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# The asymptotic behavior of fragmentation processes

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**Summary.** The fragmentation processes considered in this work are self-similar Markov processes which are meant to describe the ranked sequence of the masses of the pieces of an object that falls apart randomly as time passes. We investigate their behavior as $t \to \infty$. Roughly, we show that the rate of decay of the $\ell^p$-norm (where $p > 1$) is exponential when the index of self-similarity $\alpha$ is 0, polynomial when $\alpha > 0$, whereas the entire mass disappears in a finite time when $\alpha < 0$. Moreover, we establish a strong limit theorem for the empirical measure of the fragments in the case when $\alpha > 0$. Properties of size-biased picked fragments provide key tools for the study.

**Key words.** Fragmentation, self-similar, scattering rate, empirical measure.

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## 1 Introduction

In the recent years, there has been a renewed and growing interest amongst probabilists in random models of coalescence and the dual splitting models of fragmentation. We refer to Aldous' survey [1] for motivations, connections with other scientific areas, references, ... The present work is concerned with certain random fragmentation processes that we now introduce. The state space $\mathcal{S}^\downarrow$ consists in decreasing numerical sequences $x = (x_1, \ldots)$ with

$$x_1 \geq x_2 \geq \ldots \geq 0 \quad \text{and} \quad \sum_1^\infty x_i \leq 1 \,.$$

An element $x \in \mathcal{S}^\downarrow$ should be thought of as the ranked family of masses arising from the partition of some object with unit mass (a portion of the initial mass may be lost after the partition, which corresponds to the situation when $\sum_1^\infty x_i < 1$). In [4, 5], we considered a class of Markov processes $X = (X(t), t \geq 0)$ with values in $\mathcal{S}^{\downarrow}$, called *self-similar fragmentations*, which fulfill both the *scaling* and the so-called *fragmentation* properties. More precisely, the first means that there is an index of self-similarity $\alpha \in \mathbb{R}$ such that for every $r \in [0,1]$, if $\mathbb{P}_r$ stands for the law of $X$ started from $(r, 0, \ldots)$ (i.e. at the initial time, there is a single fragment with mass $r$), then the distribution of $(rX(r^\alpha t), t \geq 0)$ under $\mathbb{P}_1$ is $\mathbb{P}_r$. The second can be viewed as a version of the branching property. That is for every $s, t \geq 0$, conditionally on $X(t) = x = (x_1, \ldots)$, $X(t+s)$ has the same law as the variable obtained by ranking in the decreasing order the terms of the random sequences $X^{(1)}(s), X^{(2)}(s), \ldots$, where the latter are independent variables with values in $\mathcal{S}^{\downarrow}$, such that $X^{(i)}(s)$ has the same distribution as $X(s)$ under $\mathbb{P}_{x_i}$ for each $i = 1, \ldots$.

Excluding implicitly the degenerate case when $X$ remains constant, it can be seen that when $t$ goes to infinity, $X(t) = (X_1(t), \ldots)$ converges a.s. to $(0, \ldots)$, say in the sense of the uniform distance on $\mathcal{S}^{\downarrow}$. We shall first estimate the rate of decay in the sense of the $\ell^p$-norm for the so-called *homogeneous* fragmentation, that is when the index of self-similarity is $\alpha = 0$. Like for the classical result of Kesten and Stigum about the rate of growth of branching processes, our approach relies on the following simple observation. For $1 \leq p < \infty$, a combination of the Markov, scaling, and fragmentation properties entails that

$$M(t,p) := \exp(m_p t) \sum_{i=1}^{\infty} X_i^p(t) \,, \qquad t \geq 0$$

is a positive martingale, where $m_p = -\log \mathbb{E}_1 \left( \sum_{i=1}^{\infty} X_i^p(1) \right)$. The value of $m_p$ can be computed in terms of the characteristics of the fragmentation (see the forthcoming section 2 for details), so we immediately deduce that there exists some a.s. finite random variable $M(\infty, p)$ such that

$$\sum_{i=1}^{\infty} X_i^p(t) \sim M(\infty, p) \exp\left(-m_p t\right) \,, \qquad t \to \infty \,.$$

Of course, the heart of the matter is to decide whether $M(\infty, p)$ is strictly positive or zero, and essentially, this amount to find conditions on $p$ that ensure uniform integrability for the martingale $M(\cdot, p)$.

Next we shall turn our attention to the case of positive indices of self-similarity, that is $\alpha > 0$. Our approach is then essentially adapted from the work of Brennan and Durrett [7, 8], who considered a particle system with the following dynamics. Each particle is specified by its size, say $r > 0$. A particle with size $r$ waits an exponential time with parameter $r^\alpha$ and then splits into two particles with respective sizes $rV$ and $r(1-V)$, where $V$ is a random variable with values in $]0,1[$ which has a fixed distribution and is independent of the past of the system. Such a particle system is a simple example of a self-similar fragmentation with index $\alpha$; more precisely, it is a binary and discrete fragmentation, in the sense that splitting times of the particle system can only accumulate at $\infty$. Brennan and Durrett obtained strong limit theorems for the number of particles at time $t$, $N(t)$, and for the rescaled empirical measure,

$$\frac{1}{N(t)} \sum_{i=1}^{N(t)} \delta_{t^{1/\alpha} X_i(t)} \,,$$

where $X_i(t)$ denotes the size of the $i$-th largest particle at time $t$ and $\delta_r$ stands for the Dirac point mass at $r$. In general, self-similar fragmentations may produce an infinite number of fragments at any time $t > 0$, that is we may have $N(t) = \infty$ a.s. Our main result in this setting states (under certain conditions) the convergence in probability as $t \to \infty$ of the random measures

$$\sum_{i=1}^{\infty} X_i(t)\delta_{t^{1/\alpha}X_i(t)}$$

towards some deterministic measure on $]0,\infty[$ that is specified in terms of the characteristics of the fragmentation (this is closely related to a propagation of chaos type property induced by the dynamics of fragmentation; see [14]). In particular, this implies that the rate of decay of the $\ell^p$-norm of $X(t)$ is then polynomial.

Finally, the case of a negative index of self-similarity is studied via the homogeneous case, using an invertible transformation which changes a self-similar fragmentation into a homogeneous one. We shall show that the entire mass disappears in a finite time, in the sense that

$$\inf\{t \geq 0 : X(t) = (0,\ldots)\} < \infty \quad \text{a.s.}$$

If further $\alpha < -1$, then we shall also point out that at each fixed time $t > 0$, there are only a finite number of fragments with positive size, a.s.

The rest of this paper is organized as follows. The next section is devoted to the analysis of the homogeneous case $\alpha = 0$. The self-similar cases $\alpha > 0$ and $\alpha < 0$ are discussed in Section 3.

# 2 Homogeneous fragmentations

Throughout this section, we focus on homogeneous fragmentations, i.e. with index of self-similarity $\alpha = 0$. We shall further implicitly assume that the fragmentation starts from a single fragment with unit mass, i.e. we shall work under $\mathbb{P} := \mathbb{P}_1$.

## 2.1 Preliminaries

We begin by recalling some features lifted from [4] (see also [3]). We also refer to Pitman [12] and Schweinsberg [13] for related works. Roughly, homogeneous fragmentations result from the combination of two different phenomena: a continuous erosion and sudden dislocations. The erosion is a continuous deterministic mechanism, whereas the dislocations occur randomly and can be viewed as the jump-component of the process. More precisely, if $X$ is a pure-jump homogeneous fragmentation and $c > 0$ an arbitrary real number, then $(\mathrm{e}^{-ct}X(t), t \geq 0)$, is a again a homogeneous fragmentation, but now with a continuous component corresponding to an erosion. Any homogeneous fragmentation can be obtained from a pure-jump one by this elementary transformation.

Let us now focus on the sudden dislocations (i.e. the jumps). Just like in the celebrated Lévy-Itô decomposition for subordinators, their distribution can be described in terms of a certain intensity measure. More precisely, set $\mathcal{S}^* = \mathcal{S}^{\downarrow}\backslash\{(1,0,\ldots)\}$ for the space of decreasing numerical sequences $x = (x_1,\ldots)$ with $\sum x_i \leq 1$ and $x_1 < 1$. The so-called Lévy measure $\nu$ is a measure on $\mathcal{S}^*$ which fulfills the condition

$$\int_{\mathcal{S}^*} (1 - x_1)\, \nu(dx) \;<\; \infty\,.$$

One can construct a Poisson point process $((\Delta(t), k(t)), t \geq 0)$ with values in $\mathcal{S}^* \times \mathbb{N}$, with characteristic measure $\nu \otimes \#$, where $\#$ stands for the counting measure on $\mathbb{N} = \{1, 2, \ldots\}$, such that the following holds. The process $X(\cdot)$ jumps only at times $t \geq 0$ at which a point $(\Delta(t), k(t))$ occurs, and then $X(t)$ is obtained from $X(t-)$ by replacing its $k(t)$-th term $X_{k(t)}(t-)$ by the sequence $X_{k(t)}(t-)\Delta(t)$, and ranking all the terms in the decreasing order. Of course, it may happen that $X_{k(t)}(t-) = 0$, and in that case we have $X(t) = X(t-)$.

In conclusion, the dynamics of a homogeneous fragmentation are entirely characterized by its Lévy measure $\nu$ and its erosion rate $c \geq 0$. We shall always implicitly exclude the trivial case when $\nu \equiv 0$, that is when the fragmentation reduces to a pure erosion. Some information about the characteristics $c$ and $\nu$, and hence on the distribution of the homogeneous fragmentation $X$, is caught by the function

$$\Phi(q) \;:=\; c(q+1) + \int_{\mathcal{S}^*} \left(1 - \sum_{i=1}^{\infty} x_i^{q+1}\right) \nu(dx)\,, \qquad q \geq 0\,. \tag{1}$$

In general $\Phi$ does not characterize the law of the fragmentation, because one cannot always recover $c$ and $\nu$ from (1). The noticeable exception is when the fragmentation is binary, in the sense that $\nu(x_1 + x_2 < 1) = 0$, as then $\Phi$ determines the erosion coefficient $c$ and the Lévy measure $\nu$.

The function $\Phi$ has a crucial role in this work, which essentially stems from the following fact. Suppose that at the initial time, a point is picked at random (according to the mass distribution of the object and independently of the fragmentation process) and tagged. Denote by $\lambda(t)$ the size of the fragment that contains this tagged point at time $t \geq 0$. In particular, $\lambda(t)$ is a size-biased pick from the sequence $X(t) = (X_1(t), \ldots)$, that is

$$\lambda(t) \;:=\; \mathbf{1}_{\{N \geq 1\}} X_N(t)\,,$$

where $N$ is an integer valued variable whose conditional distribution given $X(t)$ is

$$\mathbb{P}(N = k \mid X(t)) \;=\; X_k(t)\,, \quad k = 0, 1, \ldots$$

with the convention that $X_0(t) = 1 - \sum_{i=1}^{\infty} X_i(t)$. Then the process $(-\log \lambda(t), t \geq 0)$ is a subordinator, i.e. it is an increasing process with independent and stationary increments, and we have

$$\mathbb{E}\left(\lambda(t)^q\right) \;=\; \exp\left(-t\Phi(q)\right)\,, \qquad t, q \geq 0\,. \tag{2}$$

See section 5 in [4], and in particular Theorem 3 and Lemma 3 there. In the sequel, we shall often use the fact that, as $\Phi : [0, \infty[ \to [0, \infty[$ is the Laplace exponent of a subordinator, it is a concave increasing analytic function.

## 2.2 Scattering rates

For every $p \in [1, \infty[$ and $t \geq 0$, we introduce the quantities

$$\Sigma(t, p) \;:=\; \sum_{i=1}^{\infty} X_i^p(t) \qquad \text{and} \qquad \ell(t, p) \;:=\; \left(\Sigma(t, p)\right)^{1/p}\,,$$

and for $p = \infty$,

$$\ell(t, \infty) := X_1(t) = \max\{X_i(t), i = 1, \ldots\} .$$

In other words, $\ell(t,p)$ is the $\ell^p$-norm of the random sequence $X(t)$, and can be thought of as a measurement of the scattering of the fragmentation at time $t$. It should be clear that each process $\ell(\cdot,p)$ decreases, and tends to 0 whenever $p > 1$ as time goes to $\infty$. An interesting question is thus to evaluate its rate of decay.

We observe that if $(\mathcal{F}_t)_{t \geq 0}$ denotes the natural filtration of the fragmentation process, then the conditional moments of a size-biased picked fragment are given by

$$\mathbb{E}\left(\lambda^{p-1}(t) \mid \mathcal{F}_t\right) = \Sigma(t,p) ,$$

We may thus reformulate the identity (2) as

$$\mathbb{E}\left(\Sigma(t,p)\right) = \exp\left(-t\Phi(p-1)\right) , \qquad p \in [1, \infty[ . \tag{3}$$

We then immediately deduce from the Markov, scaling and fragmentation properties that

$$M(t,p) := \exp\left(t\Phi(p-1)\right)\Sigma(t,p) , \qquad t \geq 0$$

is a positive martingale [1] with bounded variation (and hence $M(\cdot,p)$ is pure-jump).

We are now able to state our main result in the homogeneous case.

**Theorem 1** *Assume that the index of self-similarity is $\alpha = 0$ and that the Lévy measure $\nu$ is such that $\nu\left(\sum_1^\infty x_i < 1\right) = 0$ (which means that no mass can be lost when a sudden dislocation occurs).*

(i) *The function $p \to \Phi(p-1)/p$ has a unique maximum on $[1,\infty[$, which is reached at $\overline{p} > 1$, and is given by*

$$m = \Phi'(\overline{p}-1) = \Phi(\overline{p}-1)/\overline{p} \in ]0,\infty[.$$

(ii) *For every $p \in [1, \overline{p}[$, the martingale $M(\cdot,p)$ is bounded in $L^{\overline{p}/p}(\mathbb{P})$, and its terminal value, $M(\infty,p)$, is a strictly positive a random variable a.s. In particular the limit*

$$\lim_{t\to\infty} \exp(t\Phi(p-1)/p)\ell(t,p)$$

*exists in $]0,\infty[$, both a.s. and in $L^{\overline{p}}(\mathbb{P})$.*

(iii) *For every $p \in [\overline{p}, \infty]$, it holds with probability one that*

$$\lim_{t\to\infty} \frac{1}{t} \log \ell(t,p) = -m .$$

Before proceeding to the proof, let us comment on the assumptions and the results in Theorem 1.

[1] Alternatively, this can be derived by an optional projection on the natural filtration of the fragmentation process, from the well-known fact that if $\sigma$ is a subordinator with Laplace exponent say $\varphi$, then $\exp(-q\sigma(t) + t\varphi(q))$ is a martingale.

First, it may be interesting to recall that in general, the function $\Phi$ which is used to calculate the scattering rates, does not characterize the distribution of the fragmentation. Nonetheless, it is noteworthy that the scattering rates are able to capture even slight perturbations of the dynamics of the fragmentation. To give a simple example, pick an arbitrarily large integer $j$ and consider a sequence $x = (x_1, \ldots) \in \mathcal{S}^{\downarrow}$ with $\sum_1^\infty x_i = 1$ and $x_j > 0$. Then denote by $\tilde{x}$ the sequence obtained from $x$ by deleting the term $x_j$ and replacing it by two terms, say $(1-\varepsilon)x_j$ and $\varepsilon x_j$ for some $0 < \varepsilon < 1$. Then let $\nu$ and $\tilde{\nu}$ be the Dirac point masses at $x$ and $\tilde{x}$, respectively. Recall that the evolution of the homogeneous fragmentation $X$ with no erosion and Lévy measure $\nu$ can be described as follows. Each mass $r > 0$ is unchanged during an exponential time with parameter 1 and then it splits, giving rise to the sequence $rx$. A similar description holds for the homogeneous fragmentation $\tilde{X}$ with no erosion and Lévy measure $\tilde{\nu}$, replacing $x$ by $\tilde{x}$. One might be tempted to believe that, at least when $j$ is sufficiently large and $\varepsilon$ sufficiently small, the largest fragments $X_1(t) = \ell(t,\infty)$ and $\tilde{X}_1(t) = \tilde{\ell}(t,\infty)$ of the fragmentations $X$ and $\tilde{X}$ respectively, should have the same asymptotic behavior. However this is not the case: in the obvious notation, one has $\Phi(p) < \tilde{\Phi}(p)$ for all $p \geq 0$, and it follows that the exponential rate of decay $m = \max\{\Phi(p-1)/p, p \geq 1\}$ for $X_1(t)$ is strictly smaller than $\tilde{m} = \max\left\{\tilde{\Phi}(p-1)/p, p > 1\right\}$, the exponential rate of decay for $\tilde{X}_1(t)$.

In the same direction, as the function $\Phi$ is entirely determined by the distribution of a size-biased-picked fragment (see (2)), it may be natural to compare the asymptotic behavior of the largest fragment $X_1(t)$ at time $t$ with that of a size-biased-picked fragment, $\lambda(t)$. Using Theorem 3 in [4] and the law of large numbers for subordinators, one sees that provided that the erosion rate is $c = 0$ , it holds that

$$\lim_{t\to\infty} \frac{1}{t} \log \lambda(t) \;=\; -\Phi'(0+)\,,$$

where $\Phi'(0+) \in ]0,\infty]$ stands for the right derivative of $\Phi$ at 0 (of course, the fluctuations of $\log \lambda(t) + t\Phi'(0+)$ are given by the central limit theorem whenever $\Phi''(0+) < \infty$). By the strict concavity of $\Phi$, we have

$$\Phi'(0+) \;>\; \Phi'(\overline{p}) \;=\; m\,,$$

so $\lambda(t)$ always decays much faster than $X_1(t)$.

We mention that the hypothesis $\nu\left(\sum_1^\infty x_i < 1\right) = 0$ is just a simple condition that ensures (i), and that (ii-iii) are always valid whenever (i) holds. This hypothesis can be weakened, but not completely dropped; let us present a simple counter-example. Consider a subordinator $(\sigma_t, t \geq 0)$, say with no drift and Lévy measure $\Lambda$. It should be plain that $X(t) = (\exp(-\sigma_t), 0, \ldots)$ is a homogeneous fragmentation, and it is easy to check that its erosion coefficient is 0 and its Lévy measure $\nu$ is supported by the subspace $\{x = (x_1, 0, \ldots), 0 < x_1 < 1\}$. More precisely, $\nu(x_1 \in \cdot)$ coincides with the image measure of $\Lambda$ by the map $r \to \mathrm{e}^{-r}$. In this situation, we have $\ell(t,p) = \exp(-\sigma_t)$, and may happen that this process decays faster than any exponential (for instance when $\sigma$ is a stable subordinator with index $\beta \in ]0,1[$, it is known that $\lim_{t\to\infty} t^{-\gamma}\sigma_t = \infty$ with probability one for every $\gamma < 1/\beta$). In this direction, we point out however that in all cases, the a.s. convergence of positive martingales entails the following lower-bound for the rate of decay of $X_1(t)$: it holds a.s. that

$$\limsup_{t\to\infty} \frac{1}{t} \log 1/X_1(t) \;\geq\; \sup\{\Phi(p-1)/p, p \geq 1\} \;:=\; m \;>\; 0\,. \tag{4}$$

Finally, we observe that Markov's inequality combined with the fact that $\mathbb{E}(M(t,p)) = 1$ for every $t \geq 0$ and $p \in [1,\infty[$ entails the following concentration inequality: for every $\varepsilon, t > 0$, we have

$$\mathbb{P}\left(\ell(t,p) > \varepsilon\right) \leq \varepsilon^{-p} \exp(-t\Phi(p-1)) .$$

In the same vein, as for $p > \overline{p}$, $M(\cdot,p)$ is a martingale with no positive jumps that converges to 0 at $\infty$, the optional sampling theorem yields for every $a > 1$ the identity

$$\mathbb{P}\left(\ell(t,p) \geq a\exp(-t\Phi(p-1)/p) \text{ for some } t \geq 0\right) = a^{-p} .$$

## 2.3 Proofs

(i) It follows immediately from the fact that the second derivative $\Phi''$ of $\Phi$ is strictly negative on $]0,\infty[$ that $p \to \Phi(p-1)/p$ is strictly concave on $]1,\infty[$. By (1), the hypothesis $\nu\left(\sum_1^\infty x_i < 1\right) = 0$ ensures that $\Phi(0) = c$ is the erosion coefficient. On the other hand, it is well-known that the so-called drift coefficient is given by $\lim_{q\to\infty} \Phi(q)/q = c$. Hence the function $p \to \Phi(p-1)/p$ has the same limit at 1 and at $\infty$, so it reaches its overall maximum at a unique point $\overline{p} \in ]1,\infty[$. As the derivative must be zero at $\overline{p}$, we obtain that the overall maximum is given by $\Phi'(\overline{p}-1) = \Phi(\overline{p}-1)/\overline{p}$.

(ii) For every $q > 1$, it is known that the pure-jump martingale $M(\cdot,p)$ is bounded in $L^q(\mathbb{P})$ whenever its $q$-variation

$$V_q(p) := \sum_{t\geq 0} |M(t,p) - M(t-,p)|^q$$

is an integrable variable; see e.g. Lépingle [11]. In this direction, observe that the jumps of $M(\cdot,p)$ can be expressed in terms of the points of $(\Delta(t), k(t))$ of the Poisson point process used in Section 2.1 to describe the sudden dislocations of the fragmentation:

$$|M(t,p) - M(t-,p)| = \exp(t\Phi(p-1))X^p_{k(t)}(t-)\left(1 - \sum_{i=1}^\infty \Delta_i^p(t)\right) ,$$

where $\Delta_1(t) \geq \Delta_2(t) \geq \ldots$ is the decreasing sequence of the terms of $\Delta(t)$. Since the characteristic measure of the Poisson point process is $\nu \otimes \#$, the compensation formula yields

$$\mathbb{E}\left(V_q(p)\right) = c(p,q)\int_0^\infty \exp(tq\Phi(p-1))\mathbb{E}\left(\Sigma(t,pq)\right) dt ,$$

where

$$c(p,q) := \int_{\mathcal{S}^*}\left(1 - \sum_{i=1}^\infty x_i^p\right)^q \nu(dx) .$$

Because

$$\left(1 - \sum_{i=1}^\infty x_i^p\right)^q \leq 1 - x_1^p \leq p(1-x_1) ,$$

the condition $\int_{\mathcal{S}^*}(1-x_1)\nu(dx) < \infty$ ensures that $c(p,q)$ is finite. On the other hand, we know from (3) that $\mathbb{E}\left(\Sigma(t,pq)\right) = \exp(-t\Phi(pq-1))$, so we conclude that $M(\cdot,p)$ is bounded in $L^q(\mathbb{P})$ whenever

$$q\Phi(p-1) < \Phi(qp-1) . \tag{5}$$

Suppose now that $p < \overline{p}$ and take $q = \overline{p}/p$. By the part (i), we know that

$$\Phi(p-1)/p \;<\; \Phi(\overline{p}-1)/\overline{p} \;=\; \Phi(pq-1)/pq\,,$$

so (5) holds and $M(\cdot,p)$ is bounded in $L^q(\mathbb{P})$.

Thus all that we need is to check that its terminal value $M(\infty,p)$ is strictly positive a.s., which is straightforward. Indeed, the scaling property ensures that the quantity $\mathbb{P}_r(M(\infty,p) = 0)$ does not depend on $r > 0$. On the other hand, the fragmentation property entails

$$\mathbb{P}(M(\infty,p) = 0 \mid X(t)) \;=\; \prod_{X_i(t)>0} \mathbb{P}_{X_i(t)}\left(M(\infty,p) = 0\right)\,.$$

Since $\nu \not\equiv 0$ and $\nu\left(\sum_{i=1}^{\infty} x_i < 1\right) = 0$, the number of fragments at time $t > 0$, $\texttt{Card}\,\{i : X_i(t) > 0\}$, is always at least 1, and is greater than 1 with positive probability. Hence $\mathbb{P}(M(\infty,p) = 0)$ must be equal to 0 or 1, and the uniform integrability of $M(\cdot,p)$ impedes the case when $M(\infty,p) = 0$ a.s.

(iii) Suppose now that $\overline{p} \le p$. As $X_i(t) \le 1$ for every $t \ge 0$ and $i \in \mathbb{N}$, we have $\Sigma(t,p) \le \Sigma(t,\overline{p})$, and we get by the convergence of the martingale $M(t,\overline{p})$ that

$$\sup_{t\ge 0} \exp(t\Phi(\overline{p}-1))\Sigma(t,p) \;<\; \infty \qquad \text{a.s.}$$

On the other hand, pick any $q, \varepsilon > 0$ such that $q + \varepsilon < \overline{p}$. As

$$\Sigma(t, q+\varepsilon) \;\le\; X_1^{\varepsilon}(t)\Sigma(t,q)\,,$$

we deduce again from (ii) that

$$\liminf_{t\to\infty} \frac{1}{t}\log X_1(t) \;\ge\; \frac{\Phi(q) - \Phi(q+\varepsilon)}{\varepsilon}\,.$$

We let first $\varepsilon \to 0+$, and then $q \to \overline{p}$, and we obtain the lower-bound

$$\liminf_{t\to\infty} \frac{1}{t}\log \ell(t,p) \;\ge\; \liminf_{t\to\infty} \frac{1}{t}\log X_1(t) \;\ge\; -\Phi'(\overline{p}) \;=\; -m\,,$$

where the last equality stems from (i). □

# 3 Self-similar fragmentations

Here, we turn our attention to general self-similar fragmentations, i.e. the index of self-similarity $\alpha$ is now an arbitrary real number. As connections between the homogeneous and the self-similar cases will have an important role in our analysis, and in order to avoid a possible confusion, it is convenient from now on to denote self-similar fragmentations with index $\alpha$ by $X^{(\alpha)}(t) = \left(X_1^{(\alpha)}(t), X_2^{(\alpha)}(t), \ldots\right)$, and to keep the notation $X = X^{(0)}$ in the homogeneous case. Again, we shall implicitly assume that the fragmentation starts from a single fragment with unit mass. Let us now recall some results taken from [5].

### 3.1 Preliminaries

There is no loss of generality in assuming that the fragmentation can be represented in terms of some nested family of random open subsets of $]0,1[$; see Section 3.2 in [5]. This means that we can construct for each $t \geq 0$ a random open set $V(t) \subseteq ]0,1[$ such that $X^{(\alpha)}(t)$ coincides with the ordered sequence of the lengths of the component intervals of $V(t)$, and $V(t) \subseteq V(s)$ for every $0 \leq s \leq t$.

This representation enables us to introduce a simple transformation that changes $X^{(\alpha)}$ into a homogeneous fragmentation $X$. For every $x \in ]0,1[$, let $I_x(t)$ denote the interval component of $V(t)$ that contains $x$ if $x \in V(t)$, and $I_x(t) = \emptyset$ otherwise. We write $|I|$ for the length of an interval $I \subseteq ]0,1[$, and for every $x \in ]0,1[$ we consider the time-substitution

$$T_x(t) := \inf\left\{u \geq 0 : \int_0^u |I_x(r)|^\alpha dr > t\right\} .$$

Because the open sets $V(t)$ are nested, we see that for every $x, y \in ]0,1[$, the intervals $I_x(T_x(t))$ and $I_y(T_y(t))$ are either identical or disjoint, so the family $\{I_x(T_x(t)), 0 < x < 1\}$ can be viewed as the interval components of an open set $\tilde{V}(t)$. It is straightforward that the family $(\tilde{V}(t), t \geq 0)$ is nested. More precisely, if we write $X(t)$ for the ordered sequence of the lengths of the interval components of $\tilde{V}(t)$, then $(X(t), t \geq 0)$ is a homogeneous fragmentation. See Theorem 2 in [5].

The transformation $X^{(\alpha)} \to X$ can easily be inverted. In particular, the distribution of $X^{(\alpha)}$ is entirely determined by the index of self-similarity $\alpha$, and the erosion coefficient $c \geq 0$ and the Lévy measure $\nu$ of the homogeneous fragmentation $X$. In the sequel, the notation $\Phi$ will refer to the function defined in (1).

### 3.2 Case when the self-similarity index is positive

For the purpose of this section, it is convenient to describe the fragmentation $X^{(\alpha)}(t)$ via the following modified version of the empirical measure of its components,

$$\mu_t^{(\alpha)} := \sum_{i=1}^{\infty} X_i^{(\alpha)}(t)\, \delta_{t^{1/\alpha} X_i^{(\alpha)}(t)} .$$

In words, $\mu_t^{(\alpha)}$ assigns a mass $y$ at each point $t^{1/\alpha} y$ such that $y$ is the size of a fragment of $X^{(\alpha)}(t)$ (with the obvious convention in case of multiple fragments with identical size). Note that the total mass of $\mu_t^{(\alpha)}$ is $\sum_{i=1}^{\infty} X_i^{(\alpha)}(t)$, so $\mu_t^{(\alpha)}$ is always a sub-probability measure, and is a true probability if and only if the fragmentation process induces no loss of mass. In this case, $\mu_t^{(\alpha)}$ coincides with the conditional distribution of $t^{1/\alpha}\lambda^{(\alpha)}(t)$ given the fragmentation process, where $\lambda^{(\alpha)}(t)$ is a size-biased pick from $X^{(\alpha)}(t)$. This observation is crucial to the proof of the forthcoming limit theorem for $\mu_t^{(\alpha)}$.

Our main result of this setting is the following theorem which can be viewed as an extension of Brennan and Durrett [8]. Recall that we are dealing with a self-similar fragmentation with index $\alpha$, erosion rate $c$ and Lévy measure $\nu$, and that the notation $\Phi$ refers to the function defined in (1).

**Theorem 2** *Suppose that $\alpha > 0$, and assume that*
(i) *the erosion coefficient is $c = 0$ and $\nu\left(\sum_{i=1}^{\infty} x_i < 1\right) = 0$,*

(ii)
$$\Phi'(0+) \;=\; \sum_{i=1}^{\infty} \int_{\mathcal{S}^*} x_i \log(1/x_i)\nu(dx) \;<\; \infty\,,$$

(iii) *the fragmentation is not geometric, that is there exists no $r > 0$ such that the size of every fragment at time $t > 0$ lies in the set $\left\{ \mathrm{e}^{-kr} : k = 0, 1, \ldots \right\}$.*

*Then the random measures $\mu_t^{(\alpha)}$ are probability measures a.s., and when $t \to \infty$, they converge in probability, in the sense of weak convergence of measures. The limit $\mu_\infty^{(\alpha)}$ is deterministic; it is determined by the moments*

$$\int_{]0,\infty[} y^{-\alpha k} \mu_\infty^{(\alpha)}(dy) \;=\; \frac{1}{\alpha \Phi'(0+)} \prod_{i=1}^{k-1} \frac{1}{\Phi(i\alpha)} \quad \textit{for } k = 1, \ldots,$$

*(with the usual convention that the product above equals 1 for $k = 1$).*

**Proof:** As it has been mentioned in the preceding section, we may assume that $X^{(\alpha)}(t)$ is the sequence of the lengths of the interval components of an open set $V(t)$, where the family $(V(t), t \geq 0)$ is nested. Introduce two independent random variables, $U$ and $\tilde{U}$ which are both uniformly distributed on $]0,1[$ and independent of the fragmentation process. The length $|I_U(t)|$ of the interval component of $V(t)$ that contains $U$, can be viewed as the size of the fragment in $X^{(\alpha)}(t)$ containing a randomly tagged point. Plainly, $|I_U(t)|$ and $|I_{\tilde{U}}(t)|$ have the same distribution and are not independent.

On the one hand, for an arbitrary bounded continuous function $f : ]0,\infty[ \to \mathbb{R}$, we have the first and second moments identities

$$\begin{aligned}
\mathbb{E}\left( \int_{]0,\infty[} f(y) \mu_t^{(\alpha)}(dy) \right) &= \mathbb{E}(f(t^{1/\alpha}|I_U(t)|)) \\
\mathbb{E}\left( \left( \int_{]0,\infty[} f(y) \mu_t^{(\alpha)}(dy) \right)^2 \right) &= \mathbb{E}(f(t^{1/\alpha}|I_U(t)|) f(t^{1/\alpha}|I_{\tilde{U}}(t)|))\,,
\end{aligned}$$

In order to study the first moment, we use Corollary 2 of [5] which states that the distribution of the process $|I_U(\cdot)|$ can be described in terms of the function $\Phi$ as follows. Let $\xi = (\xi_t, t \geq 0)$ be a subordinator with Laplace exponent $\Phi$, and define implicitly $\tau(t)$ for every $t \geq 0$ by the identity

$$\int_0^{\tau(t)} \exp\left(\alpha \xi_s\right) ds \;=\; t\,.$$

Then the processes $\left(\exp(-\xi_{\tau(t)}), t \geq 0\right)$ and $(|I_U(t)|, t \geq 0)$ have the same distribution. Roughly, this is an easy consequence of the transformation that reduces a self-similar fragmentation to a homogeneous one (which was presented in the preceding section) and the fact that in the homogeneous case, the process $\lambda(\cdot)$ of mass of a fragment that contains a randomly tagged point has the same law as $\exp(-\xi_\cdot)$. In the terminology introduced by Lamperti [10], this means that $1/|I_U(\cdot)|$ is an increasing self-similar Markov process with index $1/\alpha$, which is associated with the subordinator with Laplace exponent $\Phi$.

The hypotheses (i-ii) are necessary and sufficient conditions for $\mathbb{E}(\xi_1) = \Phi'(0+) < \infty$, whereas (iii) means that the subordinator $\xi$ is not arithmetic. In this case, it has been shown in [6] (extending Brennan and Durrett [8]) that $t^{1/\alpha}\exp(-\xi_{\tau(t)})$ has a limiting distribution as $t \to \infty$, which is given by the probability measure $\mu_\infty^{(\alpha)}$ which appears in the statement. By the scaling property, we thus have that for every $r > 0$, the distribution of $t^{1/\alpha}|I_U(t)|$ under $\mathbb{P}_r$ (i.e. when at the fragmentation starts from a single fragment of size $r$) converges weakly towards $\mu_\infty^{(\alpha)}$ as $t \to \infty$.

The key of to evaluate the second moment lies in the observation that, although $|I_U(t)|$ and $|I_{\tilde{U}}(t)|$ are not independent, the fragmentation property entails that the variables $t^{1/\alpha}|I_U(t)|$ and $t^{1/\alpha}|I_{\tilde{U}}(t)|$ are asymptotically independent. Specifically, their joint law has a limit distribution as $t \to \infty$, which is $\mu_\infty^{(\alpha)} \otimes \mu_\infty^{(\alpha)}$. To see this, recall from Lemma 4 in [5] that self-similar fragmentations enjoy the Feller property, and hence the strong Markov property. The first instant when the tagged points $U$ and $\tilde{U}$ are disjoint (i.e. belong to two different fragments) is an a.s. finite randomized stopping time for which the strong Markov property thus applies. This immediately yields the stated asymptotic independence.

Putting the pieces together, we see that

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\left(\int_{]0,\infty[} f(y)\mu_t^{(\alpha)}(dy)\right) &= \int_{]0,\infty[} f(y)\mu_\infty^{(\alpha)}(dy)\\ \lim_{t\to\infty} \mathbb{E}\left(\left(\int_{]0,\infty[} f(y)\mu_t^{(\alpha)}(dy)\right)^2\right) &= \left(\int_{]0,\infty[} f(y)\mu_\infty^{(\alpha)}(dy)\right)^2,\end{aligned}$$

and we conclude that for every bounded continuous function $f$,

$$\lim_{t\to\infty} \int_{]0,\infty[} f(y)\mu_t^{(\alpha)}(dy) = \int_{]0,\infty[} f(y)\mu_\infty^{(\alpha)}(dy)$$

in $L^2(\mathbb{P})$. The convergence in probability $\mu_t^{(\alpha)}$ to $\mu_\infty^{(\alpha)}$ now follows from standard arguments. □

Let us now comment on Theorem 2, and then make the connection with related results in the literature. We first point out that the approach is close to the so-called propagation of chaos (see Sznitman [14]). More precisely, for every integer $k$, if $U_1, \ldots, U_k$ are independent uniformly distributed variables, then the argument of the proof of Theorem 2 shows that the $k$-tuple

$$t^{1/\alpha}|I_{U_1}(t)|, \ldots, t^{1/\alpha}|I_{U_k}(t)|$$

converges in law as $t \to \infty$ towards $\mu_\infty^{(\alpha)} \otimes \ldots \otimes \mu_\infty^{(\alpha)}$.

As we already pointed out, Theorem 2 is closely related to the work of Brennan and Durrett [7, 8], whose setting has been recalled in the Introduction. Roughly, these authors treated the case of discrete binary fragmentations (i.e. the Lévy measure $\nu$ has finite mass and is supported by the set of sequences $x = (x_1, \ldots) \in \mathcal{S}^*$ with $x_1 + x_2 = 1$). Then it is plain that at any time $t > 0$, the number $N(t)$ of fragments with positive size is finite, so one can investigate directly the true empirical measure

$$\frac{1}{N(t)}\sum_{i=1}^{N(t)}\delta_{X_i(t)}\,.$$

It should also be stressed that Brennan and Durrett established almost-sure convergence and not merely convergence in probability as we did.

Motivated by the study of the so-called standard additive coalescent, Aldous and Pitman [2] introduced a self-similar fragmentation with index $\alpha = 1/2$ and for which

$$\Phi(q) \;=\; \sqrt{2}\,\Gamma(q+1/2)/\Gamma(q)\,,$$

see the identity (12) in [5]. They were able to describe explicitly the distribution of $X(t)$ at any time $t>0$ in terms of the atoms of a certain conditioned Poisson measure on $]0,\infty[$, and then to investigate its asymptotic behavior for large times. More precisely, not only they obtained Theorem 2 in this case (then the limit distribution $\mu_\infty^{(\alpha)}$ has density $(2\pi y)^{-1/2}\mathrm{e}^{-y/2}$, i.e. it is the law of the square of a normal $\mathcal{N}(0,1)$ variable), but they also established Gaussian fluctuations, which does not seem easy to get in the general setting.

A striking application of Theorem 2 to statistics is the following. Suppose that the index of self-similarity $\alpha>0$ is given; then Theorem 2 yields an estimator of the function $\Phi$ evaluated at points $\alpha, 2\alpha, \ldots$, which is constructed from a *single* sample of $X^{(\alpha)}(t)$. More precisely, this estimator is consistent as we know that it converges to the exact values of $\Phi$ at such points when $t\to\infty$. On the other hand, because $\Phi$ is the Laplace exponent of some subordinator, it is easy to check that $\Phi$ is entirely determined by its values at such points. Moreover, we know that whenever the fragmentation is binary, then we can recover its Lévy measure from $\Phi$. In conclusion, if we know that a self-similar fragmentation with index $\alpha>0$ is binary and has no erosion term, then we can construct a consistent estimator of the law of the process from a single sample of the fragmentation observed at a large time.

Finally, as an easy application of Theorem 2, one can specify the asymptotic behavior of the scattering rates of $X^{(\alpha)}$. More precisely, if we set

$$\Sigma^{(\alpha)}(t,p) \;:=\; \sum_{i=1}^{\infty}(X_i^{(\alpha)}(t))^p\,,$$

then we have the following estimate.

**Corollary 1** *Under the assumptions of Theorem 2, we have for every $p\geq 1$ that*

$$\lim_{t\to\infty} t^{(p-1)/\alpha}\Sigma^{(\alpha)}(t,p) \;=\; \int_{]0,\infty[} y^{p-1}\mu_\infty^{(\alpha)}(dy) \;\in ]0,\infty[\,,$$

*where the convergence holds in probability.*

**Proof:** We have the identity

$$t^{(p-1)/\alpha}\Sigma^{(\alpha)}(t,p) \;=\; \int_{]0,\infty[} y^{p-1}\mu_t^{(\alpha)}(dy)\,;$$

however as the function $y \to y^{p-1}$ is unbounded, we cannot directly apply Theorem 2. Nonetheless, if we simply try to repeat the argument used there, we see immediately that all that is needed is to check that the family $\left(t^{1/\alpha}|I_U(t)|, t \geq 0\right)$ is bounded in $L^q(\mathbb{P})$ for every $q \geq 1$. In this direction, it is enough to exhibit a decreasing function $f : [0,\infty[\to [0,\infty[$ with $f(r) = o(r^{-q})$ at $\infty$ for every $q > 0$, such that

$$\mathbb{P}\left(t^{1/\alpha}|I_U(t)| > r\right) \leq f(r), \qquad \text{for all } t, r \geq 0.$$

For this, recall that $1/|I_U(\cdot)|$ is an increasing semi-stable Markov process corresponding to some subordinator $\xi$ via Lamperti's transformation. If we set $T_b = \inf\{t \geq 0 : 1/|I_U(t)| > b\}$, then Corollary 5 in [6] entails that for every $b > 1$, $b^{-\alpha}T_b$ is stochastically dominated by the variable $\int_0^\infty \exp(-\alpha\xi_s)ds$. We deduce that

$$\mathbb{P}\left(t^{1/\alpha}|I_U(t)| > r\right) = \mathbb{P}\left(T_{t^{1/\alpha}/r} > t\right) \leq \mathbb{P}\left(\int_0^\infty \exp(-\alpha\xi_s)ds > r^\alpha\right).$$

As the variable $\int_0^\infty \exp(-\alpha\xi_s)ds$ has finite moments of arbitrary order (these are computed e.g. in [9]), this completes the proof. □

We conclude this section with the following simple observation: when the conditions of Theorem 2 are fulfilled, the size of the largest fragment decays polynomially with power $-1/\alpha$, in the sense that

$$\log X_1^{(\alpha)}(t) \sim -\frac{1}{\alpha}\log t \qquad \text{as } t \to \infty.$$

More precisely, the upper bound follows from Corollary 1, whereas the lower bound is obtained by considering a size-biased picked fragment.

### 3.3 Case when the self-similarity index is negative

Results in the case $\alpha < 0$ are simple applications of the transformation $X^{(\alpha)} \to X$ described in Section 3.1. An informal guideline is that small masses are then subject to intense fragmentation, and this makes them vanish entirely quickly. Here is a formal statement.

**Proposition 1** (i) *For $\alpha < 0$, it holds with probability one that*

$$\inf\left\{t \geq 0 : X^{(\alpha)}(t) = (0,\ldots)\right\} < \infty.$$

(ii) *For every $t > 0$ and $\alpha < -1$, it holds with probability one that*

$$\texttt{Card}\left\{j \in \mathbb{N} : X_j^{(\alpha)}(t) > 0\right\} < \infty.$$

Of course, it would be interesting to determine the law of the lifetime

$$\zeta := \inf\left\{t \geq 0 : X^{(\alpha)}(t) = (0,\ldots)\right\},$$

but we have been able to tackle this question only in a couple of special cases. It is not even clear whether or not this distribution can be characterized just in terms of the exponent $\Phi$. On the other hand, we stress that in general, no matter what the value of $\alpha$ is, there may exist random instants $t$ at which

$$\texttt{Card}\left\{j \in \mathbb{N} : X_j^{(\alpha)}(t) > 0\right\} \;=\; \infty\,.$$

For instance in the case when the Lévy measure fulfills

$$\nu\left(x_j > 0 \text{ for all } j \in \mathbb{N}\right) \;=\; \infty\,,$$

then with probability one, there occur infinitely many sudden dislocations in the fragmentation process $X^{(\alpha)}$, each of which produces infinitely many masses. This does not induce any contradiction with Proposition 1 (ii) when $\alpha < -1$, because informally, as the index of self-similarity is negative, we know that small masses vanish quickly.

**Proof:** (i) The transformation $X^{(\alpha)} \to X$ described in Section 3.1 can be inverted as follows. Recall the notation there, and denote by $\tilde{I}_x(t)$ the interval component of the open set $\tilde{V}(t)$ that contains $x \in ]0,1[$ (again with the convention that $\tilde{I}_x(t) = \emptyset$ if $x \notin \tilde{V}(t)$. Next consider the time-substitution

$$\tilde{T}_x(t) \;:=\; \inf\left\{u \geq 0 : \int_0^u |\tilde{I}_x(r)|^{-\alpha} dr > t\right\}\,.$$

Then $\left\{\tilde{I}_x(\tilde{T}_x(t)), 0 < x < 1\right\}$ coincides with the family of the interval components of the open set $V(t)$. On the other hand, we know from (4) that for any $m' \in ]0, m[$, there exists an a.s. finite random variable $C$ such that

$$|\tilde{I}_x(r)| \;\leq\; C\mathrm{e}^{-m'r} \qquad \text{for all } 0 < x < 1 \text{ and } r \geq 0.$$

As $\alpha < 0$, we deduce that if we set $C' = C/(-\alpha m') > 0$, then

$$\int_0^\infty |\tilde{I}_x(r)|^{-\alpha} dr \;\leq\; C'\,,$$

and hence $\tilde{T}_x(C') = \infty$ for all $0 < x < 1$. This shows that $X^{(\alpha)}(t) = (0, \ldots)$ for every $t \geq C'$.

(ii) Recall the notation

$$\Sigma^{(\alpha)}(t,p) \;:=\; \sum_{i=1}^{\infty} (X_i^{(\alpha)}(t))^p\,.$$

We first point out that for every $p > -\alpha$, we have the identity

$$\int_0^\infty \Sigma^{(\alpha)}(t, p+\alpha) dt \;=\; \int_0^\infty \Sigma(t,p) dt\,. \tag{6}$$

Indeed, in the notation of Section 3.1, we have for every $t \geq 0$

$$\Sigma(t,p) \;=\; \int_0^1 |I_x(T_x(t))|^{p-1} dx\,.$$

By the change of variables $T_x(t) = s$, $dt = |I_x(s)|^\alpha ds$, in the second identity below, we get

$$\begin{aligned}
\int_0^\infty \Sigma(t,p) dt \;&=\; \int_0^1 dx \int_0^\infty dt |I_x(T_x(t))|^{p-1} \\
&=\; \int_0^1 dx \int_0^\infty ds |I_x(s)|^{p+\alpha-1} \\
&=\; \int_0^\infty \Sigma^{(\alpha)}(s, p+\alpha) ds\,.
\end{aligned}$$

As a consequence, if moreover $p > 1$, then by (3), the random variable in (6) has mean $1/\Phi(p-1)$. Then let $p$ decreases to $-\alpha$ (recall that the assumption $-\alpha > 1$), so $\Sigma^{(\alpha)}(t, p+\alpha)$ increases to

$$\Sigma^{(\alpha)}(t,0) \;=\; \texttt{Card}\left\{j \in \mathbb{N} : X_j^{(\alpha)}(t) > 0\right\} ,$$

the number of fragments at time $t$. We get by monotone convergence

$$\int_0^\infty \mathbb{E}\left(\Sigma^{(\alpha)}(t,0)\right) dt \;=\; \frac{1}{\Phi(-\alpha-1)} \;<\; \infty \,,$$

which establishes our claim. □

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