Alpha-stable (Lévy) processes through the Lamperti-Kiu transform

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§1. Self-similar Markov processes

Self-similar Markov processes (ssMp)

Definition

A regular strong Markov process $(Z_t : t \geq 0)$ on $\mathbb{R}^d$, with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}^d$, is a rssMp if there exists an index $\alpha \in (0, \infty)$ such that for all $c > 0$ and $x \in \mathbb{R}^d$,

$$(cZ_{tc^{-\alpha}} : t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal in law to } (Z_t : t \geq 0) \text{ under } \mathbb{P}_{cx}.$$

Some of your best friends are ssMp

- Write $\mathcal{N}_d(\mathbf{0}, \mathbf{\Sigma})$ for the Normal distribution with mean $\mathbf{0} \in \mathbb{R}^d$ and covariance (matrix) $\mathbf{\Sigma}$. The moment generating function of $X_t \sim \mathcal{N}_d(\mathbf{0}, \mathbf{\Sigma}t)$ satisfies, for $\theta \in \mathbb{R}^d$,
$$\mathbf{E}[e^{\theta \cdot X_t}] = e^{t\theta^{\mathrm{T}}\mathbf{\Sigma}\theta/2} = e^{(c^{-2}t)(c\theta)^{\mathrm{T}}\mathbf{\Sigma}(c\theta)/2} = E[e^{\theta \cdot cX_{c^{-2}t}}].$$

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- Thinking about the stationary and independent increments of Brownian motion, this can be used to show that $\mathbb{R}^d$**-Brownian motion:** is a ssMp with $\alpha = 2$.

SOME OF YOUR BEST FRIENDS ARE SSMP

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- Write $\underline{X}_t := \inf_{s \leq t} X_s$. Then $(X_t, \underline{X}_t)$, $t \geq 0$ is a Markov process.

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- For $c > 0$ and $\alpha = 2$,

$$\begin{pmatrix} c\underline{X}_{c^{-\alpha}t} \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} c\inf_{s \leq c^{-\alpha}t} X_s \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} \inf_{u \leq t} cX_{c^{-\alpha}u} \\ cX_{c^{-\alpha}t} \end{pmatrix}, \qquad t \geq 0,$$

and the latter is equal in law to $(X, \underline{X})$, because of the scaling property of X.

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- Markov process $Z_t := X_t - (-x \wedge \underline{X}_t)$, $t \geq 0$ is also a ssMp on $[0, \infty)$ issued from $x > 0$ with index 2.

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- Markov process $Z_t := X_t - (-x \wedge \underline{X}_t)$, $t \geq 0$ is also a ssMp on $[0, \infty)$ issued from $x > 0$ with index 2.
- $Z_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$ is also a ssMp, again on $[0, \infty)$.

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Consider $Z_t := |X_t|$, $t \geq 0$. Because of rotational invariance, it is a Markov process.
- Again the self-similarity (index 2) of Brownian motion, transfers to the case of $|X|$. Note again, this is a ssMp on $[0, \infty)$.

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- Again the self-similarity (index 2) of Brownian motion, transfers to the case of $|X|$. Note again, this is a ssMp on $[0, \infty)$.
- Note that $|X_t|$, $t \geq 0$ is a Bessel-d process. It turns out that all Bessel processes, *and* all squared Bessel processes are self-similar on $[0, \infty)$. Once can check this by e.g. considering scaling properties of their transition semi-groups.

Some of your best friends are ssMp

Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Note when $d = 3$, $|X_t|, t \geq 0$ is also equal in law to a Brownian motion conditioned to stay positive
- i.e if we define, for a 1-d Brownian motion $(B_t : t \geq 0)$,

$$\mathbb{P}^{\uparrow}_x(A) = \lim_{s\to\infty} \mathbb{P}_x(A | \underline{B}_{t+s} > 0) = \mathbb{E}_x\left[\frac{B_t}{x}\mathbf{1}_{(\underline{B}_t > 0)}\mathbf{1}_{(A)}\right]$$

where $A \in \sigma\{B_t : u \leq t\}$, then

$$(|X_t|, t \geq 0) \text{ with } |X_0| = x \text{ is equal in law to } (B, \mathbb{P}^{\uparrow}_x).$$

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- We want to find more exotic examples as most of the previous examples have been extensively studied through existing theories (of Brownian motion and continuous semi-martingales).
- All of the previous examples are functional transforms of Brownian motion and have made use of the scaling and Markov properties and (in some cases) isotropic distributional invariance.
- If we replace Brownain motion by an α-stable process, a Lévy process that has scale invariance, then all of the functional transforms still produce new examples of self-similar Markov processes.

(Killed) Lévy process

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- $(\xi_t, t \geq 0)$ is a (killed) Lévy process if it has stationary and independents with RCLL paths (and is sent to a cemetery state after and independent and exponentially distributed time).
- Process is entirely characterised by its one-dimensional transitions, which are coded by the Lévy–Khinchine formula

$$\mathbf{E}[\mathrm{e}^{\mathrm{i}\theta\cdot\xi_t}] = \mathrm{e}^{-\Psi(\theta)t}, \qquad \theta \in \mathbb{R}^d,$$

where,

$$\Psi(\theta) = q + \mathrm{i}\mathrm{a}\cdot\theta + \frac{1}{2}\theta\cdot\mathbf{A}\theta + \int_{\mathbb{R}^d}(1 - \mathrm{e}^{\mathrm{i}\theta\cdot x} + \mathrm{i}(\theta\cdot x)\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x),$$

where $\mathrm{a} \in \mathbb{R}$, $\mathbf{A}$ is a $d \times d$ Gaussian covariance matrix and Π is a measure satisfying $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\Pi(\mathrm{d}x) < \infty$. Think of Π as the intensity of jumps in the sense of

$$\mathbf{P}(X \text{ has jump at time } t \text{ of size } \mathrm{d}x) = \Pi(\mathrm{d}x)\mathrm{d}t + o(\mathrm{d}t).$$

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$$\Psi(\theta) = |\theta|^{\alpha}(\mathrm{e}^{\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

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  where $\rho = \mathrm{P}_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$
- Assume jumps in both directions ($0 < \alpha\rho, \alpha\hat{\rho} < 1$), so that the Lévy **density** takes the form

$$\frac{\Gamma(1+\alpha)}{\pi}\frac{1}{|x|^{1+\alpha}}\left(\sin(\pi\alpha\rho)\mathbf{1}_{\{x>0\}} + \sin(\pi\alpha\hat{\rho})\mathbf{1}_{\{x<0\}}\right)$$

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- which by stationary and independent increments is equivalent to saying $(cX_{c^{-\alpha}t}, t \geq 0) =^d (X_t, t \geq 0)$ when $X_0 = 0$,
- or equivalently is equivalent to saying $(cX^{(x)}_{c^{-\alpha}t}, t \geq 0) =^d (X^{(cx)}_t, t \geq 0)$, where we have indicated the point of issue as an additional index.

STABLE PROCESS PATH PROPERTIES

index	jumps	path	recurrence/transience
$\alpha \in (0,1)$			transient
$\rho = 0$	$-$	monotone decreasing	$\lim_{t\to\infty} X_t = -\infty$
$\rho = 1$	$+$	monotone increasing	$\lim_{t\to\infty} X_t = \infty$
$\rho \in (0,1)$	$+,-$	bounded variation	$\lim_{t\to\infty} \lvert X_t\rvert = \infty$
$\alpha = 1$			recurrent
$\rho = \frac{1}{2}$	$+,-$	unbounded variation	$\limsup_{t\to\infty} \lvert X_t\rvert = \infty$, $\liminf_{t\to\infty} \lvert X_t\rvert = 0$
$\alpha \in (1,2)$			recurrent
$\alpha\rho = 1$	$-$	unbounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R}$, $-\liminf_{t\to\infty} X_t = \limsup_{t\to\infty} X_t = \infty$
$\alpha\rho = \alpha - 1$	$+$	unbounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R}$, $-\liminf_{t\to\infty} X_t = \limsup_{t\to\infty} X_t = \infty$
$\alpha\rho \in (\alpha-1, 1)$	$+,-$	unbounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R}$, $-\liminf_{t\to\infty} X_t = \limsup_{t\to\infty} X_t = \infty$

Your new friends

Suppose $X = (X_t : t \geq 0)$ is within the assumed class of α-stable processes in one-dimension and let $\underline{X}_t = \inf_{s \leq t} X_s$.

Your new friends are:

- $Z = X$
- $Z = X - (-x \wedge \underline{X})$, $x > 0$.
- $Z = X \mathbf{1}_{(\underline{X} > 0)}$
- $Z = |X|$ providing $\rho = 1/2$
- $Z = X$ conditioned to stay positive

$$\mathbb{P}_x^{\uparrow}(A) = \lim_{s \to \infty} \mathbb{P}_x(A | \underline{X}_{t+s} > 0) = \mathbb{E}_x\left[\frac{X_t^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}} \mathbf{1}_{(\underline{X}_t > 0)} \mathbf{1}_{(A)}\right]$$

for $A \in \sigma(X_u : u \leq t)$

CONDITIONED α-STABLE PROCESSES

- For $c, x > 0$, $t \geq 0$ and appropriately bounded, measurable and non-negative f, we can write,

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\begin{aligned}
&\mathbb{E}_x^{\uparrow}[f(\{cX_{c^{-\alpha}s} : s \leq t\})] \\
&\quad = \mathbb{E}\left[f(\{cX^{(x)}_{c^{-\alpha}s} : s \leq t\})\frac{(X^{(x)}_{c^{-\alpha}t})^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}^{(x)}_{c^{-\alpha}t} \geq 0)}\right] \\
&\quad = \mathbb{E}\left[f(\{X^{(cx)}_{s} : s \leq t\}\frac{(X^{(cx)}_{t})^{\alpha\hat{\rho}}}{(cx)^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}^{(cx)}_{t} \geq 0)}\right] \\
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- This also makes the process $(X, \mathbb{P}_x^{\uparrow})$, $x > 0$, a self-similar Markov process on $[0, \infty)$.
- Unlike the case of Brownian motion, the conditioned stable process does not have the law of the radial part of a 3-dimensional stable process (the analogue to the Brownian case).

§2. Lamperti Transform

NOTATION

- Use $\xi := \{\xi_t : t \geq 0\}$ to denote a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time, $\mathbf{e}_q$, with rate in $q \in [0, \infty)$. The characteristic exponent of ξ is thus written

$$-\log \mathbf{E}(\mathrm{e}^{\mathrm{i}\theta\xi_1}) = \Psi(\theta) = q + \text{Lévy–Khintchine}$$

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- Define the associated integrated exponential Lévy process

$$I_t = \int_0^t \mathrm{e}^{\alpha\xi_s} \mathrm{d}s, \qquad t \geq 0. \tag{1}$$

and its limit, $I_\infty := \lim_{t\uparrow\infty} I_t$.

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- Also interested in the inverse process of I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0. \tag{2}$$

As usual, we work with the convention $\inf \emptyset = \infty$.

Lamperti transform for positive ssMp

Theorem (Part (i))

Fix $\alpha > 0$. If $Z^{(x)}$, $x > 0$, is a positive self-similar Markov process with index of self-similarity α, then up to absorption at the origin, it can be represented as follows. For $x > 0$,

$$Z_t^{(x)} \mathbf{1}_{(t<\zeta^{(x)})} = x\exp\{\xi_{\varphi(x^{-\alpha}t)}\}, \qquad t \geq 0,$$

where $\zeta^{(x)} = \inf\{t > 0 : Z_t^{(x)} = 0\}$ and either

(1) *$\zeta^{(x)} = \infty$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\limsup_{t\uparrow\infty} \xi_t = \infty$,*

(2) *$\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} = 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\lim_{t\uparrow\infty} \xi_t = -\infty$, or*

(3) *$\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} > 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process killed at an independent and exponentially distributed random time.*

In all cases, we may identify $\zeta^{(x)} = x^\alpha I_\infty$.

Lamperti transform for positive ssMp

Theorem (Part (ii))

Conversely, suppose that ξ is a given (killed) Lévy process. For each $x > 0$, define

$$Z_t^{(x)} = x\exp\{\xi_{\varphi(x^{-\alpha}t)}\}\mathbf{1}_{(t<x^{\alpha}I_\infty)}, \qquad t \geq 0.$$

Then $Z^{(x)}$ defines a positive self-similar Markov process, up to its absorption time $\zeta^{(x)} = x^{\alpha}I_\infty$, with index α.

§3. Positive self-similar Markov processes

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- Write $\xi^* = \{\xi^*_t : t \geq 0\}$ for the underlying Lévy process and denote its killing rate by q^*.

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- Write $\xi^* = \{\xi^*_t : t \geq 0\}$ for the underlying Lévy process and denote its killing rate by q^*.
- Let's try and decode the characteristics of ξ^*.

STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

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- Given that we know the value of Z^*_{t-}, on $\{\underline{X}_t > 0\}$, the stable process will pass over the origin at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\left(\int_{Z^*_{t-}}^{\infty}\frac{1}{|x|^{1+\alpha}}\mathrm{d}x\right) = \frac{\Gamma(1+\alpha)}{\alpha\pi}\sin(\pi\alpha\hat{\rho})(Z^*_{t-})^{-\alpha}.$$

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- On the other hand, the Lamperti transform says that on $\{t < \zeta\}$, as a pssMp, Z is sent to the origin at rate

$$q^* \frac{\mathrm{d}}{\mathrm{d}t} \varphi(t) = q^* \mathrm{e}^{-\alpha \xi^*_{\varphi(t)}} = q^* (Z^*_t)^{-\alpha}.$$

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$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\frac{1}{|x|^{1+\alpha}}\mathrm{d}x, \qquad x<0.$$

- Given that we know the value of Z^*_{t-}, on $\{\underline{X}_t > 0\}$, the stable process will pass over the origin at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\left(\int_{Z^*_{t-}}^{\infty}\frac{1}{|x|^{1+\alpha}}\mathrm{d}x\right)=\frac{\Gamma(1+\alpha)}{\alpha\pi}\sin(\pi\alpha\hat{\rho})(Z^*_{t-})^{-\alpha}.$$

- On the other hand, the Lamperti transform says that on $\{t<\zeta\}$, as a pssMp, Z is sent to the origin at rate

$$q^*\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)=q^*\mathrm{e}^{-\alpha\xi^*_{\varphi(t)}}=q^*(Z^*_t)^{-\alpha}.$$

- Comparing gives us

$$q^*=\Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi=\frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}.$$

Stable process killed on entry to $(-\infty, 0)$

- Referring again to the Lamperti transform, we know that, under $\mathbb{P}_1$ (so that $\xi^*_0 = 0$ almost surely),

$$Z^*_{\zeta-} = X_{\tau_0^- -} = \mathrm{e}^{\xi^*_{\mathbf{e}_{q^*}}},$$

where $\mathbf{e}_{q^*}$ is an exponentially distributed random variable with rate q^*.

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- This motivates the computation

$$\mathbb{E}_1[(Z^*_{\zeta-})^{\mathrm{i}\theta}] = \mathbf{E}_0[\mathrm{e}^{\mathrm{i}\theta\xi^*_{\mathbf{e}_{q^*}-}}] = \frac{q^*}{(\Psi^*(z) - q^*) + q^*}, \qquad \theta \in \mathbb{R},$$

where Ψ^* is the characteristic exponent of ξ^*.

STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

Remembering the “overshoot-undershoot" distributional law at first passage (well known in the literature for Lévy processes c.f. the quintuple law - Chapter 7 of my book) and deduce that, for all $v \in [0,1]$,

$$
\begin{aligned}
&\mathbb{P}_1(X_{\tau_0^- -} \in \mathrm{d}v) \\
&\quad = \hat{\mathbb{P}}_0(1 - X_{\tau_1^+ -} \in \mathrm{d}v) \\
&\quad = \frac{\sin(\alpha\hat{\rho}\pi)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}\left(\int_0^\infty\int_0^\infty \mathbf{1}_{(y\le 1\wedge v)}\frac{(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}}{(v+u)^{1+\alpha}}\mathrm{d}u\mathrm{d}y\right)\mathrm{d}v \\
&\quad = \frac{\sin(\alpha\hat{\rho}\pi)}{\pi}\frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}\left(\int_0^1 \mathbf{1}_{(y\le v)} v^{-\alpha}(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}\mathrm{d}y\right)\mathrm{d}v,
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where $\hat{\mathbb{P}}_0$ is the law of $-X$ issued from 0.

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\end{aligned}$$

where $\hat{\mathbb{P}}_0$ is the law of $-X$ issued from 0.
Note: more generally:

$$\begin{aligned}
&\mathbb{P}_1(-X_{\tau_0^-} \in \mathrm{d}u,\, X_{\tau_0^- -} \in \mathrm{d}v) \\
&\quad = \frac{\sin(\alpha\hat{\rho}\pi)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}\left(\int_0^\infty \mathbf{1}_{(y\le 1\wedge v)}\frac{(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}}{(v+u)^{1+\alpha}}\mathrm{d}y\right)\mathrm{d}v\mathrm{d}u
\end{aligned}$$

STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

We are led to the conclusion that

$$
\begin{aligned}
&\frac{q*}{\Psi^*(\theta)} \\
&= \frac{\sin(\alpha\hat{\rho}\pi)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}\int_0^1 (1-y)^{\alpha\hat{\rho}-1}\int_0^\infty \mathbf{1}_{(y\leq v)} v^{\mathrm{i}\theta-\alpha\hat{\rho}-1}\left(1-\frac{y}{v}\right)^{\alpha\rho-1} \mathrm{d}v\mathrm{d}y \\
&= \frac{\sin(\alpha\hat{\rho}\pi)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}\int_0^1 (1-y)^{\alpha\hat{\rho}-1} y^{\mathrm{i}\theta-\alpha\hat{\rho}}\mathrm{d}y \frac{\Gamma(\alpha\hat{\rho}-\mathrm{i}\theta)\Gamma(\alpha\rho)}{\Gamma(\alpha-\mathrm{i}\theta)} \\
&= \frac{\Gamma(\alpha\hat{\rho}-\mathrm{i}\theta)\Gamma(\alpha\rho)\Gamma(1-\alpha\hat{\rho}+\mathrm{i}\theta)\Gamma(\alpha\hat{\rho})\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})\Gamma(\alpha\hat{\rho})\Gamma(1+\mathrm{i}\theta)\Gamma(\alpha-\mathrm{i}\theta)},
\end{aligned}
$$

where in the first equality Fubini's Theorem has been used, in the second equality a straightforward substitution $w = y/v$ has been used for the inner integral on the preceding line together with the classical beta integral and, finally, in the third equality, the Beta integral has been used for a second time. Inserting the respective values for the constants q^* and K, we come to rest at the following result:

Stable process killed on entry to $(-\infty, 0)$

Theorem

For the pssMp constructed by killing a stable process on first entry to $(-\infty, 0)$, the underlying killed Lévy process, ξ^*, that appears through the Lamperti transform has characteristic exponent given by

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

Stable processes conditioned to stay positive

- Use the Lamperti representation of the α-stable process X to write, for $A \in \sigma(X_u : u \leq t)$,

$$\mathbb{P}^{\uparrow}_x(A) = \mathbb{E}_x\left[\frac{X_t^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}_t > 0)}\mathbf{1}_{(A)}\right] = \mathbf{E}_0\left[\mathrm{e}^{\alpha\hat{\rho}\xi^*_\tau}\mathbf{1}_{(\tau<\mathbf{e}_{q^*})}\mathbf{1}_{(A)}\right],$$

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Theorem

The underlying Lévy process, $\xi^{\uparrow}$, that appears through the Lamperti transform applied to $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, has characteristic exponent given by

$$\Psi^{\uparrow}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)}\frac{\Gamma(1 + \alpha\hat{\rho} + \mathrm{i}z)}{\Gamma(1 + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

- In particular $\Psi^{\uparrow}(z) = \Psi^*(z - \mathrm{i}\alpha\hat{\rho})$, $z \in \mathbb{R}$ so that $\Psi^{\uparrow}(0) = 0$ (i.e. no killing!)
- One can also check by hand that $\Psi^{\uparrow\prime}(0+) = \mathbf{E}_0[\xi^{\uparrow}_1] > 0$ so that $\lim_{t\to\infty}\xi^{\uparrow}_t = \infty$.

Did you spot the other root?

- In essence, the case of the stable process conditioned to stay positive boils down to an Esscher transform in the underlying (Lamperti-transformed) Lévy process.
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- And this means that

$$\mathrm{e}^{(1-\alpha\hat{\rho})\xi^*}, \qquad t \geq 0,$$

is a unit-mean Martingale, which can also be used to construct an Esscher transform:

$$\Psi^{\downarrow}(z) = \Psi^*(z - \mathrm{i}(1 - \alpha\hat{\rho})) = \Psi^{\downarrow}(z) = \frac{\Gamma(1 + \alpha\rho - \mathrm{i}z)}{\Gamma(1 - \mathrm{i}z)} \frac{\Gamma(\mathrm{i}z + \alpha\hat{\rho})}{\Gamma(\mathrm{i}z)}.$$

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- The choice of notation is pre-emptive since we can also check that $\Psi^{\downarrow}(0) = 0$ and $\Psi^{\downarrow\prime}(0) < 0$ so that if $\xi^{\downarrow}$ is a Lévy process with characteristic exponent $\Psi^{\downarrow}$, then $\lim_{t\to\infty} \xi^{\downarrow}_t = -\infty$.

Reverse engineering

- What now happens if we define for $A \in \sigma(X_u : u \leq t)$,

$$\mathbb{P}^{\downarrow}_x(A) = \mathbf{E}_0\left[\mathrm{e}^{(1-\alpha\hat{\rho})\xi^*_\tau}\mathbf{1}_{(\tau<\mathbf{e}_{q^*})}\mathbf{1}_{(A)}\right] = \mathbb{E}_x\left[\frac{X_t^{(1-\alpha\hat{\rho})}}{x^{(1-\alpha\hat{\rho})}}\mathbf{1}_{(\underline{X}_t>0)}\mathbf{1}_{(A)}\right],$$

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- In the same way we checked that $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, is a pssMp, we can also check that $(X, \mathbb{P}^{\downarrow}_x)$, $x > 0$ is a pssMp.

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- In the same way we checked that $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, is a pssMp, we can also check that $(X, \mathbb{P}^{\downarrow}_x)$, $x > 0$ is a pssMp.
- In an appropriate sense, it turns out that $(X, \mathbb{P}^{\downarrow}_x)$, $x > 0$ is the law of a stable process conditioned to continuously approach the origin from above.

ξ^*, $\xi^{\uparrow}$ AND $\xi^{\downarrow}$

- The three examples of pssMp offer quite striking underlying Lévy processes
- Is this exceptional?

Censored stable processes

- Start with X, the stable process.
- Let $A_t = \int_0^t \mathbf{1}_{(X_t>0)}\,\mathrm{d}t$.
- Let γ be the right-inverse of A, and put $\check{Z}_t := X_{\gamma(t)}$.
- Finally, make zero an absorbing state: $Z_t = \check{Z}_t \mathbf{1}_{(t<T_0)}$ where

$$T_0 = \inf\{t > 0 : X_t = 0\}.$$

  Note $T_0 < \infty$ a.s. if and only if $\alpha \in (1, 2)$ and otherwise $T_0 = \infty$ a.s.
- This is the censored stable process.

Censored stable processes

Theorem

Suppose that the underlying Lévy process for the censored stable process is denoted by $\overset{\leadsto}{\xi}$. Then $\overset{\leadsto}{\xi}$ is equal in law to $\xi^{**} \oplus \xi^{C}$, with

- ξ^{**} *equal in law to* ξ^{*} *with the killing removed,*
- ξ^{C} *a compound Poisson process with jump rate* $q^{*} = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi$.

Moreover, the characteristic exponent of $\overset{\leadsto}{\xi}$ *is given by*

$$\overset{\leadsto}{\Psi}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)} \frac{\Gamma(1-\alpha\rho+\mathrm{i}z)}{\Gamma(1-\alpha+\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

The radial part of a stable process

- Suppose that X is a symmetric stable process, i.e $\rho = 1/2$.
- We know that $|X|$ is a pssMp.

Theorem

Suppose that the underlying Lévy process for $|X|$ is written ξ, then it characteristic exponent is given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z+1))}{\Gamma(\frac{1}{2}(\mathrm{i}z+1-\alpha))}, \qquad z \in \mathbb{R}.$$

Hypergeometric Lévy processes (reminder)

Definition (and Theorem)

For $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ in

$$\left\{ \ \beta \leq 2, \ \gamma, \hat{\gamma} \in (0,1) \ \hat{\beta} \geq -1, \text{ and } 1-\beta+\hat{\beta}+\gamma\wedge\hat{\gamma} \geq 0 \ \right\}$$

there exists a (killed) Lévy process, henceforth refered to as a hypergeometric Lévy process, having the characteristic function

$$\Psi(z) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}z)}{\Gamma(1-\beta-\mathrm{i}z)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}z)}{\Gamma(\hat{\beta}+\mathrm{i}z)} \qquad z \in \mathbb{R}.$$

The Lévy measure of Y has a density with respect to Lebesgue measure is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2F_1\left(1+\gamma, \eta; \eta-\hat{\gamma}; \mathrm{e}^{-x}\right), & \text{if } x > 0, \\ -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1\left(1+\hat{\gamma}, \eta; \eta-\gamma; \mathrm{e}^{x}\right), & \text{if } x < 0, \end{cases}$$

where $\eta := 1-\beta+\gamma+\hat{\beta}+\hat{\gamma}$, for $|z|<1$, ${}_2F_1(a,b;c;z) := \sum_{k\geq 0} \frac{(a)_k (b)_k}{(c)_k k!} z^k$.

§4. Real valued self-similar Markov processes

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- What can we say about $\mathbb{R}$-valued self-similar Markov processes.

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- What can we say about $\mathbb{R}$-valued self-similar Markov processes.
- This requires us to first investigate Markov Additive (Lévy) Processes

Markov additive processes (MAPs)

- E is a finite state space
- $(J(t))_{t\geq 0}$ is a continuous-time, irreducible Markov chain on E
- process (ξ, J) in $\mathbb{R}\times E$ is called a *Markov additive process (MAP)* with probabilities $\mathbf{P}_{x,i}$, $x\in\mathbb{R}$, $i\in E$, if, for any $i\in E$, $s,t\geq 0$: Given $\{J(t)=i\}$,

  $(\xi(t+s)-\xi(t), J(t+s)) \stackrel{d}{=} (\xi(s), J(s))$ with law $\mathbf{P}_{0,i}$.
- Strictly speaking, a more general definition would allow ξ to be killed and sent to a cemetery state $\{-\infty\}$ at a rate which depends on the current state of J.

Pathwise description of a MAP

The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$,

- there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$
- and a sequence of iid random variables $(U_{ij}^n)_{n\geq 0}$, independent of the chain J,
- such that if $T_0 = 0$ and $(T_n)_{n\geq 1}$ are the jump times of J, the process ξ has the representation

$$\xi(t) = \mathbf{1}_{(n>0)}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n),$$

for $t \in [T_n, T_{n+1})$, $n \geq 0$.

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $\mathbf{Q} = (q_{ij})_{i,j\in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$ with $i \neq j$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Otherwise define $U_{i,i} \equiv 0$, for each $i \in E$.
- Write $G(z)$ for the $N \times N$ matrix whose (i, j)th element is $G_{ij}(z)$.
- Let
$$\boldsymbol{\Psi}(z) = \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) + \mathbf{Q} \circ G(z),$$
(when it exists), where $\circ$ indicates elementwise multiplication.
- The matrix exponent of the MAP (ξ, J) is given by
$$\mathbf{E}_{0,i}(e^{z\xi(t)}; J(t) = j) = \left(e^{\boldsymbol{\Psi}(z)t}\right)_{i,j}, \qquad i, j \in E,$$
(when it exists).

LAMPERTI-KIU TRANSFORM

- Take J to be irreducible on $E = \{1, -1\}$.

Lamperti-Kiu transform

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- Let
$$Z_t = |x|e^{\xi(\tau(|x|^{-\alpha}t))}J(\tau(|x|^{-\alpha}t)) \qquad 0 \leq t < T_0,$$
where
$$\tau(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))\mathrm{d}u > t\right\}$$
and
$$T_0 = |x|^{-\alpha}\int_0^\infty e^{\alpha\xi(u)}\mathrm{d}u.$$
- Then Z_t is a real-valued self-similar Markov process in the sense that the law of $(cZ_{tc^{-\alpha}} : t \geq 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{cx}$.

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- Then Z_t is a real-valued self-similar Markov process in the sense that the law of $(cZ_{tc^{-\alpha}} : t \ge 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{cx}$.
- The converse (within a special class of rssMps) is also true.

An α-stable process is a rssMp

- An α-stable process up to absorption in the origin is a rssMp.
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- When $\alpha \in (0,1]$, the process never hits the origin a.s.
- When $\alpha \in (1,2)$, the process is absorbs at the origin a.s.
- The matrix exponent of the underlying MAP is given by:

$$\begin{bmatrix} -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho}-z)\Gamma(1-\alpha\hat{\rho}+z)} & \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\[2ex] \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho-z)\Gamma(1-\alpha\rho+z)} \end{bmatrix},$$

for $\mathrm{Re}(z) \in (-1,\alpha)$. Note a matrix $\boldsymbol{A}$ in this context is arranged with the ordering

$$\begin{pmatrix} A_{1,1} & A_{1,-1} \\ A_{-1,1} & A_{-1,-1} \end{pmatrix}.$$

Esscher transform for MAPs

- If $\mathbf{\Psi}(z)$ is well defined then it has a real simple eigenvalue $\chi(z)$, which is larger than the real part of all its other eigenvalues.
- Furthermore, the corresponding right-eigenvector $\mathbf{v}(z) = (v_1(z), \cdots, v_N(z))$ has strictly positive entries and may be normalised such that $\pi \cdot \mathbf{v}(z) = 1$.

Theorem

Let $\mathcal{G}_t = \sigma\{(\xi(s), J(s)) : s \leq t\}$, $t \geq 0$, and

$$M_t := \mathrm{e}^{\gamma\xi(t) - \chi(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_i(\gamma)}, \qquad t \geq 0,$$

for some $\gamma \in \mathbb{R}$ such that $\chi(\gamma)$ is defined. Then, M_t, $t \geq 0$, is a unit-mean martingale. Moreover, under the change of measure

$$\mathrm{d}\mathbf{P}^{\gamma}_{0,i}\Big|_{\mathcal{G}_t} = M_t \, \mathrm{d}\mathbf{P}_{0,i}\Big|_{\mathcal{G}_t}, \qquad t \geq 0,$$

the process (ξ, J) remains in the class of MAPs with new exponent given by

$$\mathbf{\Psi}_\gamma(z) = \mathbf{\Delta}_v(\gamma)^{-1}\mathbf{\Psi}(z+\gamma)\mathbf{\Delta}_v(\gamma) - \chi(\gamma)\mathbf{I}.$$

Here, $\mathbf{I}$ is the identity matrix and $\mathbf{\Delta}_v(\gamma) = \mathrm{diag}(\boldsymbol{v}(\gamma))$.

Esscher and drift

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- With all of the above

$$\lim_{t \to \infty} \frac{\xi_t}{t} = \chi'(0) \qquad \text{a.s.}$$

Esscher and the stable-MAP

- For the MAP that underlies the stable process $D = (-1, \alpha)$, it can be checked that $\det\Psi(\alpha - 1) = 0$ i.e. $\chi(\alpha - 1) = 0$, remember the martingale

$$M_t := \mathrm{e}^{\gamma\xi(t) - \chi(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_i(\gamma)}, \qquad t \geq 0,$$

which makes

$$\begin{aligned}\boldsymbol{\Psi}^\circ(z) &= \boldsymbol{\Delta}^{-1}\boldsymbol{\Psi}(z + \alpha - 1)\boldsymbol{\Delta} \\ &= \begin{bmatrix} -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\rho-z)\Gamma(\alpha\rho+z)} & \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \\ \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} & -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\hat{\rho}-z)\Gamma(\alpha\hat{\rho}+z)} \end{bmatrix},\end{aligned}$$

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- When $\alpha \in (0, 1)$, $\chi'(0) > 0$ (because the stable process never touches the origin a.s.) and $\boldsymbol{\Psi}^\circ(z)$-MAP drifts to $-\infty$
- When $\alpha \in (1, 2)$, $\chi'(0) < 0$ (because the stable process touches the origin a.s.) and $\boldsymbol{\Psi}^\circ(z)$-MAP drifts to $+\infty$.

RIESZ-BOGDAN-ZAK TRANSFORM

Theorem (Riesz–Bogdan–Zak transform)

Suppose that X is an α-stable process as outlined in the introduction. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}\backslash\{0\}$, $(-1/X_{\eta(t)})_{t\geq 0}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{-1/x})$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \left(\frac{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(X_t)}{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(x)}\right)\left|\frac{X_t}{x}\right|^{\alpha-1}\mathbf{1}_{(t<\tau^{\{0\}})}$$

$\tau^{\{0\}} = \inf\{t > 0 : X_t = 0\}$ and $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$. Moreover, the process $(X, \mathbb{P}^\circ_x)$, $x \in \mathbb{R}\backslash\{0\}$ is a self-similar Markov process with underlying MAP via the Lamperti-Kiu transform given by $\boldsymbol{\Psi}^\circ(z)$.

WHAT IS THE Ψ°-MAP?

Thinking of the affect on the long term behaviour of the underlying MAP of the Esscher transform

- When $\alpha \in (0,1)$, $(X, \mathbb{P}^\circ_x)$, $x \neq 0$ has the law of the the stable process conditioned to absorb continuously at the origin in the sense,

$$\mathbb{P}^\circ_y(A) = \lim_{a\to 0} \mathbb{P}_y(A, t < T_0 \,|\, \tau_{(-a,a)} < \infty),$$

  for $A \in \mathcal{F}_t = \sigma(X_s, s \leq t)$,
  $\tau_{(-a,a)} = \inf\{t > 0 : |X_t| < a\}$ and $T_0 = \inf\{t > 0 : X_t = 0\}$.
- When $\alpha \in (1,2)$, $(X, \mathbb{P}^\circ_x)$, $x \neq 0$ has the law of the stable process conditioned to avoid the origin in the sense

$$\mathbb{P}^\circ_y(A) = \lim_{s\to \infty} \mathbb{P}_y(A \,|\, T_0 > t + s),$$

  for $A \in \mathcal{F}_t = \sigma(X_s, s \leq t)$ and $T_0 = \inf\{t > 0 : X_t = 0\}$.

§5. Isotropic stable processes in dimension $d \geq 2$ seen as Lévy processes

ISOTROPIC α-STABLE PROCESS IN DIMENSION $d \geq 2$

For $d \geq 2$, let $X := (X_t : t \geq 0)$ be a d-dimensional isotropic stable process.

- X has stationary and independent increments (it is a Lévy process)
- Characteristic exponent $\Psi(\theta) = -\log \mathbb{E}_0(e^{i\theta \cdot X_1})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha, \qquad \theta \in \mathbb{R}.$$

- Necessarily, $\alpha \in (0, 2]$, we exclude 2 as it pertains to the setting of a Brownian motion.
- Associated Lévy measure satisfies, for $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}\Pi(B) &= \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} \mathrm{d}y \\ &= \frac{2^{\alpha-1}\Gamma((d+\alpha)/2)\Gamma(d/2)}{\pi^d \big|\Gamma(-\alpha/2)\big|} \int_{\mathbb{S}_{d-1}} r^{d-1}\sigma_1(\mathrm{d}\theta) \int_0^\infty \mathbf{1}_B(r\theta) \frac{1}{r^{\alpha+d}} \mathrm{d}r,\end{aligned}$$

  where $\sigma_1(\mathrm{d}\theta)$ is the surface measure on $\mathbb{S}_{d-1}$ normalised to have unit mass.
- X is Markovian with probabilities denoted by $\mathbb{P}_x$, $x \in \mathbb{R}^d$

ISOTROPIC α-STABLE PROCESS IN DIMENSION $d \geq 2$

- Stable processes are also self-similar. For $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$\text{under } \mathbb{P}_x, \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx}.$$

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- Isotropy means, for all orthogonal transformations (e.g. rotations) $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

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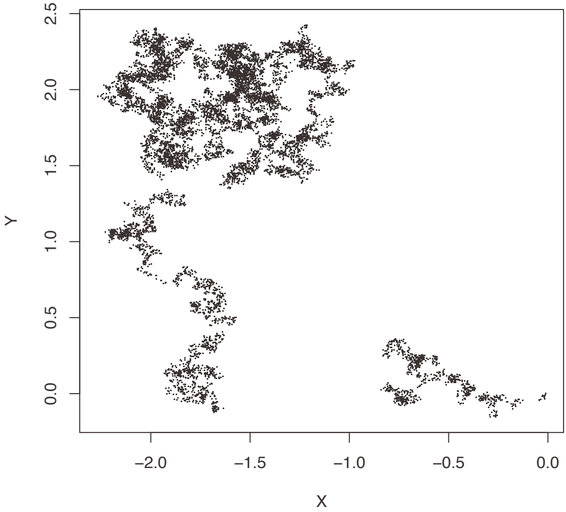
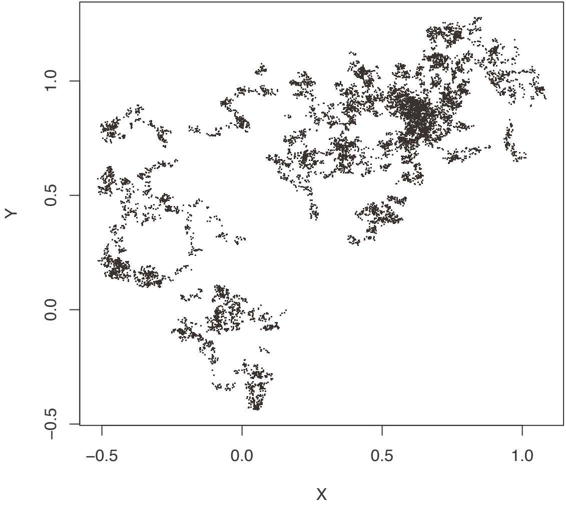
- If $(S_t, t \geq 0)$ is a stable subordinator with index $\alpha/2$ (a Lévy process with Laplace exponent $-t^{-1} \log \mathbb{E}[e^{-\lambda S_t}] = \lambda^\alpha$) and $(B_t, t \geq 0)$ for a standard (isotropic) d-dimensional Brownian motion, then it is known that $X_t := \sqrt{2} B_{S_t}, t \geq 0$, is a stable process with index α.

$$\mathbb{E}[e^{i\theta X_t}] = \mathbb{E}\left[e^{-\theta^2 S_t}\right] = e^{-|\theta|^\alpha t}, \qquad \theta \in \mathbb{R}.$$

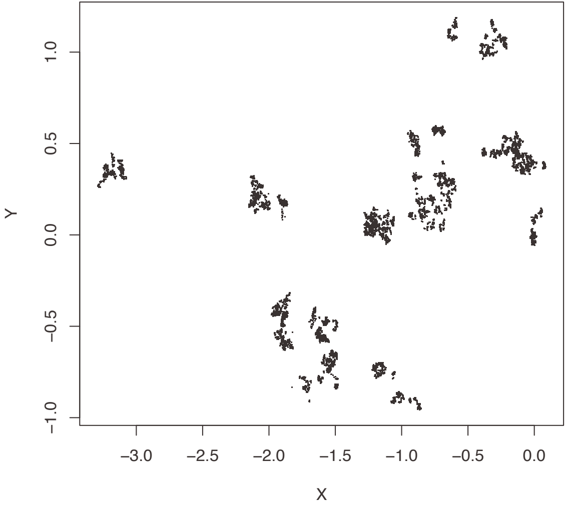
Sample path, $\alpha = 1.9$



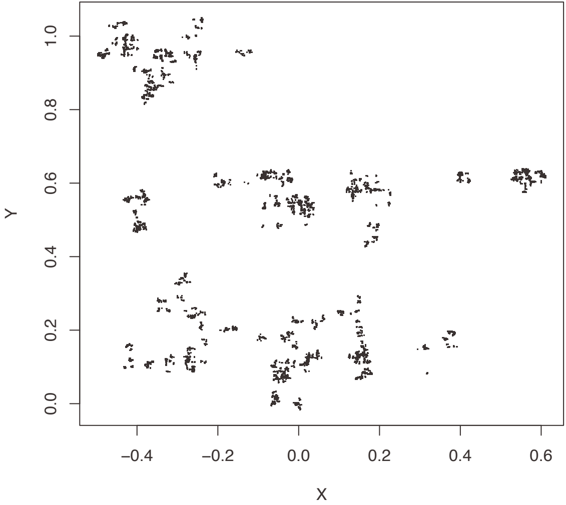
SAMPLE PATH, $\alpha = 1.7$



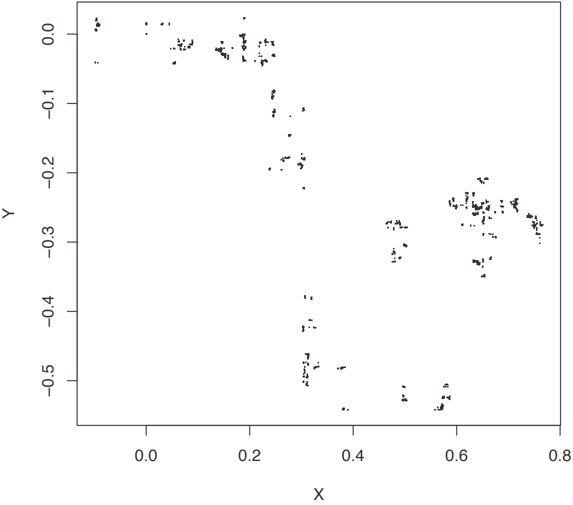
Sample path, $\alpha = 1.5$



SAMPLE PATH, $\alpha = 1.2$



Sample path, $\alpha = 0.9$



§6. Isotropic stable processes in dimension $d \geq 2$ seen as a self-similar Markov process

LAMPERTI-TRANSFORM OF $|X|$

Theorem (Caballero-Pardo-Perez (2011))

For the pssMp constructed using the radial part of an isotropic d-dimensional stable process, the underlying Lévy process, ξ that appears through the Lamperti has characteristic exponent given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z+d))}{\Gamma(\frac{1}{2}(\mathrm{i}z+d-\alpha))}, \qquad z \in \mathbb{R}.$$

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Here are some facts that can be deduced from the above Theorem

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Here are some facts that can be deduced from the above Theorem

- The fact that $\lim_{t\to\infty} |X_t| = \infty$
- The fact that
$$|X_t|^{\alpha-d}, \qquad t \geq 0,$$
is a martingale.

CONDITIONED STABLE PROCESS

- We can define the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}_x^\circ}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0, x \neq 0$$

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$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0, x \neq 0$$

- Suppose that f is a bounded measurable function then, for all $c > 0$,

$$\begin{aligned}\mathbb{E}^\circ_x[f(cX_{c^{-\alpha}s}, s \leq t)] &= \mathbb{E}_x\left[\frac{|cX_{c^{-\alpha}t}|^{\alpha-d}}{|cx|^{d-\alpha}} f(cX_{c^{-\alpha}s}, s \leq t)\right] \\ &= \mathbb{E}_{cx}\left[\frac{|X_t|^{\alpha-d}}{|cx|^{d-\alpha}} f(X_s, s \leq t)\right] = \mathbb{E}^\circ_{cx}[f(X_s, , s \leq t)]\end{aligned}$$

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- Markovian, isotropy and self-similarity properties pass through to $(X, \mathbb{P}^\circ_x)$, $x \neq 0$.
- Similarly $(|X|, \mathbb{P}^\circ_x)$, $x \neq 0$ is a positive self-similar Markov process.

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$$\mathbb{P}^\circ_x(A, t < \mathtt{k}) = \lim_{a \downarrow 0} \mathbb{P}_x(A,\ t < \mathtt{k} | \tau_a^\oplus < \infty),$$

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- Given the pathwise interpretation of $(X, \mathbb{P}_x^\circ)$, $x \neq 0$, it follows immediately that $\lim_{t \to \infty} \xi_t = -\infty$, $\mathbb{P}_x^\circ$ almost surely, for any $x \neq 0$.

$\mathbb{R}^d$-SELF-SIMILAR MARKOV PROCESSES

Definition

A $\mathbb{R}^d$-valued regular Feller process $Z = (Z_t, t \geq 0)$ is called a *$\mathbb{R}^d$-valued self-similar Markov process* if there exists a constant $\alpha > 0$ such that, for any $x > 0$ and $c > 0$,

$$\text{the law of } (cZ_{c^{-\alpha}t}, t \geq 0) \text{ under } P_x \text{ is } P_{cx},$$

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- Is there an analogue of the Lamperti representation?

Lamperti–Kiu transform

In order to introduce the analogue of the Lamperti transform in d-dimensions, we need to remind ourselves of what we mean by a Markov additive process in this context.

Definition

An $\mathbb{R} \times E$ valued regular Feller process $(\xi, \Theta) = ((\xi_t, \Theta_t) : t \geq 0)$ with probabilities $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in E$, and cemetery state $(-\infty, \dagger)$ is called a *Markov additive process* (MAP) if Θ is a regular Feller process on E with cemetery state $\dagger$ such that, for every bounded measurable function $f : (\mathbb{R} \cup \{-\infty\}) \times (E \cup \{\dagger\}) \to \mathbb{R}$, $t, s \geq 0$ and $(x, \theta) \in \mathbb{R} \times E$, on $\{t < \varsigma\}$,

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- Roughly speaking, one thinks of a MAP as a 'Markov modulated' Lévy process
- It has 'conditional stationary and independent increments'

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In order to introduce the analogue of the Lamperti transform in d-dimensions, we need to remind ourselves of what we mean by a Markov additive process in this context.

Definition

An $\mathbb{R} \times E$ valued regular Feller process $(\xi, \Theta) = ((\xi_t, \Theta_t) : t \geq 0)$ with probabilities $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in E$, and cemetery state $(-\infty, \dagger)$ is called a *Markov additive process* (MAP) if Θ is a regular Feller process on E with cemetery state $\dagger$ such that, for every bounded measurable function $f : (\mathbb{R} \cup \{-\infty\}) \times (E \cup \{\dagger\}) \to \mathbb{R}$, $t, s \geq 0$ and $(x, \theta) \in \mathbb{R} \times E$, on $\{t < \varsigma\}$,

$$\mathbf{E}_{x,\theta}[f(\xi_{t+s} - \xi_t, \Theta_{t+s}) | \sigma((\xi_u, \Theta_u), u \leq t)] = \mathbf{E}_{0,\Theta_t}[f(\xi_s, \Theta_s)],$$

where $\varsigma = \inf\{t > 0 : \Theta_t = \dagger\}$.

- Roughly speaking, one thinks of a MAP as a 'Markov modulated' Lévy process
- It has 'conditional stationary and independent increments'
- Think of the E-valued Markov process Θ as modulating the characteristics of ξ (which would otherwise be a Lévy processes).

Lamperti–Kiu transform

Theorem

Fix $\alpha > 0$. The process Z is a ssMp with index α, issued from $x \in \mathbb{R}^d$, if and only if there exists a (killed) MAP, (ξ, Θ) on $\mathbb{R} \times \mathbb{S}_{d-1}$, issued from $(\log|x|, \arg(x))$, such that

$$Z_t := \mathrm{e}^{\xi_{\varphi(t)}} \Theta_{\varphi(t)} \qquad , \qquad t \le I_\varsigma,$$

where

$$\varphi(t) = \inf\left\{ s > 0 : \int_0^s \mathrm{e}^{\alpha \xi_u}\, \mathrm{d}u > t \right\}, \qquad t \le I_\varsigma,$$

and $\int_0^\varsigma \mathrm{e}^{\alpha\xi_s} \mathrm{d}s$ is the lifetime of Z until absorption at the origin. Here, we interpret $\exp\{-\infty\} \times \dagger := 0$ and $\inf \emptyset := \infty$.

- In the above representation, the time to absorption in the origin,

$$\zeta = \inf\{t > 0 : Z_t = 0\},$$

  satisfies $\zeta = \int_0^\varsigma \mathrm{e}^{\alpha\xi_s} \mathrm{d}s$.
- Note $x \in \mathbb{R}^d$ if and only if

$$x = (|x|, \mathrm{Arg}(x)),$$

  where $\mathrm{Arg}(x) = x/|x| \in \mathbb{S}_{d-1}$. The Lamperti–Kiu decomposition therefore gives us a d-dimensional skew product decomposition of self-similar Markov processes.

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- **How do we characterise its underlying MAP** (ξ, Θ)?
- We already know that $|X|$ is a positive similar Markov process and hence ξ is a Lévy process, albeit corollated to Θ
- What properties does Θ have and what properties to the pair (ξ, Θ) have?

MAP ISOTROPY

Theorem

Suppose (ξ, Θ) is the MAP underlying the stable process. Then $((\xi, U^{-1}\Theta), \mathbf{P}_{x,\theta})$ is equal in law to $((\xi, \Theta), \mathbf{P}_{x,U^{-1}\theta})$, for every orthogonal d-dimensional matrix U and $x \in \mathbb{R}^d$, $\theta \in \mathbb{S}_{d-1}$.

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Proof.

First note that $\varphi(t) = \int_0^t |X_u|^{-\alpha} \mathrm{d}u$. It follows that

$$(\xi_t, \Theta_t) = (\log |X_{A(t)}|, \operatorname{Arg}(X_{A(t)})), \qquad t \geq 0,$$

where the random times $A(t) = \inf \left\{ s > 0 : \int_0^s |X_u|^{-\alpha} \mathrm{d}u > t \right\}$ are stopping times in the natural filtration of X.

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Now suppose that U is any orthogonal d-dimensional matrix and let $X' = U^{-1}X$. Since X is isotropic and since $|X'| = |X|$, and $\operatorname{Arg}(X') = U^{-1}\operatorname{Arg}(X)$, we see that, for $x \in \mathbb{R}$ and $\theta \in \mathbb{S}_{d-1}$

$$\begin{aligned} ((\xi, U^{-1}\Theta), \mathbf{P}_{\log|x|,\theta}) &= ((\log |X_{A(\cdot)}|, U^{-1}\operatorname{Arg}(X_{A(\cdot)})), \mathbb{P}_x) \\ &\stackrel{d}{=} ((\log |X_{A(\cdot)}|, \operatorname{Arg}(X_{A(\cdot)})), \mathbb{P}_{U^{-1}x}) \\ &= ((\xi, \Theta), \mathbf{P}_{\log|x|, U^{-1}\theta}) \end{aligned}$$

as required. □

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Theorem (Bo Li, Victor Rivero, Bertoin-Werner)

Suppose that f is a bounded measurable function on $[0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}_{d-1} \times \mathbb{S}_{d-1}$ such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, then, for all $\theta \in \mathbb{S}_{d-1}$,

$$\mathbf{E}_{0,\theta}\left(\sum_{s>0} f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right)$$
$$= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{R}} V_\theta(\mathrm{d}s, \mathrm{d}x, \mathrm{d}\vartheta)\sigma_1(\mathrm{d}\phi)\mathrm{d}y \frac{c(\alpha)\mathrm{e}^{yd}}{|\mathrm{e}^y\phi - \vartheta|^{\alpha+d}} f(s, x, y, \vartheta, \phi),$$

where

$$V_\theta(\mathrm{d}s, dx, \mathrm{d}\vartheta) = \mathbf{P}_{0,\theta}(\xi_s \in \mathrm{d}x, \Theta_s \in \mathrm{d}\vartheta)\mathrm{d}s, \qquad x \in \mathbb{R}, \vartheta \in \mathbb{S}_{d-1}, s \geq 0,$$

is the space-time potential of (ξ, Θ) under $\mathbf{P}_{0,\theta}$, $\sigma_1(\phi)$ is the surface measure on $\mathbb{S}_{d-1}$ normalised to have unit mass and

$$c(\alpha) = 2^{\alpha-1}\pi^{-d}\Gamma((d+\alpha)/2)\Gamma(d/2)/\left|\Gamma(-\alpha/2)\right|.$$

MAP OF $(X, \mathbb{P}^\circ_\cdot)$

- Recall that $(|X_t|^{\alpha-d}, t \geq 0)$, is a martingale.
- Informally, we should expect $\mathcal{L}h = 0$, where $h(x) = |x|^{\alpha-d}$ and $\mathcal{L}$ is the infinitesimal generator of the stable process, which has action

$$\mathcal{L}f(x) = \mathrm{a} \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x+y) - f(x) - \mathbf{1}_{(|y|\leq 1)} y \cdot \nabla f(x)] \Pi(\mathrm{d}y), \qquad |x| > 0,$$

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- Equivalently, the rate at which $(X, \mathbb{P}^\circ_x)$, $x \neq 0$ jumps given by

$$\Pi^\circ(x, B) := \frac{2^{\alpha-1}\Gamma((d+\alpha)/2)\Gamma(d/2)}{\pi^d |\Gamma(-\alpha/2)|} \int_{\mathbb{S}_{d-1}} \mathrm{d}\sigma_1(\phi) \int_{(0,\infty)} \mathbf{1}_B(r\phi) \frac{\mathrm{d}r}{r^{\alpha+1}} \frac{|x + r\phi|^{\alpha-d}}{|x|^{\alpha-d}},$$

for $|x| > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

MAP OF $(X, \mathbb{P}^\circ_\cdot)$

Theorem

Suppose that f is a bounded measurable function on $[0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}_{d-1} \times \mathbb{S}_{d-1}$ such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, then, for all $\theta \in \mathbb{S}_{d-1}$,

$$
\begin{aligned}
&\mathbf{E}^\circ_{0,\theta}\left(\sum_{s>0} f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right) \\
&= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{R}} V^\circ_\theta(\mathrm{d}s, \mathrm{d}x, \mathrm{d}\vartheta)\sigma_1(\mathrm{d}\phi)\mathrm{d}y \frac{c(\alpha)\mathrm{e}^{yd}}{|\mathrm{e}^y\phi - \vartheta|^{\alpha+d}} f(s, x, -y, \vartheta, \phi),
\end{aligned}
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where

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V^\circ_\theta(\mathrm{d}s, dx, \mathrm{d}\vartheta) = \mathbf{P}^\circ_{0,\theta}(\xi_s \in \mathrm{d}x, \Theta_s \in \mathrm{d}\vartheta)\mathrm{d}s, \qquad x \in \mathbb{R}, \vartheta \in \mathbb{S}_{d-1}, s \geq 0,
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is the space-time potential of (ξ, Θ) under $\mathbf{P}^\circ_{0,\theta}$.

Comparing the right-hand side above with that of the previous Theorem, it now becomes immediately clear that the the jump structure of (ξ, Θ) under $\mathbf{P}^\circ_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}_{d-1}$, is precisely that of $(-\xi, \Theta)$ under $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}_{d-1}$.

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§7. Riesz–Bogdan–Żak transform

Riesz–Bogdan–Żak transform

- Define the transformation $K : \mathbb{R}^d \mapsto \mathbb{R}^d$, by

$$Kx = \frac{x}{|x|^2}, \qquad x \in \mathbb{R}^d \backslash \{0\}.$$

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- Write $x \in \mathbb{R}^d$ in skew product form $x = (|x|, \mathrm{Arg}(x))$, and note that

$$Kx = (|x|^{-1}, \mathrm{Arg}(x)), \qquad x \in \mathbb{R}^d \backslash \{0\},$$

showing that the K-transform 'radially inverts' elements of $\mathbb{R}^d$ through $\mathbb{S}_{d-1}$.

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Theorem (d-dimensional Riesz–Bogdan–Żak Transform, $d \geq 2$)

Suppose that X is a d-dimensional isotropic stable process with $d \geq 2$. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0. \tag{3}$$

Then, for all $x \in \mathbb{R}^d \backslash \{0\}$, $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{Kx})$.

Proof of Riesz–Bogdan–Żak transform

We give a proof, different to the original proof of Bogdan and Żak (2010).

- Recall that $X_t = \mathrm{e}^{\xi_{\varphi(t)}}\Theta_{\varphi(t)}$, where

$$\int_0^{\varphi(t)} \mathrm{e}^{\alpha\xi_u}\,\mathrm{d}u = t, \qquad t \geq 0.$$

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- Note also that, as an inverse,

$$\int_0^{\eta(t)} |X_u|^{-2\alpha} du = t, \qquad t \geq 0.$$

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- Differentiating,

$$\frac{d\varphi(t)}{dt} = e^{-\alpha \xi_{\varphi(t)}} \text{ and } \frac{d\eta(t)}{dt} = e^{2\alpha \xi_{\varphi \circ \eta(t)}}, \qquad \eta(t) < \tau^{\{0\}}.$$

  and chain rule now tells us that

$$\frac{d(\varphi \circ \eta)(t)}{dt} = \left.\frac{d\varphi(s)}{ds}\right|_{s=\eta(t)} \frac{d\eta(t)}{dt} = e^{\alpha \xi_{\varphi \circ \eta(t)}}.$$

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- Said another way,

$$\int_0^{\varphi\circ\eta(t)} \mathrm{e}^{-\alpha\xi_u}\mathrm{d}u = t, \qquad t \geq 0,$$

or

$$\varphi\circ\eta(t) = \inf\{s > 0 : \int_0^s \mathrm{e}^{-\alpha\xi_u}\mathrm{d}u > t\}$$

Proof of Riesz–Bogdan–Żak transform

- Next note that
$$KX_{\eta(t)} = \mathrm{e}^{-\xi_{\varphi\circ\eta(t)}}\Theta_{\varphi\circ\eta(t)}, \qquad t \geq 0,$$
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- We have also seen that $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, is also a self-similar Markov process with underlying MAP given by $(-\xi, \Theta)$.

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$$\varphi\circ\eta(t) = \inf\{s > 0 : \int_0^s \mathrm{e}^{-\alpha\xi_u}\mathrm{d}u > t\}.$$
- It follows that $(KX_{\eta(t)}, t \geq 0)$ is a self-similar Markov process with underlying MAP $(-\xi, \Theta)$
- We have also seen that $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, is also a self-similar Markov process with underlying MAP given by $(-\xi, \Theta)$.
- The statement of the theorem follows.

§8. Other developments

HARMONIC FUNCTIONS ON THE CONE

- Lipchitz cone, $\Gamma = \{x \in \mathbb{R}^d : x \neq 0, \arg(x) \in \Omega\}$,
- Exit time from the cone i.e. $\kappa_\Gamma = \inf\{s > 0 : X_s \notin \Gamma\}$.
- Bañuelos and Bogdan (2004): There exists $M : \mathbb{R}^d \to \mathbb{R}$ such that
  - $M(x) = 0$ for all $x \notin \Gamma$.
  - M is locally bounded on $\mathbb{R}^d$
  - There is a $\beta = \beta(\Gamma, \alpha) \in (0, \alpha)$, such that

    $$M(x) = |x|^\beta M(x/|x|) = |x|^\beta M(\arg(x)), \qquad x \neq 0.$$

  - Up to a multiplicative constant, M is the unique such that

    $$M(x) = \mathbb{E}_x[M(X_{\tau_B})\mathbf{1}_{(\tau_B < \kappa_\Gamma)}], \qquad x \in \mathbb{R}^d,$$

    where B is any open bounded domain and $\tau_B = \inf\{t > 0 : X_t \notin B\}$.
- Bañuelos and Bogdan (2004) and Bogdan, Palmowski, Wang (2018): We have

  $$\lim_{a \to 0} \sup_{x \in \Gamma,\ |t^{-1/\alpha} x| \leq a} \frac{\mathbb{P}_x(\kappa_\Gamma > t)}{M(x) t^{-\beta/\alpha}} = C,$$

  where $C > 0$ is a constant.

Theorem

(i) *For any* $t > 0$, *and* $x \in \Gamma$,

$$\mathbb{P}^{\triangleleft}_x(A) := \lim_{s \to \infty} \mathbb{P}_x\left(A \,|\, \kappa_\Gamma > t + s\right), \qquad A \in \mathcal{F}_t,$$

defines a family of conservative probabilities on the space of càdlàg paths such that

$$\left.\frac{\mathrm{d}\mathbb{P}^{\triangleleft}_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} := \mathbf{1}_{(t<\kappa_\Gamma)} \frac{M(X_t)}{M(x)}, \qquad t \geq 0, \text{ and } x \in \Gamma.$$

In particular, the right-hand side above is a martingale.
(Note: this is nothing but an Esscher transform for the underlying MAP!)

(ii) *Let* $\mathbb{P}^{\triangleleft} := (\mathbb{P}^{\triangleleft}_x, x \in \Gamma)$. *The process* $(X, \mathbb{P}^{\triangleleft})$, *is a self-similar Markov process.*

Entrance law

Let $p_t^\Gamma(x,y)$, $x, y \in \Gamma$, $t \geq 0$, be the semigroup of X killed on exiting the cone Γ.

Theorem (Bogdan, Palmowski, Wang (2018))

The following limit exits,

$$n_t(y) := \lim_{\Gamma \ni x \to 0} \frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\kappa_\Gamma > t) t^{\beta/\alpha}}, \qquad x, y \in \Gamma, t > 0, \tag{4}$$

and $(n_t(y)\mathrm{d}y, t > 0)$*, serves as an entrance law to* $(\mathrm{X}, \mathbb{P}^\Gamma)$*, in the sense that*

$$n_{t+s}(y) = \int_\Gamma n_t(x) p_s^\Gamma(x,y)\mathrm{d}x, \qquad y \in \Gamma, s, t \geq 0.$$

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$$n_{t+s}(y) = \int_\Gamma n_t(x) p_s^\Gamma(x,y)\mathrm{d}x, \qquad y \in \Gamma, s,t \geq 0.$$

- Also easy to show that, in the sense of weak convergence,

$$\mathbb{P}_0^\triangleleft(X_t \in \mathrm{d}y) := \lim_{\Gamma \ni x \to 0} \frac{M(y)}{M(x)} \mathbb{P}_x(X_t \in \mathrm{d}y,\ t < \kappa_\Gamma) = CM(y)n_t(y)\mathrm{d}y.$$

- Can the process 'start from the apex of the cone' in a stronger sense?

Continuous entrance at the apex of the cone

Theorem

The limit $\mathbb{P}^{\triangleleft}_0 := \lim_{\Gamma \ni x \to 0} \mathbb{P}^{\triangleleft}_x$ *is well defined on the Skorokhod space, so that,* $(X, (\mathbb{P}^{\triangleleft}_x, x \in \Gamma \cup \{0\}))$ *is both Feller and self-similar which enters continuously at the origin, after which it never returns.*

Point of closest reach

- Recall that we can represent an isotropic Lévy process through the Lamperti transform

$$X_t := e^{\xi_{\varphi(t)}}\Theta_{\varphi(t)} \qquad t \geq 0,$$

where

$$\varphi(t) = \inf\left\{s > 0 : \int_0^s e^{\alpha \xi_u}\, du > t\right\}$$

and (ξ, Θ) with probabilities $\mathbf{P}_{x,\theta}$, $x \neq 0$, $\theta \in \mathbb{S}_d$, is a MAP. Recall also that, although corollated to Θ, ξ alone is a Lévy process.

- Define

$$\mathtt{g}_t = \sup\{s < t : \xi_s = \underline{\xi}_s\}$$

so that $\mathtt{g}_\infty = \lim_{t\to\infty} \mathtt{g}_t < \infty$ is the time of the point of closest reach.

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Theorem (Point of Closest Reach to the origin)

The law of the point of closest reach to the origin is given by

$$\mathbb{P}_x(X_{\mathtt{g}_\infty} \in \mathrm{d}y) = \pi^{-d/2}\frac{\Gamma\left(d/2\right)^2}{\Gamma\left((d-\alpha)/2\right)\Gamma\left(\alpha/2\right)}\frac{(|x|^2-|y|^2)^{\alpha/2}}{|x-y|^d|y|^\alpha}\mathrm{d}y, \qquad 0 < |y| < |x|.$$

Bedankt