Extinction Rates for Branching Processes in a Lévy Environment: The Subcritical Regime

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Natalia Cardona Tobón

Joint work with Dr. Juan Carlos Pardo Millán

Centro de Investigación en Matemáticas, A.C.

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Outline

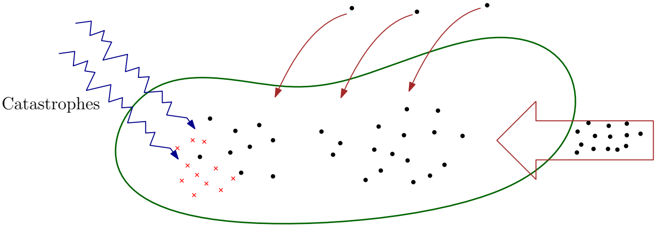
Biological Population Structure

$Z_t :=$ population size at time t

Natives + immigrants

Demographic Randomness → Reproduction, Immigration

Environmental Randomness → Catastrophes



We consider a filtered probability space
$(\Omega^{(b)}, \mathcal{F}^{(b)}, (\mathcal{F}_t^{(b)})_{t\geq 0}, \mathbb{P}^{(b)})$

Definition 1.1 (CSBP)

*A **branching process** is a* $[0,\infty]$*-valued strong Markov process* $Y = (Y_t, t \geq 0)$ *(where* 0 *and* ∞ *are absorbent states), which has càdlàg paths and their probabilities* $(\mathbb{P}_x^{(b)}, x \geq 0)$ *satisfying the branching property: for all* $\theta \geq 0$ *and* $x, y \geq 0$*,*

$$\mathbb{E}_{x+y}^{(b)}[e^{-\theta Y_t}] = \mathbb{E}_x^{(b)}[e^{-\theta Y_t}]\mathbb{E}_y^{(b)}[e^{-\theta Y_t}], \quad t \geq 0.$$

$$\mathbb{E}_x^{(b)}[e^{-\theta Y_t}] = e^{-x u_t(\theta)}, \quad \text{where } u \text{ satisfies} \quad \frac{\partial u_t}{\partial t}(\theta) = -\psi(u_t(\theta)), \quad u_0(\theta) = \theta,$$

ψ is the *branching mechanism* and satisfies the Lévy-Khintchine formula.

Let μ be a measure concentrated on $(0,\infty)$ and we assume that

$$\int_{(0,\infty)} (x \wedge x^2)\mu(\mathrm{d}x) < \infty,$$

which guarantees non-explosivity (Fu and Li, 2010).

The function ψ satisfies

$$\psi(\lambda) := \psi'(0+)\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x)\mu(\mathrm{d}x), \quad \lambda \geq 0,$$

where $\gamma \geq 0$ and we have

$$\mathbb{E}_x^{(b)}[Y_t] = x e^{-\psi'(0+)t}, \quad x,\ t \geq 0.$$

$$Y \text{ is } \begin{cases} \textbf{supercritical} & \text{if} \quad \psi'(0+) < 0 \\ \textbf{critical} & \text{if} \quad \psi'(0+) = 0 \\ \textbf{subcritical} & \text{if} \quad \psi'(0+) > 0 \end{cases}$$

The process Y can also be defined as the **unique non-negative strong solution** (Dawson and Li, 2012) of

$$Y_t = Y_0 - \psi'(0+) \int_0^t Y_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s} \mathrm{d}B_s^{(b)} + \int_0^t \int_{(0,\infty)} \int_0^{Y_{s-}} z \widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where

- $B^{(b)} = (B_t^{(b)}, t \geq 0)$ is a standard Brownian motion.
- $N^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ in $\mathbb{R}_+^3$ with intensity $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$.
- $\widetilde{N}^{(b)}$ is the compensated measure of $N^{(b)}$.

$B^{(b)}$ is independent of $N^{(b)}$.

CSBP in Lévy environment

Environmental term: $(\Omega^{(e)}, \mathcal{F}^{(e)}, (\mathcal{F}_t^{(e)})_{t\geq 0}, \mathbb{P}^{(e)})$

- $B^{(e)}$ is a standard Brownian motion.
- $N^{(e)}(\mathrm{d}s, \mathrm{d}z)$ in $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\mathrm{d}s\pi(\mathrm{d}y)$.
- π concentrated on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}} (1 \wedge x^2)\pi(\mathrm{d}x) < \infty$ and $\alpha \in \mathbb{R}, \sigma \geq 0$.

$$S_t = \alpha t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} (e^z - 1)\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)^c} (e^z - 1)N^{(e)}(\mathrm{d}s, \mathrm{d}z)$$

We consider independent processes for demography and environment.

We work on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where

$\Omega := \Omega^{(e)} \times \Omega^{(b)}, \mathcal{F} := \mathcal{F}^{(e)} \otimes \mathcal{F}^{(b)}, \mathcal{F}_t := \mathcal{F}_t^{(e)} \otimes \mathcal{F}_t^{(b)}$ for $t \geq 0$,

$\mathbb{P} := \mathbb{P}^{(e)} \otimes \mathbb{P}^{(b)}$.

A CSBP in a Lévy random environment $Z = (Z_t, t \geq 0)$ with probabilities $(\mathbb{P}_z, z \geq 0)$ is defined as the unique non-negative strong solution of

$$
\begin{aligned}
Z_t = \quad & Z_0 \quad - \int_0^t \psi'(0+) Z_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s} \mathrm{d}B_s^{(b)} \\
+ \quad & \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}} z \widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t Z_{s-} \mathrm{d}S_s.
\end{aligned}
$$

Observations:

- **Pathwise uniqueness and strong existence** were proved independently by Palau and Pardo (2018) and He et al. (2018).
- $\int_{(0,\infty)} (x \wedge x^2) \mu(dx) < \infty$ guarantees **non-explosivity** (Bansaye et al., 2019).

We define the process K, a modification of the jump structure of S, with probabilities $(\mathbb{P}_x^{(e)}, x \in \mathbb{R})$

$$K_t = \overline{\alpha}t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} z\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)^c} zN^{(e)}(\mathrm{d}s, \mathrm{d}z),$$

where $\overline{\alpha} := \alpha - \psi'(0+) - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^z - 1 - z)\pi(\mathrm{d}z)$.

Theorem 1.1 (Bansaye et al., 2019)

For $\mathbb{P}^{(e)}$ almost every $w^{(e)} \in \Omega^{(e)}$, $\left(\exp\left\{-K_t(w^{(e)}, \cdot)\right\} Z_t(w^{(e)}, \cdot), t \geq 0\right)$ is a $(\Omega^{(b)}, \mathcal{F}^{(b)}, (\mathcal{F}_t^{(b)})_{t\geq 0}, \mathbb{P}^{(b)})$-martingale and for any $t \geq 0$ and $z \geq 0$,

$$\mathbb{E}_z[Z_t|S] = ze^{K_t}, \quad \mathbb{P} - a.s.$$

$$Z \text{ is } \begin{cases} \textbf{supercritical} & \text{if} \quad K \text{ drifts to } \infty \\ \textbf{critical} & \text{if} \quad K \text{ oscillates} \\ \textbf{subcritical} & \text{if} \quad K \text{ drifts to } -\infty \end{cases}$$

Theorem 2.1 (Palau and Pardo, 2018, Li and Xu, 2018)

For every $z, \lambda, t \geq 0$, we have a.s.,

$$\mathbb{E}_{(z,0)}\left[\exp\left\{-\lambda Z_t e^{-K_t}\right\}|K\right] = \exp\left\{-z v_t(0,\lambda,K)\right\},$$

where for every $t, \lambda \geq 0$, the function $(v_t(s,\lambda,K), s \leq t)$ is the a.s. unique solution of the backward differential equation

$$\frac{\partial}{\partial s} v_t(s,\lambda,K) = e^{K_s}\psi_0(v_t(s,\lambda,K)e^{-K_s}), \quad v_t(t,\lambda,K) = \lambda,$$

and $\psi_0(\lambda) := \psi(\lambda) - \lambda\psi'(0+)$.

The survival probability if: $\psi_0(\lambda) = c\lambda^{\beta+1}, \quad \beta \in (0,1], \quad c > 0.$

$$\mathbb{P}_z\left(Z_t > 0\right) = \mathbb{E}^{(e)}\left[1 - \exp\left\{-z\left(\beta c \mathcal{I}_t(\beta K)\right)^{-1/\beta}\right\}\right],$$

where $\mathcal{I}_t(\beta K) := \int_0^t e^{-\beta K_u} \mathrm{d}u, \quad 0 \leq t \leq \infty.$

Theorem 2.2 (Palau et al., 2016, Li and Xu, 2018)

Let $(Z_t, t \geq 0)$ *be a stable CSBP in Lévy environment with* $Z_0 = z > 0$. *We denote* $\Phi_K(\lambda) = \log \mathbb{E}^{(e)}[e^{\lambda K_1}]$.

1. *Supercritical. If* $\Phi'_K(0+) > 0$, *then* $\lim_{t\to\infty} \mathbb{P}_z(Z_t > 0) > 0$.
2. *Critical. If* $\Phi'_K(0+) = 0$, *then* $\lim_{t\to\infty} \sqrt{t}\mathbb{P}_z(Z_t > 0) = c_3(z)$.
3. *Subcritical. Assume that* $\Phi'_K(0+) < 0$, *then*
   - *Strongly. If* $\Phi'_K(1) < 0$, *then there exists* $c_1 > 0$ *such that*
   $$\lim_{t\to\infty} e^{-t\Phi_K(1)}\mathbb{P}_z(Z_t > 0) = c_1 z.$$
   - *Intermediate. If* $\Phi'_K(1) = 0$, *then there exist* $c_2 > 0$ *such that*
   $$\lim_{t\to\infty} t^{1/2} e^{-t\Phi_K(1)}\mathbb{P}_z(Z_t > 0) = c_2 z.$$
   - *Weakly. If* $\Phi'_K(1) > 0$ *and* τ *satisfies* $\Phi'_K(\tau) = 0$, *then there exists* c_4 *such that*
   $$\lim_{t\to\infty} t^{3/2} e^{-t\Phi_K(\tau)}\mathbb{P}_z(Z_t > 0) = c_4(z).$$

General branching mechanism (critical case)

$$\frac{1}{t}\int_0^t \mathbb{P}^{(e)}(K_t \geq 0)\mathrm{d}t \to \rho \in (0,1), \quad \text{as} \quad t \to \infty,$$

(i) $$\int^{\infty} x \ln^2(x)\mu(\mathrm{d}x) < \infty,$$

(ii) there exists $\theta^+ > 1$ such that $\mathbb{E}^{(e)}\left[e^{\theta^+ K_1}\right] < \infty$,

(iii) there exists $\beta \in (\mathbf{0}, \mathbf{1}]$ y $\mathbf{C} > \mathbf{0}$ such that $\psi_{\mathbf{0}}(\lambda) \geq \mathbf{C}\lambda^{\mathbf{1}+\beta}$ for $\lambda \geq \mathbf{0}$.

Theorem 2.3 (Bansaye et al. (2019))

There exists a positive function c such that for any $z > 0$,

$$\mathbb{P}_z(Z_t > 0) \sim c(z)\mathbb{P}_1^{(e)}(I_t > 0) \sim b(z)t^{\rho-1}l(t), \quad \text{as } t \to \infty,$$

where $I_t = \text{ínf}_{0\leq s\leq t} K_s$, $\rho < 1$, b is other positive function and l is the slowly varying function, i.e., for $\lambda > 0$, $\lim_{t\to\infty} \frac{l(\lambda t)}{l(t)} = 1$.

Problem

Our objective is to relax the assumption that the branching mechanism is stable and to find **extinction rates** of the process Z. We focus on the **subcritical regime**, i.e., when the Lévy process K drifts to $-\infty$.

Subcritical regime: $\Phi_K'(0) < 0$, where $\Phi_K(\lambda) = \log \mathbb{E}^{(e)}[e^{\lambda K_1}]$		
Weakly	**Intermediate**	**Strongly**
$\Phi_K'(1) > 0$ $\exists \tau \in (0,1),\ \Phi_K'(\tau) = 0$	$\Phi_K'(1) = 0$	$\Phi_K'(1) < 0$

The **Esscher transform** is define on $\mathcal{F}_t$ as

$$\mathbb{P}^{(\tau)} = e^{\tau K_t - t\Phi_K(\tau)} \cdot \mathbb{P}.$$

Fluctuation Theory- Definitions

We define,

$$I_t = \inf_{0 \le s \le t} K_s, \quad M_t = \sup_{0 \le s \le t} K_s, \quad t \ge 0.$$

Let $V^{(\tau)}$ and $\widehat{V}^{(\tau)}$ be the renewal functions under $\mathbb{P}^{(\tau)}$, i.e.,

$$V^{(\tau)}(x) := \int_0^\infty \mathbb{P}^{(\tau)}(H_t \le x)\mathrm{d}t \quad \text{and} \quad \widehat{V}^{(\tau)}(x) := \int_0^\infty \mathbb{P}^{(\tau)}(\widehat{H}_t \le x)\mathrm{d}t,$$

where

$$\widehat{H}_t = -I_{\widehat{L}_t^{-1}}, \quad H_t = M_{L_t^{-1}}, \quad t \ge 0,$$

are the descending and ascending ladder processes, respectively. We recall that

$$\widehat{L}_t^{-1} = \inf\{s \ge 0 : \widehat{L}_s > t\},$$

for $\widehat{L}$ the local time at 0 of $K - I$. Similarly, we define the local time at 0 of $M - K$.

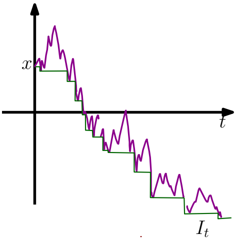
CSBP in a Lévy environment conditioned to stay positive and negative

For $\Lambda \in \mathcal{F}_t$

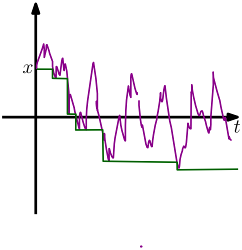
$$\mathbb{P}^{(\tau),\uparrow}_{(z,x)}(\Lambda) = \frac{1}{\widehat{V}^{(\tau)}(x)} \mathbb{E}^{(\tau)}_{(z,x)}[\widehat{V}^{(\tau)}(K_t)\mathbf{1}_{\{I_t>0\}}\mathbf{1}_\Lambda], \quad x, z > 0,$$

$$\mathbb{P}^{(\tau),\downarrow}_{(z,x)}(\Lambda) = \frac{1}{V^{(\tau)}(-x)} \mathbb{E}^{(\tau)}_{(z,x)}[V^{(\tau)}(\widehat{K}_t)\mathbf{1}_{\{M_t<0\}}\mathbf{1}_\Lambda], \quad x < 0, z > 0.$$

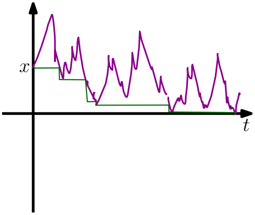
Illustration



K under $\mathbb{P}_x$
$\Phi'_K(0) < 0$



K under $\mathbb{P}^{(\tau)}_x$
$\Phi'_K(\tau) = 0$



K under $\mathbb{P}^{(\tau),\uparrow}_x$

Strongly Subcritical

Theorem 5.1 (Strongly Subcritical)

Assume condition $\int^{\infty} x\log(x)\mu(\mathrm{d}x) < \infty$ *and that the Laplace exponent of Lévy process* K *fulfills the conditions* $\Phi'_K(0) < 0$ *and* $\Phi'_K(1) < 0$. *For* $z > 0$, *we have*

$$\mathbb{P}_z(Z_t > 0) \sim b_3(z,x)\mathbb{E}_z[Z_t] \sim b_3(z,x)e^{\Phi_K(1)t}, \quad as \quad t \to \infty,$$

where $0 < b_3(z,x) < \infty$.

$$\boxed{\mathbb{P}_z(Z_t > 0) \sim b_3(z,x)\mathbb{E}_z[Z_t] \sim b_3(z,x)e^{t\Phi_K(1)},\ t \to \infty}$$

$$0 < \lim_{t\to\infty} \mathbb{E}^{(1)}_{(z,x)}[e^{-K_t} g_z(K)] < \infty$$

where $g_z(K) = \mathbb{P}_z(Z_t > 0|K)$

Bansaye et al. (2019)

$$\mathbb{E}_{(z,x)}[Z_t] = ze^x e^{t\Phi_K(1)}$$

Weakly Subcritical

Theorem 5.2 (Weakly Subcritical)

Suppose that $\Phi_K'(0) < 0 < \Phi_K'(1)$. *Let* $\tau \in (0,1)$ *be the solution of* $\Phi_K'(\tau) = 0$. *Also assume that* $\textbf{\textit{Re}}\Psi_K(\lambda) = \log \mathbb{E}^{(e)}[e^{i\lambda K_1}] > 0, \lambda \neq 0$. *For* $z > 0$, *we have, as* $t \to \infty$,

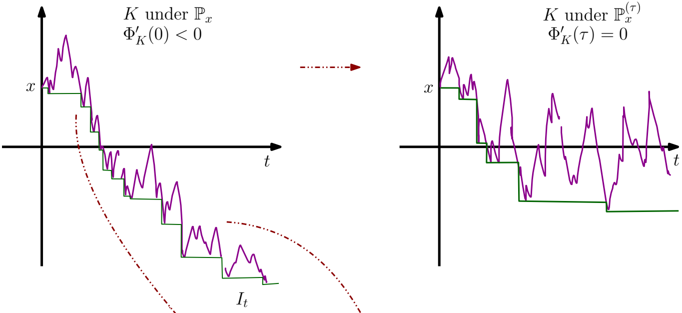
$$\begin{aligned}\mathbb{P}_{(z,x)}(Z_t > 0) \quad &\sim \quad b_1(z,x)\mathbb{P}_x^{(e)}(I_t > 0) \\ &\sim \quad b_1(z,x)Ae^{\tau x}\widehat{V}^{(\tau)}(x)t^{-3/2}e^{\Phi_K(\tau)t}\int_0^\infty e^{-\tau z}V^{(\tau)}(z)\mathrm{d}z,\end{aligned}$$

where

$$A := \frac{1}{\sqrt{2\pi\Phi_K''(\tau)}}\exp\left\{\int_0^\infty (e^{-t}-1)t^{-1}e^{-t\Phi_K(\tau)}\mathbb{P}^{(e)}(K_t = 0)\mathrm{d}t\right\},$$

and $b_1(z,x)$ *is a constant that depend on* z *and* x.

Weakly Subcritical



$$\mathbb{P}_z(Z_t > 0) = \mathbb{P}_{(z,x)}(Z_t > 0, I_t > 0) + \mathbb{P}_{(z,x)}(Z_t > 0, I_t \leq 0)$$

We study (Z, K) under $\mathbb{P}_{(z,x)}^{(\tau),\uparrow}$

$$\mathbb{P}_{(z,x)}(Z_t > 0, I_t < 0) \leq \epsilon \mathbb{P}_{(z,x)}(Z_t > 0, I_t > 0)$$

$$\mathbb{P}_{(z,x)}(Z_t > 0, I_t > 0) = \mathbb{P}_{(z,x)}(Z_t > 0 | I_t > 0)\mathbb{P}_x^{(e)}(I_t > 0)$$

Hirano (2001)

$$\mathbb{P}_x^{(e)}(I_t > 0) \sim c_3 e^{\tau x} \widehat{V}^{(\tau)}(x) t^{-3/2} e^{\Phi_K(\tau) t}, \ t \to \infty$$

$$c_3 := A \int_0^\infty e^{-\tau z} V^{(\tau)}(z) dz \text{ y } A > 0$$

Intermediate Subcritical

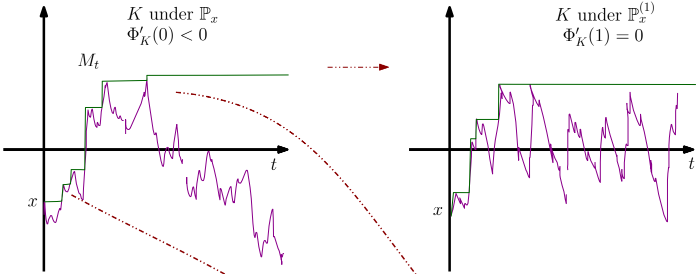
Theorem 5.3 (Intermediate Subcritical)

Assume condition $\int^{\infty} x \ln^2(x)\mu(\mathrm{d}x) < \infty$ *and that the Laplace exponent of Lévy process* K *fulfills the conditions* $\Phi_K'(0) < 0$ *and* $\Phi_K'(1) = 0$. *For* $z > 0$ *and* $x < 0$ *we have, as* $t \to \infty$

$$\begin{aligned} \mathbb{P}_z(Z_t > 0) &\sim b_2(z,x)\mathbb{P}_x^{(1)}(M_t < 0)\mathbb{E}_z[Z_t] \\ &\sim b_2(z,x)\sqrt{\frac{2}{\pi\Phi_K''(1)}}\mathbb{E}^{(1)}[H_1]V^{(1)}(-x)t^{-1/2}e^{\Phi_K(1)t}, \end{aligned}$$

where $0 < b_2(z,x) < \infty$.

Intermediate Subcritical



$$\mathbb{P}_z(Z_t > 0) = \mathbb{P}_{(z,x)}(Z_t > 0, M_t \leq 0) + \mathbb{P}_{(z,x)}(Z_t > 0, M_t > 0), \ t \to \infty$$

We study (Z, K) under $\mathbb{P}^{(1),\downarrow}_{(z,x)}$

$$\mathbb{P}_{(z,x)}(Z_t > 0, M_t > 0) \leq \epsilon \mathbb{P}_{(z,x)}(Z_t > 0, M_t < 0)$$

Hirano (2001)

$$\mathbb{P}^{(1)}(M_t \leq x) \sim \sqrt{\frac{2}{\pi \Phi''_K(1)}} \mathbb{E}^{(1)}[H_1] V^{(1)}(-x) t^{-1/2}, \ t \to \infty$$

Bansaye et al. (2019)

$$\mathbb{E}_{(z,x)}[Z_t] = z e^x e^{t\Phi_K(1)}$$

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