will allow us to introduce the notion of stable process conditioned to stay in Γ. Before doing that we make a slight digression to introduce some notation.

Let $\mathbb{D}$ be the space of càdlàg paths defined on $[0, \infty)$, with values in $\mathbb{R}^d \cup \Delta$, where Δ is a cemetery state. Each path $\omega \in \mathbb{D}$ is such that $\omega_t = \Delta$, for any $t \geq \inf\{s \geq 0 : \omega_s = \Delta\} =: \zeta(\omega)$. As usual we extend any function $f : \mathbb{R}^d \to \mathbb{R}$ to $\mathbb{R}^d \cup \Delta$ by taking $f(\Delta) = 0$. The space $\mathbb{D}$ is endowed with the Skohorod topology and its Borel σ-field. Here after we will denote by X the canonical process of the coordinates, and by $(\mathcal{F}_t, t \geq 0)$ the right-continuous filtration generated by X. We will also denote by $\mathbb{P}^\Gamma$ the law of the stable process $(X, \mathbb{P})$ killed when it leaves the cone Γ. Note, in particular, that $(X, \mathbb{P}^\Gamma)$ is also a self-similar Markov process.

Theorem 3.1.

(i) *For any $t > 0$, and $x \in \Gamma$,*

$$\mathbb{P}_x^\triangleleft(A) := \lim_{s \to \infty} \mathbb{P}_x(A \mid \kappa_\Gamma > t + s), \quad A \in \mathcal{F}_t,$$

defines a family of conservative probabilities on the space of càdlàg paths which respect the Doob h-transform

$$\left. \frac{d\mathbb{P}_x^\triangleleft}{d\mathbb{P}_x} \right|_{\mathcal{F}_t} := \mathbf{1}_{(t < \kappa_\Gamma)} \frac{M(X_t)}{M(x)}, \quad t \geq 0, \text{ and } x \in \Gamma. \tag{3.1}$$

In particular, the right-hand side of (3.1) *is a martingale.*

(ii) *Let $\mathbb{P}^\triangleleft := (\mathbb{P}_x^\triangleleft, x \in \Gamma)$. The process $(X, \mathbb{P}^\triangleleft)$, is a self-similar Markov process.*

Next, we want to extend the definition of the process $(X, \mathbb{P}^\triangleleft)$, to include the apex of the cone Γ as a point of issue in a similar spirit to the inclusion of the origin as a point of issue for positive and real-valued self-similar Markov processes (cf. Bertoin and Yor [13], Bertoin and Caballero [9], Caballero and Chaumont [24], Bertoin and Savov [11], Chaumont et al. [25], Dereich et al. [32]). Said another way, we want to show the consistent inclusion of the state 0 to the state space Γ in the definition of $(X, \mathbb{P}^\triangleleft)$ as both a self-similar and a Feller process.

Before stating our theorem in this respect, we must first provide a candidate law for $\mathbb{P}_0^\triangleleft$, which is consistent with the family $\mathbb{P}^\triangleleft$. To this end, we need to recall the following theorem which is a copy of Theorem 3.3 in Bogdan et al. [21] (see also [7,30] for earlier work pertaining to distributional identities concerning the entrance law from the apex of Brownian motion in a cone). In order to state it, we need to introduce the notation $p_t^\Gamma(x, y)$, $x, y \in \Gamma$, $t \geq 0$, which will denote the transition density of $(X, \mathbb{P}^\Gamma)$. Note the reason why this density exists is because of the existence of a transition density for X, say $p_t(x, y)$, $x, y \in \mathbb{R}^d$, $t \geq 0$ and the relation

$$p_t^\Gamma(x, y) := p_t(x, y) - \mathbb{E}_x\big[\kappa_\Gamma < t; p_{t-\kappa_\Gamma}(X_{\kappa_\Gamma}, y)\big], \quad x, y \in \Gamma, t > 0.$$

Theorem 3.2 (Bogdan et al. [21]). *The following limit exits,*

$$n_t(y) := \lim_{\Gamma \ni x \to 0} \frac{p_t^\Gamma(x, y)}{\mathbb{P}_x(\kappa_\Gamma > t)t^{\beta/\alpha}}, \quad x, y \in \Gamma, t > 0, \tag{3.2}$$

and $(n_t(y)\,dy, t > 0)$, serves as an entrance law to $(X, \mathbb{P}^\Gamma)$, in the sense that

$$n_{t+s}(y) = \int_\Gamma n_t(x) p_s^\Gamma(x, y)\,dx, \quad y \in \Gamma, s, t \geq 0.$$

The latter is a finite strictly positive jointly continuous function with the properties

$$n_t(y) = t^{-(d+\beta)/\alpha} n_1\big(t^{-1/\alpha} y\big) \quad \text{and} \quad n_1(y) \approx \frac{\mathbb{P}_y(\kappa_\Gamma > 1)}{(1 + |y|)^{d+\alpha}}, \quad y \in \Gamma, t > 0, \tag{3.3}$$

and

$$\int_\Gamma n_t(y)\,dy = t^{-\beta/\alpha}, \quad t > 0, \tag{3.4}$$

where $f \approx g$ means the ratio of the functions f and g are bounded from above and below by two positive constants, uniformly in their domains.

The existence of the entrance law $(n_t(y)\,dy, t > 0)$, is sufficient to build a candidate for a probability measure, say $\mathbb{P}_0^\triangleleft$, on $\mathbb{D}$ carried by the paths of X that start continuously from 0 and remain in Γ forever. To that end, note from (3.2) and Proposition 2.1, that for any $t > 0$ we have the following weak convergence,

$$\lim_{\Gamma \ni x \to 0} \frac{M(y)}{M(x)} \mathbb{P}_x(X_t \in dy, t < \kappa_\Gamma) = \lim_{\Gamma \ni x \to 0} C \frac{M(y)}{\mathbb{P}_x(t < \kappa_\Gamma) t^{\beta/\alpha}} p_t^\Gamma(x, y)\,dy$$

$$= CM(y)n_t(y)\,dy, \quad y \in \Gamma, \tag{3.5}$$

where C is the constant in Proposition 2.1, and it is independent of t. With a pre-emptive choice of notation, let us denote by $\mathbb{P}_0^\triangleleft(X_t \in dy)$ the measure that is obtained as a limit in the above relation, that is

$$\mathbb{P}_0^\triangleleft(X_t \in dy) := CM(y)n_t(y)\,dy, \quad y \in \Gamma, t > 0.$$

Recalling from (3.1) that M forms a martingale for the process killed at its first exit from Γ, we have that necessarily $\int_\Gamma \mathbb{P}_0^\triangleleft(X_t \in dy) = 1$. Furthermore, denote by $\mathbb{P}_0^\triangleleft$ the probability measure on $\mathbb{D}$ whose finite dimensional distributions are given by,

$$\mathbb{P}_0^\triangleleft(X_{t_i} \in A_i, i = 1, \ldots, n) := C \int_{A_1} M(y)n_{t_1}(y)\mathbb{P}_y^\triangleleft(X_{t_i - t_1} \in A_i, i = 2, \ldots, n)\,dy,$$

for $n \in \mathbb{N}, 0 < t_1 \leq \cdots t_n < \infty$ and Borel subsets of Γ, $A_1, \ldots, A_n$. The weak convergence in (3.5) extends in a straightforward way to the finite dimensional convergence $\mathbb{P}_x^\triangleleft \xrightarrow[\Gamma \ni x \to 0]{\text{f.d.}} \mathbb{P}_0^\triangleleft$. Our main result in this respect establishes that the convergence holds in the stronger sense of Skorohod's topology.

Theorem 3.3. *The limit $\mathbb{P}_0^\triangleleft := \lim_{\Gamma \ni x \to 0} \mathbb{P}_x^\triangleleft$ is well defined on the Skorokhod space, so that, $(X, (\mathbb{P}_x^\triangleleft, x \in \Gamma \cup \{0\}))$ is both Feller and self-similar which enters continuously at the origin, after which it never returns and subsequently remains in Γ.*

The proof of Theorem 3.3 leads us to a stronger understanding how stable processes enter into the wedge Γ from its apex, specifically showing that there is functional continuity on the Skorokhod space, complementing the existing works of [21] in the stable setting and [7,30] in the Brownian setting, which deal with specific functionals of the respective processes killed on exiting a cone.

Recalling Williams' classical decomposition of Brownian excursions from the origin, which shows that their entrance law can be constructed from the Brownian motion conditioned to stay positive (also identifiable as a Bessel-3 process), one may wonder, in light of Theorem 3.3, if it there is a self-similar process that behaves like $(X, \mathbb{P}^\Gamma)$, but once it exits Γ it is not absorbed but is reissued at 0 in a sensible way. In the specialised literature, a process bearing those characteristics would be called a *self-similar recurrent extension* of the stable process killed on exiting the cone Γ, $(X, \mathbb{P}^\Gamma)$, i.e. a $\Gamma \cup \{0\}$–valued process that behaves like $(X, \mathbb{P}^\Gamma)$ up to the first hitting time of $\partial\Gamma$, for which 0 is a recurrent and regular state, and that has the scaling property (1.2). If such a process exists, say $(\vec{X}_t, t \geq 0)$, the fact that 0 is regular for it, implies that there exists a local time at 0, say L, and an excursion measure from 0, say $\mathbf{N}^\Gamma$. We will see in Section 13 that either $\mathbf{N}^\Gamma(X_{0+} \neq 0) = 0$ or $\mathbf{N}^\Gamma(X_{0+} = 0) = 0$. In the former case, we say that the recurrent extension leaves 0 continuously, and in the latter that it leaves 0 by a jump. General results from excursion theory (see e.g. [17,37]), ensure that both objects together $(L, \mathbf{N}^\Gamma)$ characterize $\vec{X}$. Furthermore, the measure $\mathbf{N}^\Gamma$ is a *self-similar excursion measure compatible* with the transition semi-group of $(X, \mathbb{P}^\Gamma)$, that is, it is a measure on $(\mathbb{D}, \mathcal{F})$ such that

(i) it is carried by the set of Γ-valued paths that are sent to the cemetery state 0 at their death with lifetime ζ,

$$\{\chi \in \mathbb{D} \mid \zeta > 0, \chi_s \in \Gamma, s < \zeta, \text{ and } \chi_t = 0, \text{ for all } t \geq \zeta\},$$

i.e. where, tautologically, $\zeta = \inf\{t > 0 : \chi_t = 0\}$;

(ii) the Markov property under $\mathbf{N}^\Gamma$, is satisfied, that is, for every bounded $\mathcal{F}$-measurable variable Y and $A \in \mathcal{F}_t, t > 0$,

$$\mathbf{N}^\Gamma\big(Y \circ \theta_t, A \cap \{t < \zeta\}\big) = \mathbf{N}^\Gamma\big(\mathbb{E}_{X_t}^\Gamma[Y], A \cap \{t < \zeta\}\big);$$

(iii) the quantity $\mathbf{N}^\Gamma(1 - e^{-\zeta})$ is finite;

(iv) there exists a $\gamma \in (0, 1)$ such that for any $q, c > 0$,

$$\mathbf{N}^\Gamma \left(\int_0^\zeta e^{-qs} f(X_s) \, ds \right) = c^{(1-\gamma)\alpha} \mathbf{N}^\Gamma \left(\int_0^\zeta e^{-qc^\alpha s} f(cX_s) \, ds \right). \tag{3.6}$$

The condition (iv) above is equivalent to require that

(iv′) there exists $\gamma \in (0, 1)$ such that, for any $c > 0$ and $f : \Gamma \to \mathbb{R}^+$ measurable,

$$\mathbf{N}^\Gamma \big(f(X_s), s < \zeta \big) = c^{\alpha\gamma} \mathbf{N}^\Gamma \big(f\big(c^{-1}X_{c^\alpha s}\big), c^\alpha s < \zeta \big), \quad \text{for } s > 0.$$

The fact that $\mathbf{N}^\Gamma$ necessarily satisfies the above conditions is a consequence of a straightforward extension of the arguments in Section 2.2 in [56].

Conversely, Itô's synthesis theorem ensures that, given an excursion measure satisfying the conditions (i)–(iv) above, and a local time at zero, there is a self-similar recurrent extension of $(X, \mathbb{P}^\Gamma)$. Using this fact, and that the entrance law $(n_t(y) \, dy, t > 0)$ is intimately related to an excursion measure, we establish in the next result the existence of unique self-similar recurrent extension of $(X, \mathbb{P}^\Gamma)$ that leaves 0 (the apex of the cone) continuously. Furthermore, we will give a complete description of recurrent extensions that leave zero by a jump.

Theorem 3.4. *Let $\mathbf{N}^\Gamma$ be a self-similar excursion measure compatible with $(X, \mathbb{P}^\Gamma)$. We have that there exists a $\gamma \in (0, \beta/\alpha)$, a constant $a \geq 0$, and a measure π^Γ on Ω such that $a\pi^\Gamma \equiv 0$, $\int_\Omega \pi^\Gamma(d\theta)M(\theta) < \infty$, and $\mathbf{N}^\Gamma$ can be represented by, for any $t > 0$, and any $A \in \mathcal{F}_t$*

$$\mathbf{N}^\Gamma(A, t < \zeta) = a\mathbb{E}_0^\triangleleft \left[\frac{1}{M(X_t)} \mathbf{1}_A \right] + \int_0^\infty \frac{dr}{r^{1+\alpha\gamma}} \int_\Omega \pi^\Gamma(d\theta)\mathbb{E}_{r\theta}[A, t < \kappa_\Gamma]. \tag{3.7}$$

If $a > 0$, the process $(X, \mathbb{P}^\Gamma)$ has unique recurrent extension that leaves 0 continuously, and $\gamma = \beta/\alpha$.

Conversely, for each $\gamma \in (0, \beta/\alpha)$, and π^Γ a non-trivial measure satisfying the above conditions, there is a unique recurrent extension that leaves zero by a jump and such that

$$\mathbf{N}^\Gamma \big(|X_{0+}| \in dr, \arg(X_{0+}) \in d\theta \big) = \frac{dr}{r^{1+\alpha\gamma}} \pi^\Gamma(d\theta), \quad r > 0, \theta \in \Omega.$$

Finally, any self-similar recurrent extension with excursion measure $\mathbf{N}^\Gamma$ has an invariant measure

$$\tilde{\tilde{\pi}}^\Gamma(dx) := \mathbf{N}^\Gamma \left(\int_0^\zeta \mathbf{1}_{(X_t \in dx)} \, dt \right)$$

$$= a|x|^{\alpha-d-\beta} M\big(\arg(x)\big) \, dx + \int_0^\infty \frac{dr}{r^{1+\alpha\gamma}} \int_\Omega \pi^\Gamma(d\theta)\mathbb{E}_{r\theta}\left[\int_0^{\kappa_\Gamma} \mathbf{1}_{(X_t \in dx)} \, dt \right],$$

which is unique up to a multiplicative constant, and this measure is sigma-finite but not finite.

As a final remark, we note that, whilst we have provided a recurrent extension from the apex of the cone, one might also consider the possibility of a recurrent extension from the entire boundary of the cone. We know of no specific work in which there is a recurrent extension from a set rather than a point. That said, one may consider the work on censored stable processes as meeting this notion in some sense; see e.g. [20,45].

4. Auxiliary results for associated MAPs

By an $\mathbb{R} \times \mathbb{S}^{d-1}$-valued Markov additive process (MAP), we mean here that $(\xi, \Theta) = ((\xi_t, \Theta_t), t \geq 0)$ is a regular Strong Markov Process on $\mathbb{R} \times \mathbb{S}^{d-1}$ (possibly with a cemetery state), with probabilities $\mathbf{P} := (\mathbf{P}_{x,\theta}, x \in \mathbb{R}, \theta \in \mathbb{S}^{d-1})$, such that, for any $t \geq 0$, the conditional law of the process $((\xi_{s+t} - \xi_t, \Theta_{s+t}) : s \geq 0)$, given $\{(\xi_u, \Theta_u), u \leq t\}$, is that of (ξ, Θ) under $\mathbf{P}_{0,\theta}$, with $\theta = \Theta_t$. For a MAP pair (ξ, Θ), we call ξ the *ordinate* and Θ the *modulator*.

A very useful fact in the theory of self-similar Markov process is the so called Lamperti–Kiu transform, which is one of the main results in [1], extending the seminal work of Lamperti in [51], and establishes that there is a bijection between

self-similar Markov processes (ssMp) in $\mathbb{R}^d$, and $\mathbb{R} \times \mathbb{S}^{d-1}$-valued MAPs. Indeed, for any ssMp in $\mathbb{R}^d$, say Z, there exists a unique $\mathbb{R} \times \mathbb{S}^{d-1}$-valued MAP (ξ, Θ) such that Z can be represented as

$$Z_t = \begin{cases} \exp\{\xi_{\varphi(t)}\}\Theta_{\varphi(t)}, & t < I_\infty, \\ \Delta, & t \geq I_\infty, \end{cases} \tag{4.1}$$

where $I_t := \int_0^t e^{\alpha \xi_t} \, dt$, $t \geq 0$ (so that I_∞ is the almost sure monotone limit $= \lim_{t \to \infty} I_t$) and

$$\varphi(t) = \inf\left\{ s > 0 : \int_0^s e^{\alpha \xi_u} \, du > t \right\}, \quad t \geq 0. \tag{4.2}$$

Reciprocally, given a $\mathbb{R} \times \mathbb{S}^{d-1}$-valued MAP (ξ, Θ) the process defined in (4.1) is an $\mathbb{R}^d$-valued ssMp. This is known as the Lamperti–Kiu representation of the ssMp Z. When the lifetime of Z is infinite a.s. we say that law is conservative, which is equivalent to require that $I_\infty = \infty$ almost surely.

In Theorem 3.13 of [44], it was shown that the MAP underlying the stable process, for whom we will henceforth reserve the notation $\mathbf{P} = (\mathbf{P}_{x,\theta}, x \in \mathbb{R}, \theta \in \mathbb{S}^{d-1})$ for its probabilities, is a pure jump process, such that ξ and Θ jump simultaneously. Moreover, the instantaneous jump rate with respect to Lebesgue time dt when $(\xi_t, \Theta_t) = (x, \vartheta)$ is given by

$$c(\alpha) \frac{e^{yd}}{|e^y \phi - \vartheta|^{\alpha + d}} \, dy \sigma_1(d\phi), \quad t \geq 0, \tag{4.3}$$

where $\sigma_1(\phi)$ is the surface measure on $\mathbb{S}^{d-1}$ normalised to have unit mass and

$$c(\alpha) = 2^{\alpha-1} \pi^{-d} \frac{\Gamma((d+\alpha)/2)\Gamma(d/2)}{|\Gamma(-\alpha/2)|}.$$

More precisely, suppose that f is a bounded measurable function on $(0, \infty) \times \mathbb{R}^2 \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ such that $f(\cdot, \cdot, 0, \cdot, \cdot) = 0$, then, for all $\theta \in \mathbb{S}^{d-1}$,

$$\mathbf{E}_{0,\theta}\left(\sum_{s>0} f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s) \right)$$

$$= \int_{(0,\infty) \times \mathbb{R} \times \mathbb{S}^{d-1}} V_\theta(ds, dz, d\vartheta) \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \sigma_1(d\phi) \, dy \frac{c(\alpha)e^{yd}}{|e^y \phi - \vartheta|^{\alpha+d}} f(s, z, y, \vartheta, \phi), \tag{4.4}$$

where $\Delta\xi_s = \xi_s - \xi_{s-}$,

$$V_\theta(dt, dz, d\vartheta) = \mathbf{P}_{0,\theta}(\xi_t \in dz, \Theta_t \in d\vartheta) \, dt, \quad z \in \mathbb{R}, \vartheta \in \mathbb{S}^{d-1}, t > 0,$$

$\sigma_1(d\phi)$ is the surface measure on $\mathbb{S}^{d-1}$ normalised to have unit mass and

$$c(\alpha) = 2^{\alpha-1} \pi^{-d} \frac{\Gamma((d+\alpha)/2)\Gamma(d/2)}{|\Gamma(-\alpha/2)|}.$$

Similar calculations as those used in [44] show that the Lévy system (H, L) of the MAP (ξ, Θ), associated to the stable process $(X, \mathbb{P})$, see [28] for background, is given by the additive functional $H_t := t$, $t \geq 0$ and the kernel

$$L_\vartheta(d\phi, dy) := c(\alpha) \frac{e^{yd}}{|e^y \phi - \vartheta|^{\alpha+d}} \sigma_1(d\phi) \, dy. \tag{4.5}$$

So, for any positive predictable process $(G_t, t \geq 0)$, and any function f as above, one has

$$\mathbf{E}_{0,\theta}\left(\sum_{s>0} G_s f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s) \right)$$

$$= \mathbf{E}_{0,\theta}\left(\int_0^\infty ds \, G_s \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} L_{\Theta_s}(d\phi, dy) f(s, \xi_s, y, \Theta_s, \phi) \right). \tag{4.6}$$

We are interested in the characterisation of the Lévy system of the MAP associated to $(X, \mathbb{P}^{\triangleleft})$, via the Lamperti–Kiu transform. To this end, suppose now we write $\mathbf{P}^{\triangleleft} := (\mathbf{P}^{\triangleleft}_{x,\theta}, (x, \theta) \in \mathbb{R} \times \Omega)$ for the probabilities of the MAP that underly $(X, \mathbb{P}^{\triangleleft})$.

Proposition 4.1. *For any positive predictable process $(G_t, t \geq 0)$, and any function $f : (0, \infty) \times \mathbb{R}^2 \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to [0, \infty)$, bounded and measurable, such that $f(\cdot, \cdot, 0, \cdot, \cdot) = 0$, one has*

$$
\mathbf{E}^{\triangleleft}_{0,\theta} \left(\sum_{s>0} G_s f(s, \xi_{s-}, \Delta \xi_s, \Theta_{s-}, \Theta_s) \right)
$$

$$
= \mathbf{E}^{\triangleleft}_{0,\theta} \left(\int_0^\infty \mathrm{d}s\, G_s \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} L^{\triangleleft}_{\Theta_s}(\mathrm{d}\phi, \mathrm{d}y) f(s, \xi_s, y, \Theta_s, \phi) \right), \quad \forall \theta \in \Omega, \tag{4.7}
$$

where

$$
L^{\triangleleft}_{\theta}(\mathrm{d}\phi, \mathrm{d}y) := \mathrm{e}^{\beta y} \frac{M(\phi)}{M(\theta)} L_\theta(\mathrm{d}\phi, \mathrm{d}y), \quad \phi \in \mathbb{S}^{d-1}, y \in \mathbb{R}.
$$

That is to say that, under $\mathbf{P}^{\triangleleft}$, the instantaneous jump rate when $(\xi_t, \Theta_t) = (x, \vartheta)$ is

$$
c(\alpha) \frac{\mathrm{e}^{y(\beta+d)}}{|\mathrm{e}^y \phi - \theta|^{\alpha+d}} \frac{M(\phi)}{M(\vartheta)} \mathrm{d}y \sigma_1(\mathrm{d}\phi)\, \mathrm{d}t, \quad t \geq 0, \theta, \phi \in \Omega.
$$

A better understanding of the MAP that underlies $(X, \mathbb{P}^{\triangleleft})$, allows us to deduce e.g. the following result. For $a > 0$, define by

$$
\tau_a^{\ominus} = \inf\{t > 0 : |X_t| > a\}
$$

and $\overline{m}(\tau_a^{\ominus}-) = \sup\{t < \tau_a^{\ominus} : |X_t| = \sup_{s<t} |X_s|\}$ the last radial maximum before exiting the ball of radius a.

Theorem 4.2. *There exists a probability measure, υ^* on Ω, which is invariant in the sense that*

$$
\mathbb{P}^{\triangleleft}_{\upsilon^*}\big(\arg(X_{\tau_e^{\ominus}}) \in \mathrm{d}\theta\big) := \int_\Omega \upsilon^*(\mathrm{d}\phi) \mathbb{P}^{\triangleleft}_\phi\big(\arg(X_{\tau_e^{\ominus}}) \in \mathrm{d}\theta\big) = \upsilon^*(\mathrm{d}\theta), \quad \theta \in \Omega,
$$

such that, for all $x \in \Gamma$, under $\mathbb{P}^{\triangleleft}_{r\theta}$, the triple

$$
\left(\frac{X_{\overline{m}(\tau_a^{\ominus}-)}}{a}, \frac{X_{\tau_a^{\ominus}-}}{a}, \frac{X_{\tau_a^{\ominus}}}{a} \right)
$$

converges in distribution as $a \to \infty$ to a limit which is independent of r and θ and non-degenerate. Equivalently, by scaling, the triple

$$
(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})
$$

converges in distribution under $\mathbb{P}_x$, as $\Gamma \ni x \to 0$, to the same limit. More precisely, for any continuous and bounded $f : \Gamma^3 \to [0, \infty)$,

$$
\lim_{\Gamma \ni x \to 0} \mathbb{E}^{\triangleleft}_x \big[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)}) \big] = \frac{1}{\mathbb{E}^{\triangleleft}_{\upsilon^*}[\log |X_{\tau_e^{\ominus}}|]} \int_\Omega \int_0^\infty \upsilon^*(\mathrm{d}\phi)\, \mathrm{d}r\, G(r, \phi),
$$

where

$$
G(r, \phi) = \mathbb{E}^{\triangleleft}_{\mathrm{e}^{-r}\phi} \big[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)}) \mathbf{1}_{(\tau_1^{\ominus} \leq \tau_{\mathrm{e}^{1-r}}^{\ominus})} \big].
$$

The above theorem, although seemingly innocent and intuitively clear, offers us access to a very important result. In order to understand why, we must take a small diversion into radial excursion theory, as described in Kyprianou et al. [47].

Theorem 3.1 shows that $(X, \mathbb{P}^\triangleleft)$, is a self-similar Markov process. As mentioned above, it follows that it has a Lamperti–Kiu representation of the form (4.1), with an underlying MAP, say (ξ, Θ), with probabilities $\mathbf{P}^\triangleleft_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}^{d-1}$. For each $t > 0$, let $\overline{\xi}_t = \sup_{u \leq t} \xi_u$ and define

$$\mathrm{g}_t = \sup\{s < t : \xi_s = \overline{\xi}_t\} \quad \text{and} \quad \mathrm{d}_t = \inf\{s > t : \xi_s = \overline{\xi}_t\},$$

which code the left and right end points of excursions of ξ from its maximum, respectively. Then, for all $t > 0$, with $\mathrm{d}_t > \mathrm{g}_t$, we define the excursion process

$$\left(\epsilon_{\mathrm{g}_t}(s), \Theta^\epsilon_{\mathrm{g}_t}(s)\right) := (\xi_{\mathrm{g}_t+s} - \xi_{\mathrm{g}_t}, \Theta_{\mathrm{g}_t+s}), \quad s \leq \zeta_{\mathrm{g}_t} := \mathrm{d}_t - \mathrm{g}_t;$$

it codes the excursion of $(\overline{\xi} - \xi, \Theta)$ from the set $(0, \mathbb{S}^{d-1})$ which straddles time t. Such excursions live in the space $\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})$, the space of càdlàg paths with lifetime $\zeta = \inf\{s > 0 : \epsilon(s) < 0\} > 0$ such that $(\epsilon(0), \Theta^\epsilon(0)) \in \{0\} \times \mathbb{S}^{d-1}$, $(\epsilon(s), \Theta^\epsilon(s)) \in (0, \infty) \times \mathbb{S}^{d-1}$, for $0 < s < \zeta$, and $\epsilon(\zeta) \in (-\infty, 0]$.

For $t > 0$, let $R_t = \mathrm{d}_t - t$, and define the set $G = \{t > 0 : R_{t-} = 0, R_t > 0\} = \{\mathrm{g}_s : s \geq 0\}$ of the left extrema of excursions from 0 for $\overline{\xi} - \xi$. The classical theory of exit systems in Maisonneuve [52] now implies that there exist an additive functional $(\ell_t, t \geq 0)$ carried by the set of times $\{t \geq 0 : (\overline{\xi}_t - \xi_t, \Theta_t) \in \{0\} \times \mathbb{S}^{d-1}\}$, with a bounded 1-potential, and a family of *excursion measures*, $(\mathbb{N}^\triangleleft_\theta, \theta \in \mathbb{S}^{d-1})$, such that

(i) $(\mathbb{N}^\triangleleft_\theta, \theta \in \mathbb{S}^{d-1})$ is a kernel from $\mathbb{S}^{d-1}$ to $\mathbb{R} \times \mathbb{S}^{d-1}$, such that $\mathbb{N}^\triangleleft_\theta(1 - e^{-\zeta}) < \infty$ and $\mathbb{N}^\triangleleft_\theta$ is carried by the set $\{(\epsilon(0), \Theta^\epsilon(0) = (0, \theta)\}$ and $\{\zeta > 0\}$ for all $\theta \in \mathbb{S}^{d-1}$;

(ii) we have the *exit formula*

$$\mathbf{E}^\triangleleft_{x,\theta}\left[\sum_{\mathrm{g} \in G} F\left((\xi_s, \Theta_s) : s < \mathrm{g}\right) H\left((\epsilon_\mathrm{g}, \Theta^\epsilon_\mathrm{g})\right)\right]$$

$$= \mathbf{E}^\triangleleft_{x,\theta}\left[\int_0^\infty F\left((\xi_s, \Theta_s) : s < t\right) \mathbb{N}^\triangleleft_{\Theta_t}\left(H\left(\epsilon, \Theta^\epsilon\right)\right) \mathrm{d}\ell_t\right], \tag{4.8}$$

for $x \neq 0$, where F is continuous on the space of càdlàg paths $\mathbb{D}(\mathbb{R} \times \mathbb{S}^{d-1})$ and H is measurable on the space of càdlàg paths $\mathbb{U}(\mathbb{R} \times \mathbb{S}^{d-1})$;

(iii) for any $\theta \in \mathbb{S}^{d-1}$, under the measure $\mathbb{N}^\triangleleft_\theta$, the process $((\epsilon(s), \Theta^\epsilon(s)), s < \zeta)$ is Markovian with the same semigroup as (ξ, Θ) killed at its first hitting time of $(-\infty, 0] \times \mathbb{S}^{d-1}$.

The couple $(\ell, (\mathbb{N}^\triangleleft_\theta, \theta \in \mathbb{S}^{d-1}))$ is called an exit system. In Maisonneuve's original formulation, the pair ℓ and the kernel $(\mathbb{N}^\triangleleft_\theta, \theta \in \mathbb{S}^{d-1})$ is not unique, but once ℓ is chosen, the $(\mathbb{N}^\triangleleft_\theta, \theta \in \mathbb{S}^{d-1})$ is determined but for a ℓ-neglectable set, i.e. a set $\mathcal{A}$ such that

$$\mathbf{E}^\triangleleft_{x,\theta}\left[\int_{t \geq 0} \mathbf{1}_{((\overline{\xi}_s - \xi_s, \Theta_s) \in \mathcal{A})} \mathrm{d}\ell_s\right] = 0.$$

Let $(\ell_t^{-1}, t \geq 0)$ denote the right continuous inverse of ℓ, $H_t^+ := \xi_{\ell_t^{-1}}$ and $\Theta_t^+ = \Theta_{\ell_t^{-1}}$, $t \geq 0$. The strong Markov property tells us that $(\ell_t^{-1}, H_t^+, \Theta_t^+)$, $t \geq 0$, defines a Markov additive process, whose first two elements are ordinates that are non-decreasing. Rotational invariance of X implies that ξ, alone, is also a Lévy process, then the pair (ℓ^{-1}, H^+), without reference to the associated modulator Θ^+, are Markovian and play the role of the ascending ladder time and height subordinators of ξ. But here, we are more concerned with their dependency on Θ^+.

Taking account of the Lamperti–Kiu transform (4.1), it is natural to consider how the excursion of $(\overline{\xi} - \xi, \Theta)$ from $\{0\} \times \mathbb{S}^{d-1}$ translates into a radial excursion theory for the process

$$Y_t := e^{\xi_t} \Theta_t, \quad t \geq 0.$$

Ignoring the time change in (4.1), we see that the radial maxima of the process Y agree with the radial maxima of the stable process X. Indeed, an excursion of $(\overline{\xi} - \xi, \Theta)$ from $\{0\} \times \mathbb{S}^{d-1}$ constitutes an excursion of $(Y_t / \sup_{s \leq t} |Y_s|, t \geq 0)$, from $\mathbb{S}^{d-1}$, or equivalently an excursion of Y from its running radial supremum. Moreover, we see that, for all $t > 0$ such that $\mathrm{d}_t > \mathrm{g}_t$,

$$Y_{\mathrm{g}_t+s} = e^{\xi_{\mathrm{g}_t}} e^{\epsilon_{\mathrm{g}_t}(s)} \Theta^\epsilon_{\mathrm{g}_t}(s) = |Y_{\mathrm{g}_t}| e^{\epsilon_{\mathrm{g}_t}(s)} \Theta^\epsilon_{\mathrm{g}_t}(s) =: |Y_{\mathrm{g}_t}| \mathcal{E}_{\mathrm{g}_t}(s), \quad s \leq \zeta_{\mathrm{g}_t}.$$

Whilst a cluster of papers on the general theory of Markov additive processes exists in the literature from the 1970s and 1980s, see e.g. Çinlar [27–29] and Kaspi [39], as well as in the setting that Θ is a discrete process, see Asmussen [4] and Albrecher and Asmussen [5], as well as some recent advances, see the Appendix in Dereich at al. [32], relatively little is known about the fluctuations of MAPs in comparison to e.g. Lévy processes. Note the latter are a degenerate class of MAPs, in the sense that a Lévy process can be seen as MAP with constant driving process.

A good example of an open problem pertaining to the fluctuation theory of MAPs is touched upon in Theorem 4.2: Suppose that Θ has a stationary distribution, under what conditions does Θ^+ have a stationary distribution? This is a question that has been raised in general in Section 4 of the paper [40]. Below we give a complete answer in the present setting.

Theorem 4.3. *Under* $\mathbf{P}^{\triangleleft}$, *the modulator process* Θ^+ *has a stationary distribution, that is*

$$\pi^{\triangleleft,+}(\mathrm{d}\theta) := \lim_{t\to\infty} \mathbf{P}_{x,\theta}^{\triangleleft}\big(\Theta_t^+ \in \mathrm{d}\theta\big), \quad \theta \in \Omega, x \in \mathbb{R},$$

exists as a non-degenerate distributional weak limit.

Remark 4.1. The reader will also note that, thanks to the change of measure (7.2) combined with the fact that ℓ_t^{-1} is an almost surely finite stopping time and Theorem III.3.4 of [38], the ascending ladder MAP process (H^+, Θ^+), under $\mathbf{P}^{\triangleleft}$, has the property

$$\frac{\mathrm{d}\mathbf{P}_{0,\theta}^{\triangleleft}}{\mathrm{d}\mathbf{P}_{0,\theta}}\bigg|_{\sigma((H_s^+,\Theta_s^+),s\le t)} = \mathrm{e}^{\beta H_t^+}\frac{M(\Theta_t^+)}{M(\theta)}\mathbf{1}_{(t<\mathrm{k}^{\Omega,+})}, \quad t \ge 0, \tag{4.9}$$

where $\mathrm{k}^{\Omega,+} = \inf\{t > 0 : \Theta_t^+ \notin \Omega\}$ As a consequence of the existence of $\pi^{\triangleleft,+}$, we note that

$$\pi^{\Gamma,+}(\mathrm{d}\theta) := \frac{1}{M(\theta)}\pi^{\triangleleft,+}(\mathrm{d}\theta), \quad \theta \in \Omega,$$

is an invariant distribution for $(\Theta_t^+\mathbf{1}_{(t<\mathrm{k}^{\Omega,+})}, t \ge 0)$ under $\mathbf{P}$.

5. Auxiliary results for dual processes in the cone

In order to prove some of the results listed above, we will need to understand another type of conditioned process, namely the stable process conditioned to continuously absorb at the origin.

Theorem 5.1. *For* $A \in \mathcal{F}_t$, *on the space of càdlàg paths in* Γ *with a cemetery state at the apex of* Γ,

$$\mathbb{P}_x^{\triangleright}(A, t < \kappa_{\{0\}}) := \lim_{a\to 0} \mathbb{P}_x^{\triangleleft}\big(A, t < \tau_a^{\oplus} \mid \tau_a^{\oplus} < \infty\big),$$

is well defined as a stochastic process which is continuously absorbed at the apex of Γ, *where* $\kappa_{\{0\}} = \inf\{t > 0 : |X_t| = 0\}$ *and* $\tau_a^{\oplus} = \inf\{s > 0 : |X_s| < a\}$. *Moreover, for* $A \in \mathcal{F}_t$,

$$\mathbb{P}_x^{\triangleright}(A, t < \kappa_{\{0\}}) = \mathbb{E}_x\left[\mathbf{1}_{(A,t<\kappa_{\Gamma})}\frac{H(X_t)}{H(x)}\right], \quad t \ge 0, \tag{5.1}$$

where

$$H(x) = |x|^{\alpha-\beta-d}M\big(\arg(x)\big).$$

Next, we write $\mathbf{P}^{\triangleright} := (\mathbf{P}_{x,\theta}^{\triangleright}, x \in \mathbb{R}, \theta \in \Omega)$ *for the probability law of the MAP that underlies* $(X, \mathbb{P}^{\triangleright})$. *For any positive predictable process* $(G_t, t \ge 0)$, *and any function* $f : (0, \infty) \times \mathbb{R}^2 \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to \mathbb{R}$, *bounded and measurable, such that* $f(\cdot, \cdot, 0, \cdot, \cdot) = 0$, *one has*

$$\mathbf{E}_{0,\theta}^{\triangleright}\left(\sum_{s>0} G_s f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right)$$

$$= \mathbf{E}_{0,\theta}^{\triangleright}\left(\int_0^{\infty} \mathrm{d}s\, G_s \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} L_{\Theta_s}^{\triangleright}(\mathrm{d}\phi, \mathrm{d}y) f(s, \xi_s, y, \Theta_s, \phi)\right), \quad \forall \theta \in \mathbb{S}^{d-1}, \tag{5.2}$$

where

$$L_\theta^\triangleright(\mathrm{d}\phi, \mathrm{d}y) := \mathrm{e}^{\beta y} \frac{H(\phi)}{H(\theta)} L_\theta(\mathrm{d}\phi, \mathrm{d}y), \quad \phi \in \mathbb{S}^{d-1}, y \in \mathbb{R}.$$

That is to say, that under $\mathbf{P}^\triangleright$, the instantaneous jump rate when $(\xi_t, \Theta_t) = (x, \vartheta)$ is

$$c(\alpha) \frac{\mathrm{e}^{(\beta+d)y}}{|\mathrm{e}^y\phi - \theta|^{\alpha+d}} \frac{H(\phi)}{H(\vartheta)} \, \mathrm{d}y \sigma_1(\mathrm{d}\phi) \, \mathrm{d}t, \quad t > 0, \theta, \phi \in \Omega.$$

As alluded to, the process $(X, \mathbb{P}^\triangleright)$, is intimately related to the process $(X, \mathbb{P}^\triangleleft)$. This is made clear in our final main result which has the flavour of the Riesz–Bogdan–Żak transform; cf. Bogdan and Żak [22]. For the sake of reflection it is worth stating the Riesz–Bogdan–Żak transform immediately below first, recalling the definition of L-time and then our first main result in this section.

Theorem 5.2 (Riesz–Bogdan–Żak transform). *Suppose we write $Kx = x/|x|^2$, $x \in \mathbb{R}^d$ for the classical inversion of space through the sphere $\mathbb{S}^{d-1}$. Then, in dimension $d \geq 2$, for $x \neq 0$, $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_{Kx}$ is equal in law to $(X_t, t \geq 0)$ under $\mathbb{P}_x^\circ$, where*

$$\left.\frac{\mathrm{d}\mathbb{P}_x^\circ}{\mathrm{d}\mathbb{P}_x}\right|_{\sigma(X_s : s \leq t)} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \quad t \geq 0 \tag{5.3}$$

and $\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \, \mathrm{d}u > t\}$.

Hereafter, by an L-time we mean the following. Suppose that $\mathcal{G}$ is the sigma-algebra generated by X and write $\mathcal{G}(\mathbb{P}_\nu^\triangleleft)$ for its completion by the null sets of $\mathbb{P}_\nu^\triangleleft$, where ν is a randomised initial distribution. Moreover, write $\overline{\mathcal{G}} = \bigcap_\nu \mathcal{G}(\mathbb{P}_\nu^\triangleleft)$, where the intersection is taken over all probability measures on the state space of X. A finite random time k is called an L-time (generalized last exit time) if $\{s < \mathrm{k}(\omega) - t\} = \{s < \mathrm{k}(\omega_t)\}$ for all $t, s \geq 0$. (Normally, we must include in the definition of an L-time that $\mathrm{k} \leq \zeta$, where ζ is the first entry of the process to a cemetery state. However, this is not applicable for $(X, \mathbb{P}^\triangleleft)$.) The two most important examples of L-times are killing times and last exit times.

Theorem 5.3. *Consider again the transformation of space via the sphere inversion $Kx = x/|x|^2$, $x \in \mathbb{R}^d$.*

(i) *The process $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_x^\triangleleft$, $x \in \Gamma$, is equal in law to $(X_t, t < \kappa_{\{0\}})$ under $\mathbb{P}_x^\triangleright$, $x \in \Gamma$, where*

$$\eta(t) = \inf\left\{s > 0 : \int_0^s |X_u|^{-2\alpha} \, \mathrm{d}u > t\right\}, \quad t \geq 0 \tag{5.4}$$

and $\kappa_{\{0\}} = \inf\{t > 0 : X_t = 0\}$.

(ii) *Under $\mathbb{P}_0^\triangleleft$, the time reversed process*

$$\overleftarrow{X}_t := X_{(\mathrm{k}-t)-}, \quad t \leq \mathrm{k},$$

is a homogenous strong Markov process whose transitions agree with those of $(X, \mathbb{P}_x^\triangleright)$, $x \in \Gamma$, where k is an L-time of $(X, \mathbb{P}_x^\triangleleft)$, $x \in \Gamma \cup \{0\}$.

Our third main theorem considers the possibility of a recurrent extension from the origin of of $(X, \mathbb{P}^\triangleright)$, similar in spirit to Theorem 3.4.

Theorem 5.4. *Let $\mathbf{N}^\triangleright$ be a self-similar excursion measure compatible with $(X, \mathbb{P}^\triangleright)$. We have that there exists a $\gamma \in (0, \alpha^{-1}(d + 2\beta - \alpha) \wedge 1)$, a constant $a \geq 0$, and a measure $\pi^\triangleright$ on Ω such that $a\pi^\triangleright \equiv 0$, $\int_\Omega \pi^\triangleright(\mathrm{d}\theta) H(\theta) < \infty$, and $\mathbf{N}^\triangleright$ can be represented by, for any $t > 0$, and any $A \in \mathcal{F}_t$*

$$\mathbf{N}^\triangleright(A, t < \zeta) = a\mathbb{E}_0^\triangleleft\left[\frac{H(X_t)}{M(X_t)}\mathbf{1}_A\right] + \int_0^\infty \frac{\mathrm{d}r}{r^{1+\alpha\gamma}} \int_\Omega \pi^\triangleright(\mathrm{d}\theta)\mathbb{E}_{r\theta}^\triangleright[A, t < \kappa_\Gamma]. \tag{5.5}$$

If $0 < \beta < (2\alpha - d)/2$, and $a > 0$, the process $(X, \mathbb{P}^\triangleright)$ has unique recurrent extension that leaves 0 continuously. If $\beta \geq (2\alpha - d)/2$, then $a = 0$, and there is no recurrent extension that leaves 0 continuously.

Conversely, for each $\gamma \in (0, \alpha^{-1}(d + 2\beta - \alpha) \wedge 1)$, *and* $\pi^{\triangleright}$ *a non-trivial measure satisfying the above conditions, there is a unique recurrent extension that leaves zero by a jump and such that*

$$\mathbf{N}^{\triangleright}\big(|X_{0+}| \in dr, \arg(X_{0+}) \in d\theta\big) = \frac{dr}{r^{1+\alpha\gamma}}\pi^{\triangleright}(d\theta), \quad r > 0, \theta \in \Omega.$$

Finally, any self-similar recurrent extension of $(X, \mathbb{P}^{\triangleright})$ *with excursion measure* $\mathbf{N}^{\triangleright}$, *has an invariant measure*

$$\overrightarrow{\widetilde{\pi}}^{\triangleright}(dx) := \mathbf{N}^{\triangleright}\left(\int_0^{\zeta} \mathbf{1}_{(X_t \in dx)} \, dt\right)$$

$$= a|x|^{2(\alpha-d-\beta)}M(x)^2 \, dx + \int_0^{\infty} \frac{dr}{r^{1+\alpha\gamma}} \int_{\Omega} \pi^{\triangleright}(d\theta)\mathbb{E}_{r\theta}^{\triangleright}\left[\int_0^{\kappa_{\Gamma}} \mathbf{1}_{(X_t \in dx)} \, dt\right],$$

which is unique up to a multiplicative constant, and this measure is sigma-finite but not finite.

It is interesting to remark here that if the cone is such that $\beta \geq (2\alpha - d)/2$, or equivalently $(d + 2\beta - \alpha)/\alpha \geq 1$, there is no recurrent extension that leaves zero continuously. This is due to the fact that the closer β is to α the smaller the cone is. Because the process is conditioned to hit zero continuously, a process starting from zero should return too quickly to zero, forcing there to be many small excursions, whose lengths become increasingly difficult to glue end to end in any finite interval of time. We could understand this phenomena with heuristic language by saying that 'the conditioned stable process is unable to escape the gravitational attraction to the origin because of the lack of space needed to do so'.

The rest of this paper is organised as follows. In the next section we give the proof of Theorem 3.1. Thereafter, we prove the above stated results in an order which differs from their presentation. We prove Proposition 4.1 in Section 7 and then turn to the proof of Theorem 4.2 and Corollary 4.3 in Sections 8 and 9, respectively. This gives us what we need to construct the process conditioned to continuously absorb at the apex of Γ, i.e. Theorem 5.1, in Section 10. With all these tools in hand, we can establish the duality properties of Theorem 5.3 in Section 11. Duality in hand, in Section 12, we can return to the Skorokhod convergence of the conditioned process $(X, \mathbb{P}_x^{\triangleleft})$, $x \in \Gamma$, to the candidate for $(X, \mathbb{P}_0^{\triangleleft})$, described in (3.5), and prove Theorem 3.3. Finally, in Sections 13 and 14, we complete the paper by looking at the recurrent extension of the conditioned processes in Theorem 3.4 and 5.4 respectively.

6. Proof of Theorem 3.1

We break the proof into the constituent parts of the statement of the theorem.

6.1. *Proof of part* (i)

For $A \in \mathcal{F}_t$ and $0 \neq x \in \Gamma$,

$$\mathbb{P}_x^{\triangleleft}(A, t < \zeta) = \lim_{s \to \infty} \mathbb{E}_x\left[\mathbf{1}_{(A \cap \{t < \kappa_{\Gamma}\})}\frac{\mathbb{P}_{X_t}(\kappa_{\Gamma} > s)}{\mathbb{P}_x(\kappa_{\Gamma} > t + s)}\right]. \tag{6.1}$$

From Lemma 4.2 of [6], for $t^{-1/\alpha}|x| < 1$, we have the bound

$$\frac{\mathbb{P}_x(\kappa_{\Gamma} > t)}{M(x)t^{-\beta/\alpha}} \in [C^{-1}, C],$$

for some $C > 1$. Otherwise, if $t^{-1/\alpha}|x| > 1$ then, $\mathbb{P}_x(\kappa_{\Gamma} > t) \leq 1 < |x|^{\beta}t^{-\beta/\alpha}$. Hence, noting that M is uniformly bounded from above, we have that, for all $x \in \Gamma$ and $s > 0$, there is a constant C' such that $\mathbb{P}_x(\kappa_{\Gamma} > s) \leq C'|x|^{\beta}s^{-\beta/\alpha}$. Hence, for s sufficiently large, there is another constant C'' (which depends on x) such that

$$\frac{\mathbb{P}_{X_t}(\kappa_{\Gamma} > s)}{\mathbb{P}_x(\kappa_{\Gamma} > t + s)} \leq \frac{C'|X_t|^{\beta}s^{-\beta/\alpha}}{C^{-1}M(x)(t + s)^{-\beta/\alpha}} < C''|X_t|^{\beta}.$$

It is well known that X_t has all absolute moments of any order in $(0, \alpha)$; cf. Section 25 of Sato [60]. The identity (3.1) now follows from Proposition 2.1 and the Dominated Convergence Theorem. Furthermore, by construction, for any $x \in \Gamma$,

$$\mathbb{P}_x^{\triangleleft}(t < \Bbbk) = 1, \quad \forall t \geq 0.$$

It thus follows that under $\mathbb{P}^{\triangleleft}$, X has an infinite lifetime.

6.2. *Proof of part* (*ii*)

That $(X, \mathbb{P}^\triangleleft)$ is a ssMp is a consequence of $(X, \mathbb{P})$ having the scaling property and the strong Markov property. Indeed, $(X, \mathbb{P}^\triangleleft)$ is a strong Markov process, since it is obtained via an h-transform of $(X, \mathbb{P})$. To verify that it has the scaling property, let $c > 0$ and define $\tilde{X}_t := cX_{c^{-\alpha}t}$, $t \geq 0$. We have that

$$\tilde{\kappa}_\Gamma := \inf\{t > 0 : \tilde{X}_t \notin \Gamma\} = c^\alpha \kappa_\Gamma, \tag{6.2}$$

and by the scaling property

$$(\tilde{X}, \mathbb{P}_x) \stackrel{\text{Law}}{=} (X, \mathbb{P}_{cx}), \quad x \in \Gamma. \tag{6.3}$$

Considering the transition probabilities of $(X, \mathbb{P}^\triangleleft)$, we note with the help of (6.3) and (1.2) that, for bounded and measurable f,

$$\begin{aligned}
\mathbb{E}_x^\triangleleft[f(\tilde{X}_t)] &= \mathbb{E}_x\left[\mathbf{1}_{(c^{-\alpha}t < \kappa_\Gamma)} f(cX_{c^{-\alpha}t}) \frac{M(X_{c^{-\alpha}t})}{M(x)}\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_{(t < \tilde{\kappa}_\Gamma)} f(\tilde{X}_t) \frac{|\tilde{X}_t|^\beta M(\tilde{X}_t/|\tilde{X}_t|)}{|cx|^\beta M(cx/|cx|)}\right] \\
&= \mathbb{E}_{cx}\left[\mathbf{1}_{(t < \kappa_\Gamma)} f(X_t) \frac{|X_t|^\beta M(X_t/|X_t|)}{|cx|^\beta M(cx/|cx|)}\right] \\
&= \mathbb{E}_{cx}^\triangleleft[f(X_t)], \quad x \in \Gamma.
\end{aligned}$$

This last observation together with the Markov property ensures the required self-similarity of $(X, \mathbb{P}^\triangleleft)$.

7. Proof of Proposition 4.1

We use a method taken from Theorem I.3.14 of [44]. From the Lamperti–Kiu transformation (4.1), we have

$$\xi_t = \log(|X_{A(t)}|/|X_0|), \qquad \Theta_t = \frac{X_{A(t)}}{|X_{A(t)}|}, \quad t \geq 0, \tag{7.1}$$

where $A(t) = \inf\{s > 0 : \int_0^s |X_u|^{-\alpha}\, du > t\}$.

To show that (4.7) holds, we first note that, on account of the fact that $A(t)$ in (7.1) is an almost surely finite stopping time and the simple relation $A(\mathrm{k}^\Omega) = \kappa_\Gamma$, where $\mathrm{k}^\Omega = \inf\{t > 0 : \Theta_t \notin \Omega\}$, from the martingale property in (3.1), we have by Theorem III.3.4 of [38] that

$$\left.\frac{d\mathbf{P}_{x,\theta}^\triangleleft}{d\mathbf{P}_{x,\theta}}\right|_{\mathcal{G}_t} = e^{\beta(\xi_t - x)} \frac{M(\Theta_t)}{M(\theta)} \mathbf{1}_{(t < \mathrm{k}^\Omega)}, \quad t \geq 0, \tag{7.2}$$

where $\mathcal{G}_t = \sigma((\xi_s, \Theta_s), s \leq t)$, $t \geq 0$.

Now write

$$\mathbf{E}_{0,\theta}^\triangleleft\left(\sum_{s>0} G_s f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right) = \lim_{t\to\infty} \mathbf{E}_{0,\theta}\left(\mathcal{M}_t \sum_{0<s\leq t} G_s f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right)$$

where $\mathcal{M}_t = \mathbf{1}_{(t < \mathrm{k}^\Omega)} e^{\beta\xi_t} M(\Theta_t)/M(\theta)$, $t \geq 0$ is the martingale density from the change of measure (3.1). Suppose we write Σ_t for the sum term in the final expectation above. The semi-martingale change of variable formula tells us that

$$\mathcal{M}_t \Sigma_t = \mathcal{M}_0(\theta)\Sigma_0 + \int_0^t \Sigma_{s-}\, d\mathcal{M}_s + \int_0^t \mathcal{M}_{s-}\, d\Sigma_s + [\mathcal{M}, \Sigma]_t, \quad t \geq 0,$$

where $[\mathcal{M}, \Sigma]_t$ is the quadratic co-variation term. On account of the fact that $(\Sigma_t, t \geq 0)$, has bounded variation, the latter term takes the form $[\mathcal{M}, \Sigma]_t = \sum_{s\leq t} \Delta\mathcal{M}_t \Delta\Sigma_t$. As a consequence

$$\mathcal{M}_t \Sigma_t = \mathcal{M}_0(\theta)\Sigma_0 + \int_0^t \Sigma_{s-}\, d\mathcal{M}_s + \int_0^t \mathcal{M}_s\, d\Sigma_s, \quad t \geq 0. \tag{7.3}$$

Moreover, after taking expectations, as the first in integral in (7.3) is a martingale and $\Sigma_0 = 0$, the only surviving terms give us

$$
\begin{aligned}
&\mathbf{E}_{0,\theta}^{\triangleleft}\left(\sum_{s>0} G_s f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right) \\
&= \mathbf{E}_{0,\theta}\left(\sum_{s>0} \mathbf{1}_{(t<\mathsf{k}^\Gamma)} e^{\beta\xi_s} \frac{M(\Theta_s)}{M(\theta)} G_s f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right) \\
&= \mathbf{E}_{0,\theta}\left[\int_0^\infty \mathrm{d}s\, G_s \mathbf{1}_{(s<\mathsf{k}^\Gamma)} e^{\beta\xi_s} \frac{M(\Theta_s)}{M(\theta)} \int_\Omega \sigma_1(\mathrm{d}\phi) \int_\mathbb{R} \mathrm{d}y \frac{c(\alpha)e^{y(\beta+d)}}{|e^y\phi - \Theta_s|^{\alpha+d}} \frac{M(\phi)}{M(\Theta_s)} f(s, \xi_s, y, \Theta_s, \phi)\right] \\
&= \mathbf{E}_{0,\theta}^{\triangleleft}\left[\int_0^\infty \mathrm{d}s\, G_s \int_\Omega \sigma_1(\mathrm{d}\phi) \int_\mathbb{R} \mathrm{d}y \frac{c(\alpha)e^{y(\beta+d)}}{|e^y\phi - \Theta_s|^{\alpha+d}} \frac{M(\phi)}{M(\Theta_s)} f(s, \xi_s, y, \Theta_s, \phi)\right]
\end{aligned}
$$

where in the second equality we have used the jump rate (4.6) and in the third Fubini's theorem together with (7.2).

8. Proof of Theorem 4.2

At the root of our proof of Theorem 4.2, we will appeal to Markov additive renewal theory in the spirit of Alsmeyer [2,3], Kesten [41] and Lalley [50]. The radial excursion theory we have outlined in Section 4 is a natural mathematical pre-cursor to Markov additive renewal theory, however, one of the problems we have at this point in our reasoning, as it will be seen later, is that it is not yet clear whether there is a stationary behaviour for the process $(\Theta^+, \mathbf{P}^{\triangleleft})$, of the radial ascending ladder MAP. Indeed, as already discussed, we will deduce from our calculations here that a stationary distribution does indeed exist in Corollary 4.3.

We build instead an alternative Markov additive renewal theory around a naturally chosen discrete subset set ladder points which behave well into the hands of the scaling property of $(X, \mathbb{P}^{\triangleleft})$. As the proof is quite long, we break the remainder of this section into a number of steps, marked by subsections. The proof of Theorem 4.2 will thus be our.

8.1. *A discrete ladder MAP*

Under $\mathbb{P}^{\triangleleft}$, define the following sequence of stopping times,

$$
T_n := \inf\{t > T_{n-1} : |X_t| > e|X_{T_{n-1}}|\}, \quad n \geq 1,
$$

with $T_0 = 0$, and

$$
S_n = \sum_{k=1}^n A_k, \qquad A_n = \log\frac{|X_{T_n}|}{|X_{T_{n-1}}|} \quad \text{and} \quad \Xi_n = \arg(X_{T_n}), \quad n \geq 1.
$$

Note in particular that

$$
X_{T_n} = |x|e^{S_n}\Xi_n, \quad n \geq 1.
$$

Then, we claim that $((S_n, \Xi_n), n \geq 0)$, is a Markov additive renewal process. To verify this claim, we appeal principally to the strong Markov and scaling property of $(X, \mathbb{P}^{\triangleleft})$. Indeed, for any $x \in \Gamma$, we have that for any finite stopping time T, under $\mathbb{P}_x^{\triangleleft}$, the conditional law of $(X_{T+s}, s > 0)$ given $(X_u, u \leq T)$ equals that of $(|y|X_{s/|y|^\alpha}, s > 0)$ under $\mathbb{P}_{\arg(y)}^{\triangleleft}$, with $y = X_T$. Hence, for any $n \geq 0$, by construction, conditionally on $(X_u, u \leq T_n)$ we have that

$$
\begin{aligned}
T_{n+1} &= \inf\{t > T_n : |X_t| > e|X_{T_n}|\} \\
&= T_n + \inf\{s > 0 : |X_{s+T_n}| > e|X_{T_n}|\} \\
&\overset{\text{Law}}{=} T_n + \inf\{s > 0 : |X_{T_n}||\tilde{X}_{s/|X_{T_n}|^\alpha}| > e|X_{T_n}|\} \\
&= T_n + |X_{T_n}|^\alpha \tilde{T}_1,
\end{aligned}
$$

where $\tilde{X}$ depends on $(X_u, u \leq T_n)$ only through $\arg(X_{T_n})$, has the same law as $(X, \mathbb{P}^{\triangleleft}_{\arg(X_{T_n})})$, and $\tilde{T}_1 = \inf\{t > 0 : |\tilde{X}_t| > e\}$. From these facts, it follows that for any bounded and measurable f on $\mathbb{R} \times \Omega$,

$$\mathbb{E}^{\triangleleft}_x\big[f(S_{n+1} - S_n, \Xi_{n+1}) \mid (S_i, \Xi_i) : i \leq n\big]$$

$$= \mathbb{E}^{\triangleleft}_x\bigg[f\bigg(\log \frac{|X_{T_{n+1}}|}{|X_{T_n}|}, \arg(X_{T_{n+1}})\bigg) \,\Big|\, X_{T_i} : i \leq n\bigg]$$

$$= \mathbb{E}^{\triangleleft}_y\big[f\big(\log|X_{T_1}|, \arg(X_{T_1})\big)\big]\big|_{y=\Xi_n}.$$

These calculations ensure that $((S_n, \Xi_n), n \geq 0)$, is a Markov additive renewal process. Note, this computation also shows that, under $\mathbb{P}^{\triangleleft}$, the modulator $\Xi := (\Xi_n, n \geq 0)$ is also a Markov process.

8.2. *Application of Markov additive renewal theory*

Let us introduce the MAP renewal function associated to (S, Ξ), for Ω,

$$V_\theta(dr, d\phi) := \sum_{n=0}^{\infty} \mathbb{P}^{\triangleleft}_\theta(S_n \in dr, \Xi_n \in d\phi), \quad r \in \mathbb{R}, \phi \in \Omega. \tag{8.1}$$

We will next show that the joint law in Theorem 4.2 can be expressed in term of a renewal like equation involving V_θ.

Lemma 8.1. *For measurable $f : \Gamma^3 \to [0, \infty)$, we have, for $x \in \Gamma \cap B_1$,*

$$\mathbb{E}^{\triangleleft}_x\big[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\big] = \int_0^{-\log|x|} \int_\Omega V_{\arg(x)}(dr, d\phi) G\big(-\log|x| - r, \phi\big), \tag{8.2}$$

where, for $\phi \in \Omega$ and $y \geq 0$,

$$G(y, \phi) := \mathbb{E}^{\triangleleft}_{e^{-y}\phi}\big[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\mathbf{1}_{(\tau_1^{\ominus} \leq \tau_{e^{1-y}}^{\ominus})}\big]. \tag{8.3}$$

Proof. Noting that $|X_{T_n}| = |x|e^{S_n}$ and $\arg(X_{T_n}) = \Xi_n$. Appealing to the strong Markov property we get

$$\mathbb{E}^{\triangleleft}_x\big[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\mathbf{1}_{(\tau_1^{\ominus} < \infty)}\big]$$

$$= \mathbb{E}^{\triangleleft}_x\bigg[\sum_{n\geq 0} \mathbf{1}_{(T_n < \tau_1^{\ominus} \leq T_{n+1})} f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\bigg]$$

$$= \mathbb{E}^{\triangleleft}_x\bigg[\sum_{n\geq 0} \mathbf{1}_{(T_n < \tau_1^{\ominus})} \mathbb{E}_y\big[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\mathbf{1}_{(\tau_1^{\ominus} \leq T_1)}\big]_{y=X_{T_n}}\bigg]$$

$$= \mathbb{E}^{\triangleleft}_x\bigg[\sum_{n=0}^{\infty} \mathbf{1}_{(|x|e^{S_n} < 1)} \mathbb{E}_y\big[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\mathbf{1}_{(\tau_1^{\ominus} \leq T_1)}\big]_{y=|x|e^{S_n}\Xi_n}\bigg],$$

where in the first equality the indicator implies $T_n \leq m(\tau_1^{\ominus}-)$. We can thus write

$$\mathbb{E}^{\triangleleft}_x\big[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\big]$$

$$= \int_0^{-\log|x|} \int_\Omega V_{\arg(x)}(dr, d\phi) \mathbb{E}^{\triangleleft}_{|x|e^r\phi}\big[f(X_{\tau_1^{\ominus}}, X_{\tau_1^{\ominus}-}, X_{m(\tau_1^{\ominus}-)})\mathbf{1}_{(\tau_1^{\ominus} \leq \tau_{|x|e^r+1}^{\ominus})}\big],$$

which agrees with the statement of the lemma. $\square$

We now have all the elements to explain the strategy we will follow to prove Theorem 4.2. In light of the conclusion of Lemma 8.1, we will apply the Markov Additive Renewal Theorem, see for example Theorem 2.1 of Alsmeyer [2]. We state the result below for completion in a form that is more appropriate for our purposes.

Theorem 8.1 (Alsmeyer [2]). *Suppose we have the following conditions*:

(I) *The process Ξ is an aperiodic Harris recurrent Markov chain, in the sense that there exists a probability measure, $\rho(\cdot)$ on $\mathcal{B}(\Omega)$ (Borel sets in Ω) such that, for some $\lambda > 0$,*

$$\mathbb{P}_\theta^\lhd(\Xi_1 \in E) \geq \lambda\rho(E), \quad \text{for all } \Omega, E \in \mathcal{B}(\Omega). \tag{8.4}$$

(II) *Under $\mathbf{P}^\lhd$, $(\Xi_n, n \geq 0)$, has a stationary distribution, that is*

$$\upsilon^*(\mathrm{d}\theta) := \lim_{n\to\infty} \mathbf{P}_{0,\phi}^\lhd(\Xi_n \in \mathrm{d}\theta), \quad \theta \in \Omega, \phi \in \Omega,$$

exists as a non-degenerate distributional weak limit.

(III) *With υ^* as above*

$$\mathbb{E}_{\upsilon^*}^\lhd[S_1] := \int_\Omega \upsilon^*(\mathrm{d}\theta)\mathbb{E}_\theta^\lhd[S_1] < \infty. \tag{8.5}$$

(IV) *For any continuous $f : \Gamma^3 \to [0, \infty)$, such that f is uniformly bounded, the mapping $r \mapsto G(r, \phi)$ is a.e. continuous, for any ϕ fixed, and*

$$\int_\Omega \int_0^\infty \upsilon^*(\mathrm{d}\phi) \sum_{n\geq 0} \sup_{nh<r\leq(n+1)h} G(r, \phi) < \infty, \tag{8.6}$$

for some $h > 0$.

Then for υ^*-a.e. Ω,

$$\lim_{a\to 0} \mathbb{E}_{a\theta}^\lhd\big[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus-}, X_{m(\tau_1^\ominus-)})\big] = \frac{1}{\mathbb{E}_{\upsilon^*}^\lhd[S_1]} \int_\Omega \int_0^\infty \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}r\,G(r, \phi), \tag{8.7}$$

where G is as defined in (8.3).

Before showing the conditions (I)–(IV) above hold, let us show that, a slightly enhanced version of Theorem 8.1 holds. More precisely we will show first the following corollary.

Corollary 8.2. *Under the assumption that* (8.7) *and (I)–(II) are satisfied, we can remove the requirement that the limit is taken along the sequence of points $a\theta$, for $a \to 0$ and υ^*-a.e. θ in* (8.7) *and replace it by taking limits along $\Gamma \ni x \to 0$.*

We note that this corollary, once proved, also ends the proof of Theorem 4.2. In order to prove Corollary 8.2, we must first recall the following lemma and deduce a relevant corollary which deals with the Boundary Harnack Principle. This will also be useful later on.

In the current setting, the Boundary Harnack Principle can be formulated as follows (see e.g. Bogdan et al. (BHP) in [21] and Bogdan [19]).

Lemma 8.2. *Write $B_c := \{x \in \mathbb{R}^d : |x| < c\}$ for the ball of radius $c > 0$. Suppose that $u, v : \Gamma \to [0, \infty)$ are functions satisfying $u(x) = v(x) = 0$ whenever $x \in \Gamma^c \cap B_1$, and are regular harmonic on $\Gamma \cap B_1$, meaning that, for each $x \in \Gamma \cap B_1$,*

$$\mathbb{E}_x\big[u(X_{\tau_1^\ominus \wedge \kappa_\Gamma})\big] = u(x) \quad \text{and} \quad \mathbb{E}_x\big[v(X_{\tau_1^\ominus \wedge \kappa_\Gamma})\big] = v(x).$$

Suppose, moreover, that $u(x_0) = v(x_0)$ for some $x_0 \in \Gamma \cap B_{1/2}$. Then, there exists a constant $C_1 = C_1(\Gamma, \alpha)$ (which does not depend on the choice of u or v) such that,

$$C_1^{-1}v(x) \leq u(x) \leq C_1 v(x), \quad x \in \Gamma \cap B_{1/2}. \tag{8.8}$$

It is worth noting immediately that M is a regular harmonic function on $\Gamma \cap B_1$ according to the above definition. Indeed, from (3.1), the Optional Sampling Theorem and dominated convergence, using e.g. Theorem A in Blumenthal et al. [18] which ensures $\mathbb{E}_x[|X_{\tau_1^\ominus}|^\beta] < \infty$, we know that

$$\begin{aligned}
M(x) &= \lim_{t\to\infty} \mathbb{E}_x\big[|X_{t\wedge\tau_1^\ominus}|^\beta M(\arg(X_{t\wedge\tau_1^\ominus})\mathbf{1}_{(t\wedge\tau_1^\ominus<\kappa_\Gamma)}\big] \\
&= \mathbb{E}_x\big[M(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus<\kappa_\Gamma)}\big] \\
&= \mathbb{E}_x\big[M(X_{\tau_1^\ominus \wedge \kappa_\Gamma})\big].
\end{aligned}$$

As M can only be defined up to a multiplicative constant, without loss of generality, we henceforth assume there is a $x_0 \in \Gamma \cap B_{1/2}$, such that $M(x_0) = 1$.

Corollary 8.3. *Let x_0 be as above. For each $f \geq 0$ on $\mathbb{R}^d$ such that*

$$0 < \mathbb{E}_{x_0}\big[f(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}\big] < \infty,$$

there is a constant $C_1 = C_1(\Gamma, \alpha)$ (which does not depend on the choice of f) such that, for all $x \in \Gamma \cap B_{1/2}$,

$$C_1^{-1}M(x) \leq \frac{\mathbb{E}_x[f(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}]}{\mathbb{E}_{x_0}[f(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}]} \leq C_1 M(x).$$

Proof. The result follows from Lemma 8.2, in particular from the inequalities (8.8), as soon as we can verify that

$$g(x) := \frac{\mathbb{E}_x[f(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}]}{\mathbb{E}_{x_0}[f(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}]}, \quad x \in \mathbb{R}^d,$$

is regular harmonic on $\Gamma \cap B_1$.

To this end, note that the function g clearly vanishes on $\Gamma^C \cap B_1$ and is equal to a constant multiple of f on $\Gamma \cap B_1^c$ by construction and (recalling that M has been normalized so that $M(x_0) = 1$) we have $g(x_0) = M(x_0) = 1$ for some $x_0 \in \Gamma \cap B_{1/2}$. Finally, note $g(X_{\kappa_\Gamma}) = 0$ and

$$g(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} = \frac{f(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}}{\mathbb{E}_{x_0}[f(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}]}$$

almost surely and hence, for $x \in \Gamma \cap B_1$,

$$g(x) = \frac{\mathbb{E}_x[f(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}]}{\mathbb{E}_{x_0}[f(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}]} = \mathbb{E}_x\big[g(X_{\tau_1^\ominus})\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}\big] = \mathbb{E}_x\big[g(X_{\tau_1^\ominus \wedge \kappa_\Gamma})\big],$$

as required. $\qquad\square$

Let us now return to the promised proof of Corollary 8.2 which states that we can take the limits to the apex of Γ in (8.7) in a more natural way.

Proof of Corollary 8.2. For Borel sets $D \subseteq (0, \infty)$ and $E \in \Omega$, we have, with the help of scaling,

$$\mathbb{P}_\theta^\triangleleft(S_1 - 1 \in D, \Xi_1 \in E) = \mathbb{P}_{\theta/\mathrm{e}}^\triangleleft\big(\log|X_{\tau_1^\ominus}| \in D, \arg(X_{\tau_1^\ominus}) \in E\big)$$

$$= \frac{1}{M(\theta/\mathrm{e})}\mathbb{E}_{\theta/\mathrm{e}}\big[M(X_{\tau_1^\ominus})\mathbf{1}_{(\log|X_{\tau_1^\ominus}| \in D, \arg(X_{\tau_1^\ominus}) \in E, \tau_1^\ominus < \kappa_\Gamma)}\big]. \tag{8.9}$$

Note that, for $x \in \Gamma \cap B_1$, $D \in \mathcal{B}(0, \infty)$, $E \in \mathcal{B}(\Omega)$,

$$g(x; D, E) := \mathbb{E}_x\big[M(X_{\tau_1^\ominus})\mathbf{1}_{(\log|X_{\tau_1^\ominus}| \in D, \arg(X_{\tau_1^\ominus}) \in E, \tau_1^\ominus < \kappa_\Gamma)}\big]$$

is a regular harmonic function in $\Gamma \cap B_1$ (according to the definition in Lemma 8.2), which can be seen by applying the Strong Markov Property. Normalising M so that $M(\theta_0/\mathrm{e}) = 1$, for some $\theta_0 \in \Omega$, also tells us that

$$\tilde{\rho}(D, E) = \mathbb{E}_{\theta_0/\mathrm{e}}\big[M(X_{\tau_1^\ominus})\mathbf{1}_{(\log|X_{\tau_1^\ominus}| \in D, \arg(X_{\tau_1^\ominus}) \in E, \tau_1^\ominus < \kappa_\Gamma)}\big] = g(\theta_0/\mathrm{e}; D, E) \tag{8.10}$$

is a probability distribution on $(0, \infty) \times \Omega$. For convenience, we shall shortly write $\rho(E)$ in place of $\tilde{\rho}((0, \infty), E)$ on Ω.

Corollary 8.3 now tells us that, for all $x \in \Gamma \cap B_{1/2}$,

$$C_1^{-1}M(x) \leq \frac{g(x; D, E)}{g(x_0; D, E)} \leq C_1 M(x), \tag{8.11}$$

where the constant $C_1 \in (0, \infty)$ is universal and does not depend on the construction of g, nor x_0. Said another way, we have, for $\theta \in \Omega$ (which corresponds to $x = \theta/\mathrm{e}$ in (8.11)),

$$C_1^{-1} \tilde{\rho}(D, E) \leq \mathbb{P}_\theta^\triangleleft(S_1 - 1 \in D, \Xi_1 \in E) \leq \tilde{\rho}(D, E) C_1. \tag{8.12}$$

Let us now assume that A is a null set of υ^*. From (II), we know that $\mathbb{P}_{\upsilon^*}(\Xi_1 \in A) = \upsilon^*(A) = 0$. On the one hand, from (8.12) and (8.9), we note that $0 = \mathbb{P}_{\upsilon^*}(\Xi_1 \in A) \geq C_1^{-1} \rho(A)$, and hence that $\rho(A) = 0$. On the other hand, we know from (8.12) again together with the latter fact, that, $\mathbb{P}_\theta(\Xi_1 \in A) \leq C_1 \rho(A) = 0$, for all $\theta \in \Omega$. We have thus shown that the very first step of the process Ξ positions it randomly so that it is in the concentration set of the support of υ^*.

From (8.2) and (8.1), writing I for the right-hand side of (8.7), we have, for $x \in \Gamma$ such that

$$\begin{aligned}
&\left| \mathbb{E}_x^\triangleleft \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \right] - I \right| \\
&= \left| \mathbb{E}_{\arg(x)}^\triangleleft \left[\sum_{n \geq 0} \mathbf{1}_{(S_n \leq -\log|x|)} G(-\log|x| - S_n, \Xi_n) \right] - I \right| \\
&\leq G(-\log|x|, \arg(x)) + \left| \mathbb{E}_{\arg(x)}^\triangleleft \left[\sum_{n \geq 1} \mathbf{1}_{(S_n \leq -\log|x|)} G(-\log|x| - S_n, \Xi_n) \right] - I \right| \\
&\leq G(-\log|x|, \arg(x)) + \mathbb{P}_{\arg(x)}^\triangleleft(S_1 > -\log|x|) I \\
&\quad + \mathbb{E}_{\arg(x)}^\triangleleft \left[\mathbf{1}_{(S_1 \leq -\log|x|)} \left| \mathbb{E}_{\Xi_1 \mathrm{e}^{S_1}}^\triangleleft \left[\sum_{n \geq 0} \mathbf{1}_{(S_n \leq -\log|x|)} G(-\log|x| - S_n, \Xi_n) \right] - I \right| \right].
\end{aligned} \tag{8.13}$$

Taking care to note that

$$\mathbf{1}_{(\tau_1^\ominus \leq \tau_{\mathrm{e}|x|}^\ominus)} = \begin{cases} \mathbf{1}_{(\tau_1^\ominus = \tau_{\mathrm{e}|x|}^\ominus)} = \mathbf{1}_{(|X_{\tau_{\mathrm{e}|x|}^\ominus}| > 1)} & \text{if } \mathrm{e}|x| < 1, \\ 1 & \text{if } \mathrm{e}|x| \geq 1, \end{cases}$$

we have for $\mathrm{e}|x| < 1$ that

$$\begin{aligned}
G(-\log|x|, \arg(x)) &= \mathbb{E}_x^\triangleleft \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \mathbf{1}_{(\tau_1^\ominus \leq \tau_{\mathrm{e}|x|}^\ominus)} \right] \\
&\leq \|f\|_\infty \mathbb{P}_x^\triangleleft(X_{\tau_{\mathrm{e}|x|}^\ominus} > 1) \\
&= \|f\|_\infty \mathbb{P}_{\arg(x)}^\triangleleft(X_{\tau_{\mathrm{e}}^\ominus} > 1/|x|) \\
&= \|f\|_\infty \mathbb{P}_{\arg(x)}^\triangleleft(S_1 > -\log|x|).
\end{aligned}$$

Hence, back in (8.13) we obtain

$$\begin{aligned}
&\left| \mathbb{E}_x^\triangleleft \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \right] - I \right| \\
&\leq (\|f\|_\infty + I) \mathbb{P}_{\arg(x)}^\triangleleft(S_1 > -\log|x|) \\
&\quad + \mathbb{E}_{\arg(x)}^\triangleleft \left[\mathbf{1}_{(S_1 \leq -\log|x|)} \left| \mathbb{E}_{\Xi_1 \mathrm{e}^{S_1}}^\triangleleft \left[\sum_{n \geq 0} \mathbf{1}_{(S_n \leq -\log|x|)} G(-\log|x| - S_n, \Xi_n) \right] - I \right| \right].
\end{aligned} \tag{8.14}$$

Appealing to (8.12) we can develop the right-hand side of (8.14) to get

$$\begin{aligned}
&\left| \mathbb{E}_x^\triangleleft \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \right] - I \right| \\
&\leq C_1 (\|f\|_\infty + I) \rho((-\log|x|, \infty)) \\
&\quad + C_1 \int_0^\infty \int_\Omega \tilde{\rho}(\mathrm{d}r, \mathrm{d}\phi) \mathbf{1}_{(r \leq -\log|x|)} \left| \mathbb{E}_{|x|\mathrm{e}^r \phi}^\triangleleft \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \right] - I \right|.
\end{aligned}$$

Using that $\tilde{\rho}$ is a proper distribution, dominated convergence, the fact that $\mathbb{P}_\theta(\Xi_1 \in \cdot) \ll \upsilon^*(\cdot)$ and (8.7) gives us that

$$\lim_{\Gamma \ni x \to 0} \mathbb{E}_x^\triangleleft \left[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus -}, X_{m(\tau_1^\ominus -)}) \right] = \frac{1}{\mathbb{E}_{\upsilon^*}^\triangleleft[S_1]} \int_\Omega \int_0^\infty \upsilon^*(\mathrm{d}\phi) \, \mathrm{d}r \, G(r, \phi),$$

without restriction on $\arg(x)$ in relation to υ^*, as $\Gamma \ni x \to 0$. $\qquad \square$

8.3. *Verification of conditions* (*I*) *and* (*II*)

We have already verified in (8.12) that

$$\mathbb{P}_\theta^\triangleleft(\Xi_1 \in E) \geq C_1^{-1} \rho(E), \tag{8.15}$$

which implies that (8.4) holds, i.e. condition (I) is holds.

The following lemma follows directly from (8.15) and is a rewording of e.g. Theorems VII.3.2 and VII.3.6 of Asmussen [4], and it addresses precisely condition (II).

Lemma 8.3. *Under* $\mathbf{P}^\triangleleft$, $(\Xi_n, n \geq 0)$, *has a stationary distribution, that is*

$$\upsilon^*(\mathrm{d}\theta) := \lim_{n \to \infty} \mathbf{P}_{0,\phi}^\triangleleft(\Xi_n \in \mathrm{d}\theta), \quad \theta \in \Omega, \phi \in \Omega,$$

exists as a non-degenerate distributional weak limit. Hence, the condition (*II*) *is satisfied.*

Remark 8.1. Note that

$$\int_\Omega \upsilon^*(\mathrm{d}\theta) \mathbb{P}_\theta(\Xi_n \in \mathrm{d}\phi, T_n < \kappa_\Gamma) \frac{M(\phi)}{M(\theta)} = \upsilon^*(\mathrm{d}\phi),$$

which makes $\upsilon^\Gamma(\mathrm{d}\phi) = \upsilon^*(\mathrm{d}\phi)/M(\phi)$, $\phi \in \Omega$, an invariant measure for the killed semigroup $\mathbb{P}_\theta(\Xi_n \in \mathrm{d}\phi, T_n < \kappa_\Gamma)$, $n \geq 0$.

Note that, under the assumptions (I) and (II), the limiting distribution (8.7) is proper, which can be seen by taking $f = 1$, in which case

$$\begin{aligned}
\int_\Omega \int_0^\infty \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}r\, G(r, \phi) &= \int_\Omega \int_0^\infty \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}r\, \mathbb{P}_{\mathrm{e}^{-r}\phi}^\triangleleft\big(\tau_1^\ominus \leq \tau_{\mathrm{e}^{1-r}}^\ominus\big) \\
&= \int_\Omega \int_0^1 \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}r + \int_\Omega \int_1^\infty \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}r\, \mathbb{P}_{\mathrm{e}^{-r}\phi}^\triangleleft\big(\tau_1^\ominus = \tau_{\mathrm{e}^{1-r}}^\ominus\big) \\
&= 1 + \int_\Omega \int_1^\infty \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}r\, \mathbb{P}_{\mathrm{e}^{-r}\phi}^\triangleleft\big(\log|X_{\tau_{\mathrm{e}^{1-r}}^\ominus}| > 0\big) \\
&= 1 + \int_\Omega \int_1^\infty \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}r\, \mathbb{P}_\phi^\triangleleft\big(\log|\mathrm{e}^{-r} X_{\tau_{\mathrm{e}}^\ominus}| > 0\big) \\
&= 1 + \int_\Omega \int_1^\infty \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}r\, \mathbb{P}_\phi^\triangleleft(S_1 > r) \\
&= 1 + \mathbb{E}_{\upsilon^*}^\triangleleft[S_1 - 1] \\
&= \mathbb{E}_{\upsilon^*}^\triangleleft[S_1] \tag{8.16}
\end{aligned}$$

and hence the limit on the right-hand side of (8.7) is equal to unity (for $f = 1$).

8.4. *Verification of conditions* (*III*) *and* (*IV*)

We do this with two individual lemmas.

Lemma 8.4. *Condition* (*III*) *holds, i.e.* $\mathbb{E}_{\upsilon^*}^\triangleleft[S_1] < \infty$.

Proof. We can appeal to the law of first exit from a sphere given in Theorem A of Blumenthal et al. [18] to deduce that, up to constant C, which is irrelevant for our computations, and may take different values in each line of the below computation, we have the following inequalities

$$\sup_{|x| < 1/2} \mathbb{E}_x\big[M(X_{\tau_1^\ominus})\big(1 + \log|X_{\tau_1^\ominus}|\big)\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)}\big]$$

$$\leq \sup_{|x| < 1/2} \mathbb{E}_x\big[M(X_{\tau_1^\ominus})\big(1 + \log|X_{\tau_1^\ominus}|\big)\big]$$

$$= C \sup_{|x|<1/2} \int_{|y|>1} dy (1 - |x|^2)^{\alpha/2} (|y|^2 - 1)^{-\alpha/2} M(y) \frac{1 + \log |y|}{|y - x|^d}$$

$$\leq C \sup_{|x|<1/2} \int_\Omega d\theta M(\theta) \int_1^\infty dr (r^2 - 1)^{-\alpha/2} r^{d-1+\beta} \frac{1 + \log r}{|r\theta - x|^d}$$

$$= C \sup_{|x|<1/2} \int_\Omega d\theta M(\theta) \int_1^2 dr (r^2 - 1)^{-\alpha/2} r^{d-1+\beta} \frac{1 + \log r}{|r\theta - x|^d}$$

$$+ C \sup_{|x|<1/2} \int_\Omega d\theta M(\theta) \int_2^\infty dr (r^2 - 1)^{-\alpha/2} r^{d-1+\beta} \frac{1 + \log r}{|r\theta - x|^d}$$

$$=: B_1 + B_2. \tag{8.17}$$

Using that $|r\theta - x| \geq r - |x| \geq 1/2$ we can bound the first term as follows:

$$B_1 \leq 2^{2d-1+\beta} (1 + \log 2) C \int_\Omega d\theta M(\theta) \int_1^2 dr (r^2 - 1)^{-\alpha/2} < \infty.$$

To verify that the second term in (8.17) is finite also, we use that $|r\theta - x| \geq 3r/4$, by the triangle inequality, and that necessarily $\beta < \alpha$, to obtain that

$$B_2 \leq \left(\frac{4}{3}\right)^{d+\frac{\alpha}{2}} C \int_\Omega d\theta M(\theta) \int_2^\infty dr r^{\beta-\alpha-1} (1 + \log r) < \infty.$$

We can now apply the Boundary Harnack Principle in Corollary 8.3 and the scaling property to deduce that

$$\int_\Omega \upsilon^*(d\theta) \mathbb{E}_\theta^\lhd[S_1] = \int_\Omega \upsilon^*(d\theta) \mathbb{E}_\theta^\lhd[\log |X_{\tau_e^\ominus}|]$$

$$= \int_\Omega \upsilon^*(d\theta) \mathbb{E}_{\theta/e}^\lhd[\log |X_{\tau_1^\ominus}| + 1]$$

$$= \int_\Omega \upsilon^*(d\theta) \frac{\mathbb{E}_{\theta/e}[\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} M(X_{\tau_1^\ominus})(\log |X_{\tau_1^\ominus}| + 1)]}{M(\theta/e)}$$

$$< \int_\Omega \upsilon^*(d\theta) C_1 \sup_{|x|<1/2} \mathbb{E}_x[\mathbf{1}_{(\tau_1^\ominus < \kappa_\Gamma)} M(X_{\tau_1^\ominus})(1 + \log |X_{\tau_1^\ominus}|)]$$

$$< \infty,$$

where finiteness follows from (8.17). □

Lemma 8.5. *The conditions in (IV) holds.*

Proof. Let $f : \Gamma^3 \to [0, \infty)$ be a continuous and bounded function. On account of continuity of M and standard Skorokhod continuity properties of the stable process with killing at first passage times, together with the dominated convergence theorem, imply that for any $\phi \in \Gamma$ fixed, the function

$$y \mapsto G(y, \phi) := \mathbb{E}_{e^{-y}\phi}^\lhd[f(X_{\tau_1^\ominus}, X_{\tau_1^\ominus-}, X_{m(\tau_1^\ominus-)}) \mathbf{1}_{(\tau_1^\ominus \leq \tau_{e^{1-y}}^\ominus)}], \quad y > 0,$$

is continuous and bounded. Since f is assumed to be bounded it is enough to check that (8.6) holds with $f \equiv 1$. But this follows from a straightforward modification of the computation in (8.16), using that for any $\theta \in \Gamma$ fixed, the function $r \mapsto \mathbb{P}_\phi^\lhd(S_1 > r)$ is non-increasing, together with the conclusion of Lemma 8.4. □

9. Proof of Theorem 4.3

Let us define a new family of stopping times with respect to the filtration generated by $((H_t^+, \theta_t^+), t \geq 0)$. Set $\chi_0 = 0$ and

$$\chi_{n+1} = \inf\{s > \chi_n : H_s^+ - H_{\chi_n}^+ > 1\}, \quad n \geq 0.$$

We should also note that these stopping times have the property that the sequence of pairs $((S_n, \Xi_n), n \geq 0)$, agrees precisely with $((H_{\chi_n}, \Theta^+_{\chi_n}), n \geq 0)$. Moreover, it is easy to show that $((\chi_n, \Xi_n), n \geq 0)$, is a Markov additive process, and we known $(\Xi_n, n \geq 0)$ is Harris recurrent, in the sense of (I) above.

Let,

$$\mathcal{U}^\lhd_\theta(\mathrm{d}s, \mathrm{d}\phi) := \sum_{n \geq 0} \mathbf{P}^\lhd_\theta(\chi_n \in \mathrm{d}s, \Xi_n \in \mathrm{d}\phi), \quad s \geq 0, \Omega.$$

Appealing to the Markov property, we have, for Ω and bounded measurable f on Ω,

$$\mathbf{E}^\lhd_{0,\theta}[f(\Theta^+_t)] = \mathbf{E}^\lhd_{0,\theta}\left[\sum_{n \geq 0} \mathbf{1}_{(\chi_n \leq t < \chi_{n+1})} f(\Theta^+_t)\right]$$

$$= \mathbf{E}^\lhd_{0,\theta}\left[\sum_{n \geq 0} \mathbf{1}_{(\chi_n \leq t)} \mathbf{E}^\lhd_{0,\phi}[\mathbf{1}_{(u < \chi_1)} f(\Theta^+_u)]_{\phi = \Theta^+_{\chi_n}, u = t - \chi_n}\right]$$

$$= \int_0^t \int_\Omega \mathcal{U}^\lhd_\theta(\mathrm{d}s, \mathrm{d}\phi) F(t - s, \phi),$$

with $F(s, \phi) = \mathbf{E}^\lhd_{0,\phi}[\mathbf{1}_{(s \leq \chi_1)} f(\Theta^+_s)]$ which is bounded and continuous in both its arguments. Note, moreover, that

$$\int_0^\infty \int_\Omega \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}s\,F(s, \phi) = \int_0^\infty \int_\Omega \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}s\,\mathbf{E}^\lhd_{0,\phi}[\mathbf{1}_{(s \leq \chi_1)} f(\Theta^+_s)]$$

$$= \int_0^\infty \int_\Omega \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}s\,\mathbf{E}^\lhd_{0,\phi}[\mathbf{1}_{(H^+_s < 1)} f(\Theta^+_s)]\,\mathrm{d}s$$

$$= \int_\Omega \int_\Omega \upsilon^*(\mathrm{d}\phi) U^\lhd_\phi([0, 1), \mathrm{d}\theta) f(\theta),$$

where $U^\lhd_\phi(\mathrm{d}x, \mathrm{d}\theta)$, $x \geq 0$, Ω, is the ascending ladder MAP potential

$$U^\lhd_\phi(\mathrm{d}x, \mathrm{d}\theta) = \int_0^\infty \mathbf{P}^\lhd_{0,\phi}(H^+_s \in \mathrm{d}x, \Theta^+_s \in \mathrm{d}\theta)\,\mathrm{d}s.$$

As such, whenever f is bounded, we have that $\int_0^\infty \int_\Omega \upsilon^*(\mathrm{d}\phi)\,\mathrm{d}s\,F(s, \phi) < \infty$.

We also note that

$$\mathbf{E}^\lhd_{0,\upsilon^*}[\chi_1] := \int_\Omega \upsilon^*(\mathrm{d}\phi)\mathbf{E}^\lhd_{0,\phi}[\chi_1]$$

$$= \int_\Omega \upsilon^*(\mathrm{d}\phi) \int_0^\infty \mathbf{P}^\lhd_{0,\phi}(\chi_1 > t)\,\mathrm{d}t$$

$$= \int_\Omega \upsilon^*(\mathrm{d}\phi) \int_0^\infty \mathbf{P}^\lhd_{0,\phi}(H^+_t < 1)\,\mathrm{d}t$$

$$= \int_\Omega \upsilon^*(\mathrm{d}\phi) U^\lhd_\phi([0, 1), \Omega) < \infty.$$

Arguing as in the proof of Lemma 8.5, it follows that whenever f is continuous and bounded, the mapping $(s, \phi) \mapsto F(s, \phi)$, satisfies the conditions in Theorem 2.1 in [2].

As such, and on account of the fact that $(\Xi_n, n \geq 0)$ has been proved to have a stationary distribution, υ^*, we can again invoke the Markov additive renewal theorem [2] and conclude that, for υ^*-almost every Ω,

$$\lim_{t \to \infty} \mathbf{E}^\lhd_{0,\theta}[f(\Theta^+_t)] = \frac{1}{\mathbf{E}^\lhd_{0,\upsilon^*}[\chi_1]} \int_\Omega \int_\Omega \upsilon^*(\mathrm{d}\phi) U^\lhd_\phi([0, 1), \mathrm{d}\theta) f(\theta).$$

We can upgrade the previous statement to allow for all Ω by appealing to reasoning similar in fashion to the proof of Corollary 8.2. For the sake of brevity, we thus leave this as an exercise for the reader.

In conclusion, $(\Theta_t^+, t \geq 0)$ has a non-degenerate stationary distribution, which is given by

$$\pi^{\lhd,+}(\mathrm{d}\theta) = \frac{\int_\Omega \int_\Omega \upsilon^*(\mathrm{d}\phi) U_\phi^\lhd([0,1), \mathrm{d}\theta)}{\int_\Omega \upsilon^*(\mathrm{d}\phi) U_\phi^\lhd([0,1), \Omega)}, \quad \theta \in \Omega$$

as required.

10. Proof of Theorem 5.1

We first need a technical lemma. Recall that $\tau_a^\oplus := \inf\{t > 0 : |X_t| < a\}$, $a > 0$.

Lemma 10.1. *We have the following convergence,*

$$\lim_{\Gamma \ni aKx \to 0} \frac{\mathbb{P}_x^\lhd(\tau_a^\oplus < \infty)}{(|x|/a)^{\alpha-2\beta-d}} = \frac{1}{\mathbb{E}_{\upsilon^*}^\lhd[\log|X_{\tau_e^\ominus}|]} \int_\Omega \int_0^\infty \upsilon^*(\mathrm{d}\phi) \, \mathrm{d}r \mathbb{E}_{\mathrm{e}^{-r}\phi}^\lhd\big[|X_{\tau_1^\ominus}|^{\alpha-2\beta-d}\big] < \infty.$$

Proof. We first use properties from the Riesz–Bogdan–Żak transform in Theorem 5.2 and the scaling properties (1.2) and (6.2) (employed similarly for $\tau_{1/a}^\ominus$) to deduce that

$$\begin{aligned}
\mathbb{P}_x^\lhd\big(\tau_a^\oplus < \infty\big) &= \mathbb{E}_x\left[\frac{M(X_{\tau_a^\oplus})}{M(x)}; \tau_a^\oplus < \kappa_\Gamma\right] \\
&= \mathbb{E}_{Kx}^\circ\left[\frac{M(KX_{\tau_{1/a}^\ominus})}{M(\arg(x))|x|^\beta}; \tau_{1/a}^\ominus < \kappa_\Gamma\right] \\
&= \mathbb{E}_{Kx}\left[\frac{|X_{\tau_{1/a}^\ominus}|^{\alpha-\beta-d} M(\arg(X_{\tau_{1/a}^\ominus}))}{|x|^{-(\alpha-\beta-d)} M(\arg(x))}; \tau_{1/a}^\ominus < \kappa_\Gamma\right] \\
&= \mathbb{E}_{aKx}\left[\frac{|X_{\tau_1^\ominus}/a|^{\alpha-\beta-d} M(\arg(X_{\tau_1^\ominus}))}{|x|^{-(\alpha-\beta-d)} M(\arg(x))}; \tau_1^\ominus < \kappa_\Gamma\right] \\
&= \frac{|x|^{\alpha-2\beta-d}}{a^{\alpha-2\beta-d}} \mathbb{E}_{aKx}^\lhd\big[|X_{\tau_1^\ominus}|^{\alpha-2\beta-d}\big].
\end{aligned}$$

Using Theorem 4.2 with $f(x) = |x|^{\alpha-2\beta-d}\mathbf{1}_{(|x|\geq 1)}$, we thus have that

$$\begin{aligned}
&\lim_{\Gamma \ni aKx \to 0} \frac{\mathbb{P}_x^\lhd(\tau_a^\oplus < \infty)}{(|x|/a)^{\alpha-2\beta-d}} \\
&= \lim_{\Gamma \ni aKx \to 0} \mathbb{E}_{aKx}^\lhd\big[|X_{\tau_1^\ominus}|^{\alpha-2\beta-d}\big] \\
&= \frac{1}{\mathbb{E}_{\upsilon^*}^\lhd[\log|X_{\tau_e^\ominus}|]} \int_\Omega \int_0^\infty \upsilon^*(\mathrm{d}\phi) \, \mathrm{d}r \mathbb{E}_{\mathrm{e}^{-r}\phi}^\lhd\big[|X_{\tau_1^\ominus}|^{\alpha-2\beta-d}\mathbf{1}_{(\tau_1^\ominus \leq \tau_{\mathrm{e}^{1-r}}^\ominus)}\big] < \infty
\end{aligned} \tag{10.1}$$

where we have used that $\alpha - 2\beta - d < 0$, $|X_{\tau_1^\ominus}| \geq 1$ and Theorem 4.2 with $f(x) = |x|^{\alpha-2\beta-d}\mathbf{1}_{(|x|\geq 1)}$, for $x \in \Gamma$. The result now follows. $\qquad\square$

Returning now to the proof of Theorem 5.1, the usual application of the strong Markov property means we need to evaluate, for $x \in \Gamma$,

$$\mathbb{P}_x^\rhd(A, t < \kappa_{\{0\}}) = \lim_{a \to 0} \mathbb{E}_x^\lhd\left[\mathbf{1}_{(A, t < \tau_a^\oplus)} \frac{\mathbb{P}_{X_t}^\lhd(\tau_a^\oplus < \infty)}{\mathbb{P}_x^\lhd(\tau_a^\oplus < \infty)}\right], \tag{10.2}$$

where $A \in \mathcal{F}_t$. In order to do so, we first note from Lemma 10.1 that,

$$\lim_{a \to 0} \frac{\mathbb{P}_{X_t}^\lhd(\tau_a^\oplus < \kappa_\Gamma)}{\mathbb{P}_x^\lhd(\tau_a^\oplus < \kappa_\Gamma)} = \frac{|X_t|^{\alpha-2\beta-d}}{|x|^{\alpha-2\beta-d}}.$$

Moreover, from (10.1), we also see that, for each $\varepsilon > 0$, there exists a constant $\Delta > 0$ such that, when $|aKx| = (a/|x|) < \Delta$,

$$(1 - \varepsilon)C_1 \leq \frac{\mathbb{P}_x^{\triangleleft}(\tau_a^{\oplus} < \infty)}{(|x|/a)^{\alpha - 2\beta - d}} \leq (1 + \varepsilon)C_1.$$

With C_1 as in the previous lemma. On the other hand, if $(a/|x|) \geq \Delta$, then

$$\mathbb{P}_x^{\triangleleft}(\tau_a^{\oplus} < \infty) \leq 1 \leq (a/|x|)^{2\beta + d - \alpha} \Delta^{\alpha - 2\beta - d}.$$

From this we conclude that there is an appropriate choice of constant C such that for $a \ll 1$,

$$\frac{\mathbb{P}_{X_t}^{\triangleleft}(\tau_a^{\oplus} < \infty)}{\mathbb{P}_x^{\triangleleft}(\tau_a^{\oplus} < \infty)} \leq C|X_t|^{\alpha - 2\beta - d}. \tag{10.3}$$

We want to show that

$$\mathbb{E}_x^{\triangleleft}\big[|X_t|^{\alpha - 2\beta - d}\big] = \mathbb{E}_x\left[\frac{M(\arg(X_t))}{M(x)}|X_t|^{\alpha - \beta - d}\mathbf{1}_{(t < \kappa_\Gamma)}\right]$$

$$\leq \frac{C}{M(x)}\mathbb{E}_x\big[|X_t|^{\alpha - \beta - d}\mathbf{1}_{(t < \kappa_\Gamma)}\big] < \infty. \tag{10.4}$$

To this end, we note that, since $\alpha - \beta - d < 0$, $\mathbb{E}_x[|X_t|^{\alpha - \beta - d}\mathbf{1}_{(|X_t| \geq 1, t < \kappa_\Gamma)}] \leq 1$. The problem thus lies in showing that $\mathbb{E}_x[|X_t|^{\alpha - \beta - d}\mathbf{1}_{(|X_t| < 1, t < \kappa_\Gamma)}] < \infty$. To this end, let us recall from Bogdan et al. [21] that $(X_t^\Gamma, t \geq 0)$, the stable process killed on exiting Γ, has a semigroup density, say $p_t^\Gamma(x, y)$, $x, y \in \Gamma$. Moreover, in equation (10) and (53) of the aforesaid reference, they showed that

$$p_t^\Gamma(x, t) \approx \mathbb{P}_x^\Gamma(\kappa_\Gamma > t)\mathbb{P}_y^\Gamma(\kappa_\Gamma > t)\left(t^{1/\alpha} \wedge \frac{t}{|y|^{\alpha + d}}\right), \quad x, y \in \Gamma,$$

where $p_t^\Gamma(x, y)$ is the transition density of $(X, \mathbb{P}^\Gamma)$ and $f(x, t) \approx g(x, t)$ means that, uniformly in the domains of f and g, there exists a constant $c > 0$ such that $c^{-1} \leq f/g \leq c$. It thus follows that

$$\mathbb{E}_x\big[|X_t|^{\alpha - \beta - d}\mathbf{1}_{(|X_t| < 1, t < \kappa_\Gamma)}\big]$$

$$\leq C\mathbb{P}_x^\Gamma(\kappa_\Gamma > t)\int_{|y| \leq 1}\mathbb{P}_y^\Gamma(\kappa_\Gamma > t)\left(t^{1/\alpha} \wedge \frac{t}{|y|^{\alpha + d}}\right)|y|^{\alpha - \beta - d}\,dy$$

$$\leq Ct^{1/\alpha}\mathbb{P}_x^\Gamma(\kappa_\Gamma > t)\int_0^1 r^{\alpha - \beta - 1}\,dr < \infty,$$

using $\alpha > \beta$, where the constant C has a different value in each line of the calculation above, but otherwise is unimportant.

The bound (10.3) and the finite moment (10.4) can now be used in conjunction with the Dominated Convergence Theorem (for the inner limit) followed by the Monotone Convergence Theorem (for the outer limit) in (10.2) to deduce

$$\mathbb{P}_x^{\triangleright}(A, t < \kappa_{\{0\}}) = \mathbb{E}_x^{\triangleleft}\left[\mathbf{1}_A\frac{|X_t|^{\alpha - 2\beta - d}}{|x|^{\alpha - 2\beta - d}}\right] = \mathbb{E}_x\left[\mathbf{1}_{(A, t < \kappa_\Gamma)}\frac{H(X_t)}{H(x)}\right], \quad t \geq 0, \tag{10.5}$$

with $H(x) = |x|^{\alpha - \beta - d}M(\arg(x))$ as required.

We must also show that this process is continuously absorbed at 0. Noting that $H(x) = |x|^{\alpha - d}M(Kx)$, applying the Riesz–Bogdan–Żak transform (cf. Theorem 5.2), for continuous and bounded $f : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ and $0 < a < |x|$,

$$\mathbb{E}_x^{\triangleright}\big[f(X_{\underline{m}(\tau_a^{\oplus}-)}, X_{\tau_a^{\oplus}})\big] = \mathbb{E}_x^{\circ}\left[f(X_{\underline{m}(\tau_a^{\oplus}-)}, X_{\tau_a^{\oplus}})\mathbf{1}_{(\tau_a^{\oplus} < \kappa_\Gamma)}\frac{M(KX_{\tau_a^{\oplus}})}{M(Kx)}\right]$$

$$= \mathbb{E}_{Kx}\left[f(KX_{\overline{m}(\tau_{1/a}^{\ominus}-)}, KX_{\tau_{1/a}^{\ominus}})\mathbf{1}_{(\tau_{1/a}^{\ominus} < \kappa_\Gamma)}\frac{M(X_{\tau_{1/a}^{\ominus}})}{M(Kx)}\right]$$

$$= \mathbb{E}_{Kx}^{\triangleleft}\big[f(KX_{\overline{m}(\tau_{1/a}^{\ominus}-)}, KX_{\tau_{1/a}^{\ominus}})\big],$$

where, for $a > 0$, $\underline{m}(\tau_a^\oplus-) = \sup\{t < \tau_a^\oplus : |X_t| = \inf_{s<t}|X_s|\}$ and $\overline{m}(\tau_a^\ominus-) = \sup\{t < \tau_a^\ominus : |X_t| = \inf_{s<t}|X_s|\}$. From Theorem 4.2 it follows that the limit on the right-hand side above is equal to $f(0,0)$. This shows $(X, \mathbb{P}_x^\triangleright)$, $x \in \Gamma$ is almost surely absorbed continuously at 0.

Finally, reconsidering the proof of Proposition 4.1, the remaining statement is straightforward to prove in the same way. This concludes the proof of Theorem 5.1.

11. Proof of Theorem 5.3

It turns out more convenient to prove Theorem 5.3 before we deal with Theorem 3.3. Indeed, it will play a crucial role in its proof.

11.1. *Proof of Theorem 5.3 (i)*

We can verify the statement of this part of the theorem by first noting that the transformation $(KX_{\eta(t)}, t \geq 0)$ maps $(X, \mathbb{P}^\triangleleft)$, to a new self-similar process. Then we verify it has the transitions of $(X, \mathbb{P}^\triangleright)$.

For the first of the aforesaid, we refer back to the Lamperti–Kiu transform. As already observed in Alili et al. [1] and Kyprianou [44], from the Lamperti–Kiu representation of $(X, \mathbb{P}^\triangleleft)$,

$$KX_{\eta(t)} = e^{-\xi_{\varphi\circ\eta(t)}}\Theta_{\varphi\circ\eta(t)}, \quad t \geq 0.$$

Note however that

$$\int_0^{\varphi(t)} e^{\alpha\xi_s}\,ds = t \quad \text{and} \quad \int_0^{\eta(t)} e^{-2\alpha\xi_{\varphi(u)}}\,du = t, \quad t \geq 0.$$

A straightforward differentiation of the last two integrals shows that, respectively,

$$\frac{d\varphi(t)}{dt} = e^{-\alpha\xi_{\varphi(t)}} \quad \text{and} \quad \frac{d\eta(t)}{dt} = e^{2\alpha\xi_{\varphi\circ\eta(t)}}, \quad t \geq 0,$$

and so the chain rule now tells us

$$\frac{d(\varphi\circ\eta)(t)}{dt} = \frac{d\varphi(s)}{ds}\bigg|_{s=\eta(t)}\frac{d\eta(t)}{dt} = e^{\alpha\xi_{\varphi\circ\eta(t)}}, \tag{11.1}$$

and hence, $\varphi\circ\eta(t) = \inf\{s > 0 : \int_0^s e^{-\alpha\xi_u}\,du > t\}$. It is thus clear that $(KX_{\eta(t)}, t \geq 0)$ is a self-similar Markov process with underlying MAP equal to $(-\xi, \Theta)$.

To verify it has the same transitions as $(X, \mathbb{P}^\triangleright)$, we note that $(\eta(t), t \geq 0)$, is a sequence of stopping times

$$\mathbb{E}_{Kx}^\triangleleft\left[f(KX_{\eta(t)})\right]$$
$$= \mathbb{E}_{Kx}\left[\frac{M(X_{\eta(t)})}{M(Kx)}f(KX_{\eta(t)}); \eta(t) < \kappa_\Gamma\right]$$
$$= \mathbb{E}_{Kx}\left[\frac{|KX_{\eta(t)}|^{-\beta}M(\arg(KX_{\eta(t)}))}{|x|^{-\beta}M(\arg(x))}f(KX_{\eta(t)}); \eta(t) < \kappa_\Gamma\right]$$
$$= \mathbb{E}_x\left[\frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}\frac{|X_t|^{-\beta}M(\arg(X_t))}{|x|^{-\beta}M(\arg(x))}f(X_t); t < \kappa_\Gamma\right]$$
$$= \mathbb{E}_x\left[\frac{|X_t|^{\alpha-d-\beta}M(\arg(X_t))}{|x|^{\alpha-d-\beta}M(\arg(x))}f(X_t); t < \kappa_\Gamma\right]$$
$$= \mathbb{E}_x^\triangleright\left[f(X_t)\right]$$

where in the third equality we have applied the regular Riesz–Bogdan–Żak transform (cf. Theorem 5.2) and in the final equality we have used Theorem 5.1.

11.2. *Proof of Theorem* 5.3 (*ii*)

The proof of this part appeals to Theorem 3.5 of Nagasawa [54]. The aforesaid classical result gives directly the conclusion of part (ii) as soon as a number of conditions are satisfied. Most of the conditions are trivially satisfied thanks to the fact that $(X, \mathbb{P}^\triangleleft)$, is a regular Markov process (see for example the use of this theorem in Bertoin and Savov [11] or Döring and Kyprianou [33]). However the two most important conditions stand out as non-trivial and require verification here.

In the current context, the first condition requires the existence of a sigma-finite measure μ such that the duality relation is satisfied, for any $f, g : \Gamma \to \mathbb{R}$ measurable and bounded, one has

$$\int_\Gamma \mu(dx) f(x) \int_\Gamma dy p_t^\triangleleft(x, y) g(y) = \int_\Gamma \mu(dx) g(x) \int_\Gamma dy p_t^\triangleright(x, y) f(y), \quad \forall t \geq 0, \tag{11.2}$$

and the second requires that

$$\mu(dx) = G^\triangleleft(0, dx) := \int_0^\infty \mathbb{P}_0^\triangleleft(X_t \in dx) \, dt, \quad x \in \Gamma. \tag{11.3}$$

Our immediate job is thus to understand the analytical shape of the measure μ. To this end, we prove the following intermediary result, the conclusion of which automatically deals with (11.3).

Lemma 11.1. *We have for bounded and measurable $f : \Gamma \to [0, \infty)$, which is compactly supported in Γ, up to a multiplicative constant,*

$$\int_\Gamma f(x) G^\triangleleft(0, dx) = \int_\Gamma f(x) M(x) H(x) \, dx.$$

Proof. Referring to some of the facts displayed in Theorem 3.2, we have with the help of Fubini's Theorem, the scaling properties of the transition density p^Γ, and (3.5) that

$$\int_\Gamma f(y) \int_0^\infty \mathbb{P}_0^\triangleleft(X_t \in dy) \, dt$$

$$= \int_0^\infty dt \int_\Gamma f(y) M(y) n_t(y) \, dy$$

$$= \int_0^\infty dt \int_\Gamma f(y) M(y) \lim_{x \to 0} \frac{p_t^\Gamma(x, y)}{\mathbb{P}_x(\kappa_\Gamma > 1)} \, dy$$

$$= \int_0^1 dt \int_\Gamma f(y) M(y) \lim_{x \to 0} \frac{p_t^\Gamma(x, y)}{\mathbb{P}_x(\kappa_\Gamma > 1)} \, dy + \int_1^\infty dt \int_\Gamma f(y) M(y) \lim_{x \to 0} \frac{p_t^\Gamma(x, y)}{\mathbb{P}_x(\kappa_\Gamma > 1)} \, dy$$

$$= \int_0^1 dt \int_\Gamma f(y) M(y) \lim_{x \to 0} \frac{p_t^\Gamma(x, y)}{\mathbb{P}_x(\kappa_\Gamma > 1)} \, dy$$

$$\quad + \int_1^\infty dt \int_\Gamma f(y) M(y) \lim_{x \to 0} \frac{t^{-d/\alpha} p_1^\Gamma(t^{-1/\alpha} x, t^{-1/\alpha} y)}{\mathbb{P}_x(\kappa_\Gamma > 1)} \, dy. \tag{11.4}$$

We wish to use Dominated Convergence theorem to pull the limit out of each of the integrals. Referring again to Theorem 3.2, and recalling the compactness of the support of f, the integrand in the first term on the righthand side of (11.4) is uniformly bounded.

For the second term on the right-hand side of (11.4), we can assume without loss of generality that the support of f lies in $\Gamma \cap \{x \in \mathbb{R}^d : |x| < 1\}$. Recall again from the bound in Lemma 4.2 of [6], which states that, for $t > 1$ and $|x| < 1$, there exists a constant $C > 0$ such that

$$C^{-1} t^{-\beta/\alpha} M(x) < \mathbb{P}_{t^{-1/\alpha} x}(\kappa_\Gamma > 1) < C t^{-\beta/\alpha} M(x).$$

Using the above, and appealing in particular to equation (4.16) of Bogdan et al. [21], for $t, |x| > 1$ and $y \in \Gamma$,

$$\frac{t^{-d/\alpha} p_1^\Gamma(t^{-1/\alpha} x, t^{-1/\alpha} y)}{\mathbb{P}_x(\kappa_\Gamma > 1)} < \frac{t^{-d/\alpha} p_1^\Gamma(t^{-1/\alpha} x, t^{-1/\alpha} y)}{M(x)}$$

$$< t^{-(d+\beta)/\alpha} C \frac{\mathbb{P}_{t^{-1/\alpha} y}(\kappa_\Gamma > 1)}{(1 + t^{-1/\alpha}|y|)^{d+\alpha}}$$

$$< t^{-(d+2\beta)/\alpha}.$$

The right-hand side above can now be used as part of a dominated convergence argument for the second term in (11.4), noting in particular that f is compactly supported.

In conclusion, we have

$$\int_\Gamma f(y) \int_0^\infty \mathbb{P}_0^\triangleleft(X_t \in \mathrm{d}y)\,\mathrm{d}t$$

$$= \lim_{x \to 0} \int_\Gamma f(y) M(y) \int_0^1 \mathrm{d}t\, \frac{p_t^\Gamma(x, y)}{\mathbb{P}_x(\kappa_\Gamma > 1)}\,\mathrm{d}y$$

$$+ \lim_{x \to 0} \int_\Gamma f(y) M(y) \int_1^\infty \mathrm{d}t\, \frac{p_t^\Gamma(x, y)}{\mathbb{P}_x(\kappa_\Gamma > 1)}\,\mathrm{d}y$$

$$= \lim_{x \to 0} \frac{\int_\Gamma f(y) M(y) G^\Gamma(x, y)}{M(x)}. \tag{11.5}$$

The above limit has already been computed in Lemma 3.5 of Bogdan et al. [21] and agrees with the conclusion of this lemma. $\qquad\square$

To complete the proof of part (ii) of Theorem 5.3, we must show (11.2). To this end, let us start by recalling Hunt's switching identity for X as a symmetric process and κ_Γ as a hitting time of an open domain. It ensures that for any $f, g : \Gamma \to \mathbb{R}$ measurable and bounded one has

$$\int_\Gamma \mathrm{d}x f(x) \int_\Gamma \mathrm{d}y p_t^\Gamma(x, y) g(y) = \int_\Gamma \mathrm{d}x g(x) \int_\Gamma \mathrm{d}y p_t^\Gamma(x, y) f(y), \quad \forall t \geq 0.$$

With this in hand, it is easy to check that

$$\int_\Gamma \mu(\mathrm{d}x) f(x) \int_\Gamma \mathrm{d}y p_t^\triangleleft(x, y) g(y) = \int_\Gamma \mathrm{d}x f(x) M(x) H(x) \int_\Gamma p_t^\Gamma(x, y) g(y) \frac{M(y)}{M(x)}$$

$$= \int_\Gamma \mathrm{d}x g(x) M(x) H(x) \int_\Gamma p_t^\Gamma(x, y) f(y) \frac{H(y)}{H(x)}$$

$$= \int_\Gamma \mu(\mathrm{d}x) g(x) \int_\Gamma \mathrm{d}y p_t^\triangleright(x, y) f(y), \quad t \geq 0,$$

as required.

12. Proof of Theorem 3.3

To prove the weak convergence on the Skorokhod space of $\mathbb{P}_x$, as $x \to 0$, to $\mathbb{P}_0$, we appeal to the following proposition, lifted from Dereich et al. [32] and written in the language of the present context.

Proposition 12.1. *Recall* $\tau_\varepsilon^\ominus = \inf\{t : |X_t| \geq \varepsilon\}$, $\varepsilon > 0$. *Suppose that the following conditions hold*:

(a) $\lim_{\varepsilon \to 0} \limsup_{\Gamma \ni z \to 0} \mathbb{E}_z^\triangleleft[\tau_\varepsilon^\ominus] = 0$
(b) $\lim_{\Gamma \ni z \to 0} \mathbb{P}_z^\triangleleft(X_{\tau_\varepsilon^\ominus} \in \cdot) =: \mu_\varepsilon(\cdot)$ *exists for all* $\varepsilon > 0$
(c) $\mathbb{P}_0^\triangleleft$-*almost surely,* $X_0 = 0$ *and* $X_t \neq 0$ *for all* $t > 0$
(d) $\mathbb{P}_0^\triangleleft((X_{\tau_\varepsilon^\ominus + t})_{t \geq 0} \in \cdot) = \int_\Gamma \mu_\varepsilon(\mathrm{d}y) \mathbb{P}_y^\triangleleft(\cdot)$ *for every* $\varepsilon > 0$.

Then the mapping

$$\Gamma \ni z \mapsto \mathbb{P}_z^\triangleleft$$

is continuous in the weak topology on the Skorokhod space.

Verification of Condition (a). Define $G^\triangleleft(x, y)$ via the relation

$$\int_\Gamma f(y)G^\triangleleft(x, y)\,\mathrm{d}y = \mathbb{E}_x^\triangleleft\left[\int_0^\infty f(X_t)\,\mathrm{d}t\right],$$

and note that $G^\triangleleft(x, y) = M(y)G^\Gamma(x, y)/M(x)$, $x, y \in \Gamma$. Then, for f positive, bounded, measurable and compactly supported and $x \in \Gamma \cup \{0\}$, then (11.5) and Lemma 11.1 tells us that

$$\lim_{\Gamma \ni z \to 0}\int_\Gamma f(y)G^\triangleleft(z, y)\,\mathrm{d}y = \int_\Gamma f(y)G^\triangleleft(0, y)\,\mathrm{d}y.$$

Now note that

$$\begin{aligned}
\lim_{\varepsilon \to 0}\limsup_{\Gamma \ni z \to 0}\mathbb{E}_z^\triangleleft\left[\tau_\varepsilon^\ominus\right] &= \lim_{\varepsilon \to 0}\limsup_{\Gamma \ni z \to 0}\mathbb{E}_z^\triangleleft\left[\int_0^{\tau_\varepsilon^\ominus}\mathbf{1}_{(|X_t|<\varepsilon)}\,\mathrm{d}t\right]\\
&\le \lim_{\varepsilon \to 0}\limsup_{\Gamma \ni z \to 0}\mathbb{E}_z^\triangleleft\left[\int_0^\infty\mathbf{1}_{(|X_t|<\varepsilon)}\,\mathrm{d}t\right]\\
&\le \lim_{\varepsilon \to 0}\limsup_{\Gamma \ni z \to 0}\int_{|y|<\varepsilon}G^\triangleleft(z, y)\,\mathrm{d}y\\
&\le C\lim_{\varepsilon \to 0}\int_{|y|<\varepsilon}H(y)M(y)\,\mathrm{d}y\\
&\le C\lim_{\varepsilon \to 0}\int_\Omega\sigma_1(\mathrm{d}\theta)M(\theta)^2\int_0^\varepsilon r^{\alpha-\beta-1}\,\mathrm{d}r\\
&\le C\lim_{\varepsilon \to 0}\varepsilon^{\alpha-\beta}\\
&= 0,
\end{aligned}$$

where $C \in (0, \infty)$ is an unimportant constant which changes its value in each line and $\sigma_1(\mathrm{d}\theta)$ is the surface measure on $\mathbb{S}^{d-1}$ normalised to have unit mass.

Verification of Condition (b). This condition is covered by Theorem 4.2. Note, moreover, that μ_ε does not depend on ε.

Verification of Condition (c). This condition is covered by Theorem 5.3.

Verification of Condition (d). We have that, for $|x| < \eta < \varepsilon$, from the Strong Markov Property,

$$\mathbb{E}_x^\triangleleft\left[f(X_{\tau_\varepsilon^\ominus+t} : t \ge 0)\right] = \mathbb{E}_x^\triangleleft\left[\mathbb{E}_{X_{\tau_\varepsilon^\ominus}}\left[f(X_t : t \ge 0)\right]\right] = \mathbb{E}_x^\triangleleft\left[g(X_{\tau_\eta^\ominus})\right] \tag{12.1}$$

for bounded, measurable f, where

$$g(y) = \mathbb{E}_y^\triangleleft\left[\mathbb{E}_{X_{\tau_\varepsilon^\ominus}}^\triangleleft\left[f(X_t : t \ge 0)\right]\right]$$

is bounded and measurable. From Theorem 4.2 and the Skorokhod continuity of X, $\mathbb{P}^\triangleleft$, which follows from the Lamperti–Kiu representation (4.1), we can take limits in (12.1) to get

$$\mathbb{E}_0^\triangleleft\left[f((X_{\tau_\varepsilon^\ominus+t} : t \ge 0)\right] = \mathbb{E}_0^\triangleleft\left[\mathbb{E}_{X_{\tau_\varepsilon^\ominus}}\left[f(X_t : t \ge 0)\right]\right] = \mathbb{E}_0^\triangleleft\left[g(X_{\tau_\eta^\ominus})\right]. \tag{12.2}$$

Now appealing to Theorem 5.3 (ii), thanks to càdlàg paths, we know that $X_{\tau_\eta^\ominus} \to 0$ almost surely under $\mathbb{P}_0^\triangleleft$. As a consequence, we can appeal to the Dominated Convergence Theorem in (12.2), together with condition (a) and, again, the Skorokhod continuity of X under $\mathbb{P}_y^\triangleleft$, $y \in \Gamma$, and deduce the statement in condition (d).

13. Itô synthesis and proof of Theorem 3.4

The basis of Theorem 3.4 is the classical method of Itô synthesis of Markov processes and an extension of the main ideas in [56]. That is to say, the technique of piecing together excursions end to end under appropriate conditions, whilst ensuring that the strong Markov property holds. In our case, we are also charged with ensuring that self-similarity is preserved as well. We split the proof of Theorem 3.4 into the construction of the recurrent extension and the existence and characterisation of a stationary distribution.

13.1. *Some general facts on self-similar recurrent extensions*

As described before the statement of Theorem 3.4, according to Itô's synthesis theory a self-similar recurrent extension of $(X, \mathbb{P}^{\Gamma})$ can be build from a self-similar excursion measure, i.e. a measure on $\mathbb{D}$ satisfying the conditions (i)–(iv) stated just before Theorem 3.4.

Suppose that $\mathbf{N}^{\Gamma}$ is a self-similar excursion measure compatible with the semigroup of $(X, \mathbb{P}^{\Gamma})$. Define a Poisson point process $((s, \chi_s), s > 0)$ on $(0, \infty) \times \mathbb{D}$ with intensity $\mathrm{d}t \times \mathbf{N}^{\Gamma}(\mathrm{d}\chi)$ and let each excursion length be denoted by

$$\zeta_s := \inf\{t > 0 : \chi_s(t) = 0\} > 0.$$

Then, via the subordinator

$$\varsigma_t = \sum_{s \leq t} \zeta_s, \quad t \geq 0,$$

we can define a local time process at 0 by

$$L_t = \inf\{r > 0 : \varsigma_r > t\}, \quad t \geq 0.$$

Note, for each $t \geq 0$, by considering the Laplace transform of ς_t, Campbell's formula and the assumption that $\mathbf{N}^{\Gamma}(1 - e^{-\zeta}) < \infty$ ensures that $(\varsigma_t, t \geq 0)$ is well defined as a subordinator with jump measure given by $\nu(\mathrm{d}s) = \mathbf{N}^{\Gamma}(\zeta \in \mathrm{d}s)$, $s > 0$.

Now, we define $(\widetilde{\overrightarrow{X}}_t, t \geq 0)$ with the following pathwise construction. For $t \geq 0$, let $L_t = s$, then $\varsigma_{s-} \leq t \leq \varsigma_{s-}$ and define

$$\widetilde{\overrightarrow{X}}_t := \begin{cases} \Delta_s(t - \varsigma_{s-}), & \text{if } \varsigma_{s-} < \varsigma_s, \\ 0, & \text{if } \varsigma_{s-} = \varsigma_s \text{ or } s = 0. \end{cases}$$

Salisbury [58,59] demonstrates how the process constructed above preserves the Markov property. In fact, one can easily adapt the arguments provided by Blumenthal [17], who considers only $[0, \infty)$ valued processes, to show that, under some regularity hypotheses on the semigroup of the minimal process $(X, \mathbb{P}^{\Gamma})$, the process constructed above is a Feller process. This is due to the fact that, here we are considering an extension from Γ to $\Gamma \cup \{0\}$, for $(X, \mathbb{P}^{\Gamma})$, which, by decomposing this process into polar coordinates, is equivalent to extend the radial part from $(0, \infty)$ to $[0, \infty)$.

To thus verify the Feller property, suppose that $C_0(\Gamma)$ is the space of continuous functions on Γ vanishing at 0 and ∞, and we write $(\mathcal{P}_t^{\Gamma}, t \geq 0)$ for the semigroup of $(X, \mathbb{P}^{\Gamma})$. The aforesaid regularity hypothesis needed to adapt the argument given by Blumenthal [17] are:

(i) If $f \in C_0(\Gamma)$, then $\mathcal{P}_t^{\Gamma} f \in C_0(\Gamma)$ and $\mathcal{P}_t^{\Gamma} f \mapsto f$ uniformly as $t \to 0$;
(ii) For each $q > 0$, the mapping $x \mapsto \mathbb{E}_x^{\Gamma}[e^{-q\zeta}]$ is continuous in Γ;
(iii) The following limits hold;

$$\lim_{\Gamma \ni x \to 0} \mathbb{E}_x^{\Gamma}[e^{-\zeta}] = 1 \quad \text{and} \quad \lim_{x \in \Gamma, |x| \to \infty} \mathbb{E}_x^{\Gamma}[e^{-\zeta}] = 0.$$

All of these are easily verified using the Lamperti–Kiu representation of $(X, \mathbb{P}^{\Gamma})$.

Now that we know that the process $(\widetilde{\overrightarrow{X}}_t, t \geq 0)$ defined above is a strong Markov process, in fact a Feller process, we should verify that such a process has the scaling property. But this is a consequence of the condition (iv) above, as can be easily verified using the arguments in the proof of Lemma 2 in [56].

We will next describe all the excursion measures $\mathbf{N}^{\Gamma}$, compatible with $(X, \mathbb{P}^{\Gamma})$. To that end, we recall that the entrance law $(\mathbf{N}_t^{\Gamma}(\mathrm{d}y), t > 0)$ of an excursion measure $\mathbf{N}^{\Gamma}$, is defined by

$$\mathbf{N}_t^{\Gamma}(\mathrm{d}y) := \mathbf{N}^{\Gamma}(X_t \in \mathrm{d}y, t < \zeta), \quad t > 0.$$

Lemma 13.1. *Let* $\mathbf{N}^{\Gamma}$ *be a self-similar excursion measure compatible with* $(X, \mathbb{P}^{\Gamma})$, *and* γ *the index appearing in* (iv). *Then, its entrance law admits the following representation: there is a constant* $a \geq 0$, *such that for all* $t > 0$ *and any* $f : \Gamma \mapsto \mathbb{R}^+$ *continuous and bounded*

$$\mathbf{N}^{\Gamma}\big(f(X_t), t < \zeta\big) = a \lim_{|x| \to 0} \frac{\mathbb{E}_x[f(X_t), t < \kappa_{\Gamma}]}{M(x)}$$

$$+ \int_{|y| > 0} \mathbf{N}^{\Gamma}(X_{0+} \in \mathrm{d}y) \mathbb{E}_y\big[f(X_t), t < \kappa_{\Gamma}\big]. \tag{13.1}$$

Furthermore, there is a measure π^Γ on Ω such that

$$\mathbf{N}^\Gamma\left(|X_{0+}| \in \mathrm{d}r, \arg(X_{0+}) \in \mathrm{d}\theta\right) = \frac{\mathrm{d}r}{r^{1+\alpha\gamma}}\pi^\Gamma(\mathrm{d}\theta), \tag{13.2}$$

and $\int_\Omega \pi^\Gamma(\mathrm{d}\theta)M(\theta) < \infty$. Finally, necessarily $\gamma \in (0,1)$, and $\gamma \le \beta/\alpha$; if $\gamma = \beta/\alpha$ then the measure $\pi^\Gamma \equiv 0$, whilst if $\gamma < \beta/\alpha$, then $a \equiv 0$.

Proof. In order to prove the decomposition (13.1) we start by noticing that for all $s, t > 0$, we have

$$\mathbf{N}^\Gamma\left(f(X_t), t < \zeta\right) = \mathbf{N}^\Gamma\left(\lim_{s\to 0} f(X_{s+t}), s + t < \zeta\right),$$

which is a consequence of the dominated convergence theorem, since

$$\mathbf{N}^\Gamma\left(f(X_{s+t}), s + t < \zeta\right) \le \|f\|\mathbf{N}^\Gamma(s + t < \zeta) \le \|f\|\mathbf{N}^\Gamma(t < \zeta) < \infty,$$

since $\mathbf{N}^\Gamma(t < \zeta)$ is always finite for any $t > 0$ because

$$\infty > \mathbf{N}^\Gamma\left(1 - \mathrm{e}^{-\zeta}\right) > \mathbf{N}^\Gamma\left(1 - \mathrm{e}^{-\zeta}, t < \zeta\right) > \left(1 - \mathrm{e}^{-t}\right)\mathbf{N}^\Gamma(t < \zeta).$$

The former, together with the Markov property under $\mathbf{N}^\Gamma$, implies that

$$\begin{aligned}
\mathbf{N}^\Gamma\left(f(X_t), t < \zeta\right) &= \mathbf{N}^\Gamma\left(\lim_{s\to 0} f(X_{s+t}), s + t < \zeta\right) \\
&= \lim_{s\to 0}\mathbf{N}^\Gamma\left(f(X_{s+t}), s + t < \zeta\right) \\
&= \lim_{s\to 0}\mathbf{N}^\Gamma\left(\mathbb{E}_{X_s}\left[f(X_t), t < \kappa_\Gamma\right], s < \zeta\right) \\
&= \lim_{\epsilon\to 0}\lim_{s\to 0}\mathbf{N}^\Gamma\left(\mathbb{E}_{X_s}\left[f(X_t), t < \kappa_\Gamma\right], |X_s| < \epsilon, s < \zeta\right) \\
&\quad + \lim_{\epsilon\to 0}\lim_{s\to 0}\mathbf{N}^\Gamma\left(\mathbb{E}_{X_s}\left[f(X_t), t < \kappa_\Gamma\right], |X_s| > \epsilon, s < \zeta\right) \\
&= \lim_{\epsilon\to 0}\lim_{s\to 0}\mathbf{N}^\Gamma\left(M(X_s)\frac{\mathbb{E}_{X_s}[f(X_t), t < \kappa_\Gamma]}{M(X_s)}, |X_s| < \epsilon, s < \zeta\right) \\
&\quad + \int_{y\in\Gamma, |y|>0}\mathbf{N}^\Gamma(X_{0+} \in \mathrm{d}y)\mathbb{E}_y\left[f(X_t), t < \kappa_\Gamma\right];
\end{aligned}$$

where in the final equality we used the continuity of the mapping $y \mapsto \mathbb{E}_y[f(X_t), t < \kappa_\Gamma]$, $y \in \Gamma$. Corollary 3.2 and Theorem 3.3 in [21] implies that the limit $\lim_{|x|\to 0}\mathbb{E}_x[f(X_t), t < \kappa_\Gamma]/M(x)$ exists. This implies from the right hand side above that

$$\mathbf{N}^\Gamma\left(f(X_t), t < \zeta\right) = a \lim_{|x|\to 0}\frac{\mathbb{E}_x[f(X_t), t < \kappa_\Gamma]}{M(x)} + \int_{y\in\Gamma, |y|>0}\mathbf{N}^\Gamma(X_{0+} \in \mathrm{d}y)\mathbb{E}_y\left[f(X_t), t < \kappa_\Gamma\right],$$

where

$$\begin{aligned}
a &= \lim_{\epsilon\to 0}\lim_{s\to 0}\mathbf{N}^\Gamma\left(M(X_s), |X_s| < \epsilon, s < \zeta\right) \\
&= \lim_{\epsilon\to 0}\lim_{s\to 0}\mathbf{N}^\Gamma\left(\frac{M(X_s)}{\mathbb{P}_{X_s}(\kappa_\Gamma > 1)}\mathbb{P}_{X_s}(\kappa_\Gamma > 1), |X_s| < \epsilon, s < \zeta\right) \\
&= \left(\lim_{|x|\to 0}\frac{M(x)}{\mathbb{P}_x(\kappa_\Gamma > 1)}\right)\lim_{\epsilon\to 0}\lim_{s\to 0}\mathbf{N}^\Gamma\left(|X_s| < \epsilon, 1 + s < \zeta\right) < \infty.
\end{aligned}$$

This finishes the proof of the identity (13.1). We will next prove the identity (13.2). The latter decomposition together with the convergence (3.5), applied to $f(x) = M(x)g(\arg(x))\mathbf{1}_{(|x|\in(0,1))}$ with g any continuous and bounded function on

Γ, implies that

$$\mathbf{N}^\Gamma\big(M(X_t)g\big(\arg(X_t)\big)\mathbf{1}_{(|X_t|\in(0,1))}, t < \zeta\big)$$

$$= aC\int_{|y|<1} g\big(\arg(y)\big)n_t(y)\,\mathrm{d}y$$

$$+ \int_{y\in\Gamma,|y|>0} \mathbf{N}^\Gamma(X_{0+}\in\mathrm{d}y)\mathbb{E}_y\big[M(X_t)g\big(\arg(X_t)\big)\mathbf{1}_{(|X_t|\in(0,1))}, t < \kappa_\Gamma\big].$$

By the scaling property (iv$'$) applied to $f(x) = M(x)g(\arg(x))\mathbf{1}_{(|x|\in(0,1))}$.

$$\mathbf{N}^\Gamma\big(M(X_{0+})g\big(\arg(X_{0+})\big), |X_{0+}|\in(0,1)\big) = c^{\alpha\gamma}\mathbf{N}^\Gamma\big(M\big(c^{-1}X_{0+}\big)g\big(\arg(X_{0+})\big), |X_{0+}|\in(0,c)\big)$$

$$= c^{\alpha\gamma-\beta}\mathbf{N}^\Gamma\big(M(X_{0+})g\big(\arg(X_{0+})\big), |X_{0+}|\in(0,c)\big).$$

Notice that this is always finite because for $|x| < 1$, $M(x) < K\mathbb{P}_x(\kappa_\Gamma > 1)$, for some $K > 0$. So, by the Markov property, the latter is bounded by $c^{\alpha\gamma-\beta}K\|g\|\mathbf{N}^\Gamma(\zeta > 1)$. Differentiating in $c > 0$, one gets

$$(\beta - \alpha\gamma)r^{\beta-\alpha\gamma-1}\,\mathrm{d}r\mathbf{N}^\Gamma\big(M(X_{0+})g\big(\arg(X_{0+})\big), |X_{0+}|\in(0,1)\big)$$

$$= \mathbf{N}^\Gamma\big(M(X_{0+})g\big(\arg(X_{0+})\big), |X_{0+}|\in\mathrm{d}r\big).$$

Observe that since the right hand side is positive as soon as g is positive, we get as a side consequence that $\beta - \alpha\gamma \geq 0$. Using again that $M(x) = |x|^\beta M(\arg(x))$, one gets

$$(\beta - \alpha\gamma)r^{-\alpha\gamma-1}\,\mathrm{d}r\mathbf{N}^\Gamma\big(|X_{0+}|^\beta M\big(\arg(X_{0+})\big)g\big(\arg(X_{0+})\big), |X_{0+}|\in(0,1)\big)$$

$$= \mathbf{N}^\Gamma\big(M\big(\arg(X_{0+})\big)g\big(\arg(X_{0+})\big), |X_{0+}|\in\mathrm{d}r\big).$$

Since this identity holds for any g continuous and bounded, we derive that, when $\beta > \alpha\gamma$, the equality of measures

$$\mathbf{N}^\Gamma\big(|X_{0+}|\in\mathrm{d}r, \arg(X_{0+})\in\mathrm{d}\theta\big) = \frac{\mathrm{d}r}{r^{\alpha\gamma+1}}\pi^\Gamma(\mathrm{d}\theta),$$

holds, where,

$$\pi^\Gamma(\mathrm{d}\theta) = \frac{\mathbf{N}^\Gamma(|X_{0+}|^\beta, |X_{0+}|\in(0,1), \arg(X_{0+})\in\mathrm{d}\theta))}{\beta - \alpha\gamma}.$$

Whilst if $\beta = \alpha\gamma$, $\mathbf{N}^\Gamma(|X_{0+}| > 0) \equiv 0$, and thus $\pi^\Gamma \equiv 0$. We are just left to prove that when $\beta > \alpha\gamma$, then $\pi^\Gamma M < \infty$ and $a \equiv 0$. Indeed, that $\pi^\Gamma M < \infty$ follows from the following estimates

$$\infty > n\big(1 - \mathrm{e}^\zeta, X_{0+}\neq 0\big)$$

$$= \int_0^\infty \mathrm{d}s\,n(s < \zeta, X_{0+}\neq 0)\mathrm{e}^{-s}$$

$$= \int_0^\infty \mathrm{d}s\,\mathrm{e}^{-s}\int_\Omega \pi^\Gamma(\mathrm{d}\theta)\int_0^\infty \frac{\mathrm{d}r}{r^{1+\alpha\gamma}}\mathbb{P}_{r\theta}(\kappa_\Gamma > s)$$

$$\geq \int_\Omega \pi^\Gamma(\mathrm{d}\theta)\int_0^1 \frac{\mathrm{d}r}{r^{1+\alpha\gamma}}\int_{r^\alpha}^\infty \mathrm{d}s\,\mathrm{e}^{-s}M(r\theta)s^{-\beta/\alpha}$$

$$\geq \int_\Omega \pi^\Gamma(\mathrm{d}\theta)M(\theta)\int_0^1 \frac{\mathrm{d}r}{r^{1+\alpha\gamma-\beta}}\int_1^\infty \mathrm{d}s\,\mathrm{e}^{-s}s^{-\beta/\alpha},$$

where we used the estimate in Proposition 2.1. As claimed we derive that $\int_\Omega \pi^\Gamma(\mathrm{d}\theta)M(\theta) < \infty$. To finish, we observe that the identity (13.1) together with the scaling property (iv$'$) implies that for any $t > 0$

$$t^{-\gamma}\mathbf{N}^\Gamma(\zeta > 1) = \mathbf{N}^\Gamma(\zeta > t) \geq a\lim_{|x|\to 0}\frac{\mathbb{P}_x(t < \kappa_\Gamma)}{M(x)} = at^{-\beta/\alpha}C,$$

with $C > 0$ the constant appearing in Proposition 2.1. Since by assumption $\beta > \alpha\gamma$, we obtain by making $t \downarrow 0$, that $a \equiv 0$. We have thus finished the proof of Lemma 13.1. □

To finish the proof of Theorem 3.4 one should notice that, from the equation (3.5), for every $f : \Gamma \to \mathbb{R}^+$ continuous with compact support, one has the convergence

$$\lim_{|x| \to 0} \frac{\mathbb{E}_x[f(X_t), t < \kappa_\Gamma]}{M(x)} = C \int_\Gamma f(y) n_t(y) \, \mathrm{d}y, \quad t > 0.$$

This together with the decomposition in Lemma 13.1 implies that we necessarily have the following representation for the entrance law of any self-similiar excursion measure $\mathbf{N}^\Gamma$. There is a measure π^Γ on Ω, such that $\int_\Omega \pi^\Gamma(\mathrm{d}\theta) M(\theta) < \infty$, and a constant $a \geq 0$ such that

$$\mathbf{N}^\Gamma(X_t \in \mathrm{d}y, t < \zeta) = a n_t(y) \, \mathrm{d}y + \int_0^\infty \frac{\mathrm{d}r}{r^{1+\alpha\gamma}} \int_\Omega \pi^\Gamma(\mathrm{d}\theta) \mathbb{E}_{r\theta}[X_t \in \mathrm{d}y, t < \kappa_\Gamma], \tag{13.3}$$

$a\pi^\Gamma \equiv 0$, if $a > 0$ then $\gamma = \beta/\alpha$, and if $\pi^\Gamma \neq 0$ then $\gamma < \beta/\alpha$, and $\pi^\Gamma M < \infty$.

Furthermore, via cylinder sets, one can check that for $t > 0$ and $A \in \mathcal{F}_t$

$$\mathbf{N}^\Gamma(A, t < \zeta) = a \mathbb{E}_0^\lhd \left[\frac{1}{M(X_t)} \mathbf{1}_A \right] + \int_0^\infty \frac{\mathrm{d}r}{r^{1+\alpha\gamma}} \int_\Omega \pi^\Gamma(\mathrm{d}\theta) \mathbb{E}_{r\theta}[A, t < \kappa_\Gamma] \tag{13.4}$$

with a and π^Γ as above. As a side consequence, we have that the measure $\tilde{\mathbf{N}}^\Gamma$ on $\mathbb{D}$, defined by the relation

$$\tilde{\mathbf{N}}^\Gamma(A, t < \zeta) := \mathbb{E}_0^\lhd \left[\frac{1}{M(X_t)} \mathbf{1}_A \right], \quad \text{for } A \in \mathcal{F}_t, t > 0, \tag{13.5}$$

is a self-similar excursion measure, whose entrance law is $(n_t, t > 0)$, and such that $\tilde{\mathbf{N}}^\Gamma(X_{0+} \neq 0) = 0$. Finally this is the unique self-similar excursion measure bearing this property, and hence the self-similar recurrent extension associated to it leaves zero continuously, and it is the unique self-similar recurrent extension having this property.

13.2. *Invariant measure*

We start by computing the invariant measure according to Chapter XIX.46 of Dellacherie and Meyer [31]. There it is shown that the invariant measure $\overset{\rightrightarrows}{\pi}{}^\Gamma(\mathrm{d}y)$, $y \in \Gamma$, defined up to a multiplicative constant, is given by the excursion occupation measure so that

$$\int_\Gamma f(y) \overset{\rightrightarrows}{\pi}{}^\Gamma(\mathrm{d}y) = \mathbf{N}^\Gamma \left(\int_0^\zeta f(\chi_t) \, \mathrm{d}t \right),$$

for all bounded measurable f on Γ. Note, however, the computations in Lemma 11.1 can be used to show that

$$\tilde{\mathbf{N}}^\Gamma \left(\int_0^\zeta f(\chi_t) \, \mathrm{d}t \right) = \int_0^\infty \int_\Gamma f(z) n_t(z) \, \mathrm{d}z \, \mathrm{d}t = \int_\Gamma \frac{f(z)}{M(z)} G^\lhd(0, \mathrm{d}z) = \int_\Gamma f(z) H(z) \, \mathrm{d}z.$$

It is then straight forward to prove the final identity in the statement of Theorem 3.4.

Finally, to see that $\overset{\rightrightarrows}{\pi}$ is not a finite measure, we can compute its total mass, after converting to generalised polar coordinates (see e.g. Blumenson [16]), by

$$\int_\Gamma H(x) \, \mathrm{d}x = C \int_\Omega \sigma_1(\mathrm{d}\theta) M(\theta) \int_0^\infty r^{\alpha-\beta-1} \, \mathrm{d}r = \infty, \tag{13.6}$$

where $C > 0$ is an unimportant constant attached to the Jacobian in the change of variables to generalised polar coordinates, and $\sigma_1(\mathrm{d}\theta)$ is the surface measure on $\mathbb{S}^{d-1}$ normalised to have unit total mass. Moreover, we also have that if π^Γ is not trivial then

$$\int_0^\infty \frac{\mathrm{d}r}{r^{1+\alpha\gamma}} \int_\Omega \pi^\Gamma(\mathrm{d}\theta) \mathbb{E}_{r\theta}[\kappa_\Gamma] = \infty,$$

because by Proposition 2.1, $\mathbb{E}_{r\theta}[\kappa_\Gamma] = \infty$, for any $r > 0$, and Ω.

The failure of this measure to normalise to have unit mass means that a stationary distribution cannot exist, cf. Chapter XIX.46 of Dellacherie and Meyer [31] and hence $\overset{\rightrightarrows}{X}$ is a null-recurrent process.

14. Proof of Theorem 5.4

The proof of this result follows verbatim that of Theorem 3.4, albeit for some of the estimates that are used. Indeed, to establish Theorem 3.4, we used that for $(X, \mathbb{P}^\Gamma)$ we have

(a) for $t > |x|^\alpha$,

$$\mathbb{P}_x(\kappa_\Gamma > t) \approx M(x)t^{-\beta/\alpha};$$

(b) for any $t > 0$ and $f : \Gamma \to \mathbb{R}^+$ continuous and bounded

$$\lim_{|x| \to 0} \frac{\mathbb{E}_x(f(X_t), t < \kappa_\Gamma)}{M(x)} \quad \text{exists.}$$

These conditions are replaced by the following conditions on $(X, \mathbb{P}^\triangleright)$

(a′) for $t > |x|^\alpha$,

$$\mathbb{P}_x^\triangleright(\kappa_\Gamma > t) \approx H(x)t^{(\alpha-d-2\beta)/\alpha},$$

(b′) for any $t > 0$ and $f : \Gamma \to \mathbb{R}^+$ continuous and bounded

$$\lim_{|x| \to 0} \frac{\mathbb{E}_x^\triangleright(f(X_t), t < \kappa_\Gamma)}{H(x)} \quad \text{exists.}$$

Moreover, in proving Theorem 5.4, where one reads β in the proof of Theorem 3.4, one should use $\beta' := 2\beta + d - \alpha$. From here we have the restriction $0 < \beta' = d + 2\beta - \alpha < \alpha$, which restricts β to the interval $((\alpha - d)/2) \vee 0 < \beta < (2\alpha - d)/2$.

Let us finish by noticing that the finiteness of $\mathbf{N}^\triangleright(1 - e^{-\zeta})$ is equivalent to $d < 2(\alpha - \beta)$. To this end, we can appeal to Lemma 4.3 of [6] to see that, there exists a constant $C > 0$, such that, for $x \in \Gamma$,

$$C^{-1}s^{(\alpha-2\beta-d)/\alpha} < \lim_{x \to 0} |x|^{\alpha-2\beta-d}\mathbb{P}_x^\triangleright(s < \kappa_{\{0\}}) < Cs^{(\alpha-2\beta-d)/\alpha}, \tag{14.1}$$

where $\kappa_{\{0\}} = \inf\{t > 0 : |X_t| = 0\}$. Recall from (10.5) that, for $x \in \Gamma$, and $A \in \mathcal{F}_t$, $t \geq 0$,

$$\mathbb{E}_x^\triangleleft\left[\mathbf{1}_A \frac{|X_t|^{\alpha-2\beta-d}}{|x|^{\alpha-2\beta-d}}\right] = \mathbb{E}_x\left[\frac{H(X_t)}{H(x)}\mathbf{1}_{(A \cap \{t < \kappa_\Gamma\})}\right] = \mathbb{P}_x^\triangleright(A, t < \kappa_{\{0\}}). \tag{14.2}$$

Hence, using (14.2), we have

$$\mathbf{N}^\triangleright(1 - e^{-\zeta}) = \int_0^\infty e^{-s}\mathbf{N}^\triangleright(\zeta > s)\,ds$$

$$= \int_0^\infty ds\,e^{-s}\mathbb{E}_0^\triangleleft[J(X_s)]$$

$$= \int_0^\infty ds\,e^{-s}\lim_{x \to 0} |x|^{\alpha-2\beta-d}\mathbb{P}_x^\triangleright(s < \kappa_{\{0\}}),$$

and, thanks to (14.1), the right hand side either converges or explodes depending on whether $d < 2(\alpha - \beta)$. So, remember that by Campbell's theorem, the sum of the lengths $\sum_{s \leq t} \zeta_s$ is finite a.s. for any $t > 0$, if and only if $\mathbf{N}^\triangleright(1 - e^{-\zeta}) < \infty$, which is equivalent to $d < 2(\alpha - \beta)$. This justifies our comment following the statement of Theorem 5.4.

References

[1] L. Alili, L. Chaumont, P. Graczyk and T. Żak. Inversion, duality and Doob *h*-transforms for self-similar Markov processes. *Electron. J. Probab.* **22** (2017) 20. MR3622890 https://doi.org/10.1214/17-EJP33

[2] G. Alsmeyer. On the Markov renewal theorem. *Stochastic Process. Appl.* **50** (1) (1994) 37–56. MR1262329 https://doi.org/10.1016/0304-4149(94)90146-5

[3] G. Alsmeyer. Quasistochastic matrices and Markov renewal theory. *J. Appl. Probab.* **51A** (Celebrating 50 Years of The Applied Probability Trust) (2014) 359–376. MR3317369 https://doi.org/10.1239/jap/1417528486

[4] S. Asmussen. *Applied Probability and Queues*, 2nd edition. *Stochastic Modelling and Applied Probability* **51**. Springer-Verlag, New York, 2003. MR1978607

[5] S. Asmussen and H. Albrecher. *Ruin Probabilities*, 2nd edition. *Advanced Series on Statistical Science & Applied Probability* **14**. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010. MR2766220 https://doi.org/10.1142/9789814282536

[6] R. Bañuelos and K. Bogdan. Symmetric stable processes in cones. *Potential Anal.* **21** (3) (2004) 263–288. MR2075671 https://doi.org/10.1023/B:POTA.0000033333.72236.dc

[7] R. Bañuelos and R. G. Smits. Brownian motion in cones. *Probab. Theory Related Fields* **108** (3) (1997) 299–319. MR1465162 https://doi.org/10.1007/s004400050111

[8] J. Bertoin, T. Budd, N. Curien and I. Kortchemski. Martingales in self-similar growth-fragmentations and their connections with random planar maps. *Probab. Theory Related Fields* **172** (3–4) (2018) 663–724. MR3877545 https://doi.org/10.1007/s00440-017-0818-5

[9] J. Bertoin and M.-E. Caballero. Entrance from 0+ for increasing semi-stable Markov processes. *Bernoulli* **8** (2) (2002) 195–205. MR1895890

[10] J. Bertoin, N. Curien and I. Kortchemski. Random planar maps and growth-fragmentations. *Ann. Probab.* **46** (1) (2018) 207–260. MR3758730 https://doi.org/10.1214/17-AOP1183

[11] J. Bertoin and M. Savov. Some applications of duality for Lévy processes in a half-line. *Bull. Lond. Math. Soc.* **43** (1) (2011) 97–110. MR2765554 https://doi.org/10.1112/blms/bdq084

[12] J. Bertoin and A. R. Watson. A probabilistic approach to spectral analysis of growth-fragmentation equations. *J. Funct. Anal.* **274** (8) (2018) 2163–2204. MR3767431 https://doi.org/10.1016/j.jfa.2018.01.014

[13] J. Bertoin and M. Yor. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.* **17** (4) (2002) 389–400. MR1918243 https://doi.org/10.1023/A:1016377720516

[14] P. Biane, P. Bougerol and N. O'Connell. Littelmann paths and Brownian paths. *Duke Math. J.* **130** (1) (2005) 127–167. MR2176549 https://doi.org/10.1215/S0012-7094-05-13014-9

[15] P. Biane, P. Bougerol and N. O'Connell. Continuous crystal and Duistermaat–Heckman measure for Coxeter groups. *Adv. Math.* **221** (5) (2009) 1522–1583. MR2522427 https://doi.org/10.1016/j.aim.2009.02.016

[16] L. E. Blumenson. Classroom notes: A derivation of *n*-dimensional spherical coordinates. *Amer. Math. Monthly* **67** (1) (1960) 63–66. MR1530579 https://doi.org/10.2307/2308932

[17] R. M. Blumenthal. On construction of Markov processes. *Z. Wahrsch. Verw. Gebiete* **63** (4) (1983) 433–444. MR0705615 https://doi.org/10.1007/BF00533718

[18] R. M. Blumenthal, R. K. Getoor and D. B. Ray. On the distribution of first hits for the symmetric stable processes. *Trans. Amer. Math. Soc.* **99** (1961) 540–554. MR0126885 https://doi.org/10.2307/1993561

[19] K. Bogdan. The boundary Harnack principle for the fractional Laplacian. *Studia Math.* **123** (1) (1997) 43–80. MR1438304 https://doi.org/10.4064/sm-123-1-43-80

[20] K. Bogdan, K. Burdzy and Z.-Q. Chen. Censored stable processes. *Probab. Theory Related Fields* **127** (1) (2003) 89–152. MR2006232 https://doi.org/10.1007/s00440-003-0275-1

[21] K. Bogdan, Z. Palmowski and L. Wang. Yaglom limit for stable processes in cones. *Electron. J. Probab.* **23** (2018) 11. MR3771748 https://doi.org/10.1214/18-EJP133

[22] K. Bogdan and T. Żak. On Kelvin transformation. *J. Theoret. Probab.* **19** (1) (2006) 89–120. MR2256481 https://doi.org/10.1007/s10959-006-0003-8

[23] M. E. Caballero and L. Chaumont. Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.* **43** (4) (2006) 967–983. MR2274630 https://doi.org/10.1239/jap/1165505201

[24] M. E. Caballero and L. Chaumont. Weak convergence of positive self-similar Markov processes and overshoots of Lévy processes. *Ann. Probab.* **34** (3) (2006) 1012–1034. MR2243877 https://doi.org/10.1214/009117905000000611

[25] L. Chaumont, A. E. Kyprianou, J. C. Pardo and V. M. Rivero. Fluctuation theory and exit systems for positive self-similar Markov processes. *Ann. Probab.* **40** (1) (2012) 245–279. MR2917773 https://doi.org/10.1214/10-AOP612

[26] L. Chaumont, H. Pantí and V. M. Rivero. The Lamperti representation of real-valued self-similar Markov processes. *Bernoulli* **19** (5B) (2013) 2494–2523. MR3160562 https://doi.org/10.3150/12-BEJ460

[27] E. Çinlar. Markov additive processes. I, II. *Z. Wahrsch. Verw. Gebiete* **24** (1972) 85–93. Ibid. **24** (1972) 95–121. MR0329047 https://doi.org/10.1007/BF00532536

[28] E. Çinlar. Lévy systems of Markov additive processes. *Z. Wahrsch. Verw. Gebiete* **31** (1974/75) 175–185. MR0370788 https://doi.org/10.1007/BF00536006

[29] E. Çinlar. Entrance-exit distributions for Markov additive processes. *Math. Program. Stud.* **5** (1976) 22–38. MR0445621 https://doi.org/10.1007/bfb0120761

[30] R. D. DeBlassie. The first exit time of a two-dimensional symmetric stable process from a wedge. *Ann. Probab.* **18** (3) (1990) 1034–1070. MR1062058

[31] C. Dellacherie and P.-A. Meyer. *Probabilités et potentiel. Chapitres XII–XVI*, 2nd edition. *Publications de l'Institut de Mathématiques de l'Université de Strasbourg [Publications of the Mathematical Institute of the University of Strasbourg], XIX*. Hermann, Paris, 1987. Théorie du potentiel associée à une résolvante. Théorie des processus de Markov. [Potential theory associated with a resolvent. Theory of Markov processes], Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], 1417. MR0898005

[32] S. Dereich, K. Döring and A. E. Kyprianou. Real self-similar processes started from the origin. *Ann. Probab.* **45** (3) (2017) 1952–2003. MR3650419 https://doi.org/10.1214/16-AOP1105

[33] L. Döring and A. E. Kyprianou. Entrance and exit at infinity for stable jump diffusions. *Ann. Probab.* **48** (3) (2020) 1220–1265. MR4112713 https://doi.org/10.1214/19-AOP1389

[34] L. Döring, A. E. Kyprianou and P. Weissmann. Stable processes conditioned to avoid an interval. *Stochastic Process. Appl.* **130** (2) (2020) 471–487. MR4046505 https://doi.org/10.1016/j.spa.2019.01.004

[35] Y. Doumerc and N. O'Connell. Exit problems associated with finite reflection groups. *Probab. Theory Related Fields* **132** (4) (2005) 501–538. MR2198200 https://doi.org/10.1007/s00440-004-0402-7

[36] P. J. Fitzsimmons. On the existence of recurrent extensions of self-similar Markov processes. *Electron. Commun. Probab.* **11** (2006) 230–241. MR2266714 https://doi.org/10.1214/ECP.v11-1222

[37] P. J. Fitzsimmons and R. K. Getoor. Excursion theory revisited. *Illinois J. Math.* **50** (1–4) (2006) 413–437. MR2247835

[38] J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes*, 2nd edition. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **288**. Springer-Verlag, Berlin, 2003. MR1943877 https://doi.org/10.1007/978-3-662-05265-5

[39] H. Kaspi. On the symmetric Wiener–Hopf factorization for Markov additive processes. *Z. Wahrsch. Verw. Gebiete* **59** (2) (1982) 179–196. MR0650610 https://doi.org/10.1007/BF00531742

[40] H. Kaspi. On invariant measures and dual excursions of Markov processes. *Z. Wahrsch. Verw. Gebiete* **66** (2) (1984) 185–204. MR0749221 https://doi.org/10.1007/BF00531528

[41] H. Kesten. Renewal theory for functionals of a Markov chain with general state space. *Ann. Probab.* **2** (1974) 355–386. MR0365740 https://doi.org/10.1214/aop/1176996564

[42] A. Kuznetsov, A. E. Kyprianou, J. C. Pardo and A. R. Watson. The hitting time of zero for a stable process. *Electron. J. Probab.* **19** (2014) 30. MR3183574 https://doi.org/10.1214/EJP.v19-2647

[43] A. E. Kyprianou. Deep factorisation of the stable process. *Electron. J. Probab.* **21** (2016) 23. MR3485365 https://doi.org/10.1214/16-EJP4506

[44] A. E. Kyprianou. Stable Lévy processes, self-similarity and the unit ball. *ALEA Lat. Am. J. Probab. Math. Stat.* **15** (1) (2018) 617–690. MR3808900 https://doi.org/10.30757/alea.v15-25

[45] A. E. Kyprianou, J. C. Pardo and A. R. Watson. Hitting distributions of α-stable processes via path censoring and self-similarity. *Ann. Probab.* **42** (1) (2014) 398–430. MR3161489 https://doi.org/10.1214/12-AOP790

[46] A. E. Kyprianou, V. M. Rivero and W. Satitkanitkul. Conditioned real self-similar Markov processes. *Stochastic Process. Appl.* **129** (3) (2019) 954–977. MR3913275 https://doi.org/10.1016/j.spa.2018.04.001

[47] A. E. Kyprianou, V. M. Rivero and W. Satitkanitkul. Deep factorisation of the stable process III: Radial excursion theory and the point of closest reach. *Potential Anal.* **53** (2020) 1347–1375.

[48] A. E. Kyprianou, V. M. Rivero and B. Şengül. Deep factorisation of the stable process II: Potentials and applications. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** (1) (2018) 343–362. MR3765892 https://doi.org/10.1214/16-AIHP806

[49] A. E. Kyprianou and S. M. Vakeroudis. Stable windings at the origin. *Stochastic Process. Appl.* **128** (12) (2018) 4309–4325. MR3906986 https://doi.org/10.1016/j.spa.2018.02.004

[50] S. P. Lalley. Conditional Markov renewal theory. I. Finite and denumerable state space. *Ann. Probab.* **12** (4) (1984) 1113–1148. MR0757772

[51] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrsch. Verw. Gebiete* **22** (1972) 205–225. MR0307358 https://doi.org/10.1007/BF00536091

[52] B. Maisonneuve. Exit systems. *Ann. Probab.* **3** (3) (1975) 399–411. MR0400417 https://doi.org/10.1214/aop/1176996348

[53] P. J. Méndez-Hernández. Exit times from cones in $\mathbf{R}^n$ of symmetric stable processes. *Illinois J. Math.* **46** (1) (2002) 155–163. MR1936081

[54] M. Nagasawa. Time reversions of Markov processes. *Nagoya Math. J.* **24** (1964) 177–204. MR0169290

[55] N. O'Connell. Random matrices, non-colliding processes and queues. In *Séminaire de Probabilités, XXXVI* 165–182. *Lecture Notes in Math.* **1801**. Springer, Berlin, 2003. MR1971584 https://doi.org/10.1007/978-3-540-36107-7_3

[56] V. M. Rivero. Recurrent extensions of self-similar Markov processes and Cramér's condition. *Bernoulli* **11** (3) (2005) 471–509. MR2146891 https://doi.org/10.3150/bj/1120591185

[57] V. M. Rivero. Recurrent extensions of self-similar Markov processes and Cramér's condition. II. *Bernoulli* **13** (4) (2007) 1053–1070. MR2364226 https://doi.org/10.3150/07-BEJ6082

[58] T. S. Salisbury. Construction of right processes from excursions. *Probab. Theory Related Fields* **73** (3) (1986) 351–367. MR0859838 https://doi.org/10.1007/BF00776238

[59] T. S. Salisbury. On the Itô excursion process. *Probab. Theory Related Fields* **73** (3) (1986) 319–350. MR0859837 https://doi.org/10.1007/BF00776237

[60] K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics* **68**. Cambridge University Press, Cambridge, 2013. Translated from the 1990 Japanese original. Revised edition of the 1999 English translation. MR3185174

[61] R. Stephenson. On the exponential functional of Markov additive processes, and applications to multi-type self-similar fragmentation processes and trees. *ALEA Lat. Am. J. Probab. Math. Stat.* **15** (2) (2018) 1257–1292. MR3867206 https://doi.org/10.30757/alea.v15-47