The ant in the labyrinth

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BUC-Chile Probability meeting

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- $\Phi_{\mathfrak{T}}(\mathfrak{T})$ is the Integrated Super Brownian excursion.
- The ISE is the CRT embedded in $\mathbb{R}^d$ using Brownian motions.
- If $d \geq 8$, $\Phi_{\mathfrak{T}}$ is injective and the ISE is a tree.

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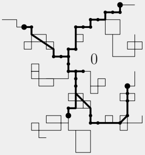

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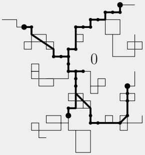

Three abstract conditions on abstract random graphs which imply the convergence of the ant to $(B_t^{ISE})_{t\geq 0}$.
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$$\mathbb{P}\left[\left|\frac{R_{\text{eff}}(\mathbf{o}, U_n)}{d_{G_n}(\boldsymbol{o}, U_n)} - \rho\right| > \epsilon\right] \to 0$$

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   for all $\epsilon > 0$, where U_n is a uniformly chosen vertex.
3. Volume condition: The edge volume of G_n is uniformly distributed in $\mathcal{T}_n^{(K)}$

Theorem. Ben Arous, C., Fribergh

If the conditions are satisfied, then

$$(n^{-1/4} X^{G_n}_{n^{3/2}t})_{t \geq 0} \to (B^{ISE}_t)_{t \geq 0}.$$

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  - $n^{-1/4} \max_{x \in G_n} d_{\mathbb{Z}^d}(x, \mathcal{T}_n^{(K)}) \to 0$ in probability.

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  - Transfer results.

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Sketch of proof of the abstract theorem

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- Let $B^{\mathfrak{T}^{(K)}}$ be the Brownian motion in $\mathfrak{T}^{(K)}$
- For K large, we have that $B^{\mathfrak{T}^{(K)}}$ is close to B^{ISE}
- By the Resistance condition we have that $(n^{-1/4}X_{nt}^{\mathcal{T}_n^{(K)}})_{t\geq 0} \to (B^{\mathfrak{T}^{(K)}})_{t\geq 0}$.

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- By the Resistance condition we have that $(n^{-1/4}X_{nt}^{\mathcal{T}_n^{(K)}})_{t\geq 0} \to (B^{\mathfrak{T}^{(K)}})_{t\geq 0}$.
- By the Volume condition, X^{G_n} is related to $X^{\mathcal{T}_n^{(K)}}$ by a time-change which is asymptotically linear (after proper rescaling and K large).

Thanks!