The ant in the labyrinth

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BUC-Chile Probability meeting

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Introduction

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- At $p = p_c$ it is expected that there is no infinite cluster, but there are large clusters at all scales.

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- de Gennes 1976: Popularized the model in the article *Percolation, a unifying concept*. He coined the term *The ant in the labyrinth*.

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For X^{IIC} we have $d_w > 2$, subdiffusivity.

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- The conjecture is expected to be false for $d < 6$.

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Contour Process

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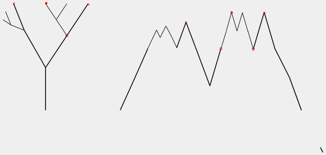

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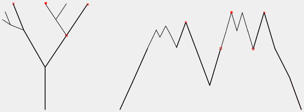

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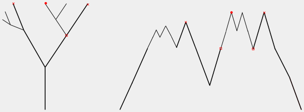

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- This explains the critical exponents for percolation on a tree discussed above.