The differential equation method

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Describing elements of a random graph

Consider a graph G on n vertices where edges are present independently at random

Vertex set $[n] = \{1,2,\dots,n\}$

Edges r.v.'s $X_{ij} = \mathbb{1}_{\{i \leftrightarrow j\}}$ independent

Some interesting statistics:

- number of edges
- isolated vertices
- number of components (of given size)
- degree sequence
- chromatic number
- independence number

Intuitively changing one variable in $\{X_{ij}\}$ won't change statistics by much.

This course: conditions to 'see' trends ... in a (graph) discrete process

Chernoff bounds $t > 0$

$$\mathbb{P}(X \geq a) = \mathbb{P}(e^{tX} \geq e^{ta}) \leq \frac{\mathbb{E}[e^{tX}]}{\underbrace{e^{ta}}_{\text{optimize } t}}$$

$$t = 4a / \Sigma(b_i - a_i)^2$$

If $S_n = \sum_{i=1}^{n} X_i$ independent X_i

$$\mathbb{P}(S_n \geq \mathbb{E}[S_n] + \varepsilon) \leq e^{-t\varepsilon} \prod_{i=1}^{n} \mathbb{E}[e^{t(X - \mathbb{E}[X])}]$$

+ bounded X_i so that $a_i \leq X_i \leq b_i$

$$\mathbb{P}(S_n \geq \mathbb{E}[S_n] + \varepsilon) \leq e^{-2\varepsilon^2 / \Sigma(b_i - a_i)^2}$$

This gives quantitative bounds for LLN $\quad P(X_i \in [0,1])$

$$\sum_{n \geq 1} \mathbb{P}\left(\left|\frac{S_n - \mathbb{E}[S_n]}{n}\right| > u\right) \leq \sum_{n \geq 1} 2e^{-nu^2/2} < \infty$$

$$\underset{\text{B.C.}}{\Longrightarrow} \mathbb{P}\left(\left|\frac{S_n - \mathbb{E}[S_n]}{n}\right| > u \text{ infinitely often}\right) = 0$$

$$\frac{S_n - \mathbb{E}[S_n]}{n} \xrightarrow{a.s.} 0$$

Two generalization

① If $\{Z_i\}_{i\geq 0}$ martingale, $Z_0 = 0$

$|Z_{i+1} - Z_i| < c_i$

$$\mathbb{P}(Z_n \geq \mathbb{E}[Z_n] + \varepsilon) \leq e^{-2\varepsilon^2/\Sigma c_i^2}$$

Proof Sketch: $\mathcal{F}_i = \sigma\text{-algebra}(Z_1 \dots Z_i)$

$$Z_n = \sum_{i=1}^{n} Z_i - Z_{i-1} = \sum_{i=1}^{n} V_i$$

$$\mathbb{E}[e^{tZ_n}] = \mathbb{E}\left[\prod_{i=1}^{n} e^{tV_i}\right]$$

$$= \mathbb{E}\left[\prod_{n=1}^{n-1} e^{tV_i}\, \mathbb{E}[e^{tV_n} \mid \mathcal{F}_{n-1}]\right]$$

Now V_n satisfies conditions of Hoeffding's lemma.

② Let $\{X_i\}$ independent $X_i \in \mathcal{X}$

$$f: \mathcal{X}^n \to \mathbb{R}$$

Conditions

$$|f(x_1, x_2, \dots x_n) - f(y_1, \dots, y_n)| \leq \sum c_i \mathbb{1}_{\{x_i \neq y_i\}}$$

$$\mathbb{P}(f(X) \geq \mathbb{E}[f] + \varepsilon) \leq e^{-2\varepsilon^2/\Sigma c_i^2}$$

$$V_i = \mathbb{E}[f \mid \mathcal{F}_i] - \mathbb{E}[f \mid \mathcal{F}_{i-1}]$$

$$\mathcal{F}_i = \sigma\text{-alge}(X_1 \dots X_i)$$

$\exists$ r.v.'s L_i, U_i measurable wrt $\mathcal{F}_{i-1}$

$$L_i \leq V_i \leq U_i \qquad U_i - L_i \leq c_i$$

so can apply Hoeffding's lemma

Extension in Hamming distance

$\underline{X} = (X_1, \dots, X_n) \in S^n \qquad A \subset S^n$

Hamming dist $d_H(\underline{x}, \underline{y}) = \sum \mathbb{1}_{\{x_i \neq y_i\}}$

$$d_H(\underline{x}, A) = \inf_{y \in A} d_H(\underline{x}, \underline{y})$$

$$[A]_r = \{ \underline{x} \in S^n : d_H(\underline{x}, A) \leq r \}$$

How big becomes $[A]_{\varepsilon n}$

$$\mathbb{P}(\underline{X} \in [A]_{\varepsilon n}) = \mathbb{P}(d(\underline{X}, A) \leq \varepsilon n)$$
$$= 1 - \mathbb{P}(d(\underline{X}, A) > \varepsilon n)$$

$*\ \varepsilon n > \mathbb{E}[d(\underline{X}, A)] \qquad u = \varepsilon n - \mathbb{E}[d(X, A)]$

$$\mathbb{P}(d(\underline{X}, A) > \varepsilon n) = \mathbb{P}(d(\underline{X}, A) > \mathbb{E}[-] + u) \leq e^{-2nu^2}$$

Problem: $\mathbb{E}[d(\underline{X}, A)]$ unknown

$$\varepsilon n \geq B \geq \mathbb{E}[d(\underline{X}, A)]$$

$$\mathbb{P}(d(\underline{X}, A) \geq \varepsilon n) \leq \mathbb{P}(d(\underline{X}, A) \geq \mathbb{E}[\cdot] + u)$$

$$u = \varepsilon n - B$$

Trick: from Hoeffding lemma' proof

$$\mathbb{E}\left[e^{t(f - \mathbb{E}[f])}\right] \leq e^{nt^2/8}$$

$f = -d(\underline{X}, A) \qquad \mathbb{E}\left[e^{-t\mathbb{E}[f]}\right] = e^{-t\mathbb{E}[f]}$

$$\mathbb{P}(A) = \mathbb{E}\left[\mathbb{1}_{\{A\}} e^{tf}\right] \leq \mathbb{E}\left[e^{tf}\right] = \mathbb{E}\left[e^{-t \cdot d(X, A)}\right]$$

$$\mathbb{P}(A)\ e^{-t\mathbb{E}d(X, A)} \leq e^{nt^2/8}$$

Take logarithm + optimize t

$$\mathbb{E}[d(\underline{X}, A)] \leq \sqrt{\frac{-n \log \mathbb{P}(A)}{2}} = B$$

Examples of analysis of algorithms

① Matchings in $G_{n,c/n}$ (Erdös-Rényi)

Graph on n vertices edges independently present w.p. $\frac{c}{n}$

Matching: set of edges that do not share common vertices

Algorithm:

1. Delete isolated vertices
2. Select a uniformly random vertex of min-degree v
3. Include in matching one of its edges
4. Delete v and all other incident edges.

Thm: $c>0$. The algorithm obtains a maximum matching a.a.s.

$(\mathbb{P}(\leftrightarrow) \to 1 \quad n\to\infty)$

② 3-coloring regular $G_{n,d}$

uniformly chosen graph on those with n vertices and $\deg(v)=d$ all v.

Proper coloring: assign colors to vertices so that no two adjacent vertices have same color

Algorithm: Classify uncolored vertices according to # available color

$$S_i(t) = \{v : i \text{ possible colors for } v\}$$

while $S_0(t)=0$

1. Select minimum i with $S_i(t)>0$
2. select uniform vertex in $S_i(t)$
3. Color v randomly, update $S_i(t+1)$.

Failure if $S_0(t)>0$

Thm: $d<4.003$. The algorithm obtains a proper 3-col. a.a.s.

③ 3-SAT formulae

$$(\bar{x}_1 \vee x_1 \vee x_2) \wedge (\underbrace{\bar{x}_2 \vee \bar{x}_1 \vee x_3}_{\text{clause}})$$

$\bar{x}_1 \to$ literal: + or − variable

$0 \vee 1 = 1$

$0 \wedge 1 = 0$

Assign values to variables, to satisfy formula

Pure literal: when formula only contains literals x_i (or only $\bar{x}_i$)

Algorithm: while $\exists$ pure literal available

1. choose one at random x_i (or $\bar{x}_i$)
2. assign 'positive' value $x_i = 1$ (or $x_i = 0$)
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3 literals per clause

m clause

n variables

density $\frac{m}{n}$

Thm: $\exists\, d_c > 0$. Construct a random 3-SAT where (# literals $= x_i$) $\simeq \mathrm{Poi}(\lambda_i)$ and density is d:

$d < d_c$ formula is satisf. a.a.s.

$d > d_c$ formula is not satisf. a.a.s.

An introductory example

There are n bins and $m = cn$ balls, sequentially place balls independ. into bins

How many $\underbrace{\text{empty bins}}_{B}$ there are left?

$$B = \sum_{j=1}^{n} B_j \qquad B_j = \mathbb{1}_{\{j\text{-th bin is empty}\}} \qquad \mathbb{P}(B_j = 1) = (1 - 1/n)^{cn} \Rightarrow \mathbb{E}[B] = n e^{-c}(1+o(1))$$

Let $\{X_i\}$ iid. $X_i \sim \mathrm{Unif}[n] \longrightarrow X_i = j$ if i-th ball goes to j-th bin

Now B satisfies the bounded differences condition

$$\mathbb{P}(|B - \mathbb{E}[B]| \geq \varepsilon n) \leq 2 e^{-2\varepsilon^2 n^2 / \sum_{i=1}^{cn} 1^2} = 2 e^{-2n\varepsilon^2/c}$$

If need to estimate $\mathbb{E}[B]$

Y_i = # empty bins when i balls have been located

$$|Y_{i+1} - Y_i| \leq 1$$

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Local changes, given current information $\mathcal{F}_i$ = generated by "info" at time i

$$\mathbb{E}[Y_{i+1} - Y_i \mid \mathcal{F}_i] = -1\, \mathbb{P}\left(\begin{smallmatrix}\text{ball into}\\ \text{empty bin}\end{smallmatrix}\right) = \frac{-Y_i}{n}$$

Idealized ODE:

$$\frac{Y(tn)}{n} \approx y(t) \qquad \frac{Y(0)}{n} = y(0) = 1 \qquad \Delta Y_i = Y_{i+1} - Y_i$$

$$\frac{\mathbb{E}\left[\frac{\Delta Y(tn)}{n} \mid \mathcal{F}_{tn}\right]}{1/n} \cong y'(t) = -y(t)$$

Solution $y(t) = e^{-t}$

$$\frac{\mathbb{E}[B]}{n} = \mathbb{E}\left[\frac{Y_{cn}}{n}\right] \approx e^{-c}$$

A toy deterministic question

How much can two collections of functions can differ if they have similar derivatives and initial values?

$\exists\, \lambda, \delta > 0$ small perturbations such that

$$y_k(0) = \hat{y}_k \quad \text{cte} \qquad\qquad y_k'(t) = F_k(t, \underline{y(t)})$$

$$|z_k(0) - y_k(0)| \leq \lambda \qquad\qquad |z_k'(t) - F_k(t, \underline{z(t)})| \leq \delta$$

Condition: F is L-Lipschitz cont. in norm $\|\cdot\|_\infty$

in particular $\quad |F_k(t, y(t)) - F_k(t, z(t))| \leq L \max_{k \in I} |y_k(t) - z_k(t)|$

$$\Longrightarrow \max_{k \in I} |y_k(t) - z_k(t)| \leq (\lambda + \delta T)\, e^{Lt} \qquad \forall t \leq T$$

$$|y'(s) - z'(s)| < |y'(s) - F(s, z(s))| + |\underbrace{F(s, z(s)) - z'(s)}|$$

$$< |F(s, y(s)) - F(s, z(s))| + \delta$$

$$\leq L \cdot |y(s) - z(s)| + \delta$$

$$\underbrace{|y(t) - z(t)|}_{x(t)} \leq |y(0) - z(0)| + \int_0^t |y'(s) - z'(s)| ds$$

$$\leq (\lambda + \delta T) + \int_0^t L\, x(s)\, ds$$

By Grönwall's ineq. $\Rightarrow$ $x(t) \leq (\lambda + \delta T)\, e^{Lt}$

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A first analysis of the d-process

Consider a graph process on n vertices

Edges added, sequentially at random from missing ones that are 'valid':

! $\exists\ m(v) \in \mathbb{N}$ so that vertex v cannot exceed degree

$$d_t(v) \leq m(v) \quad \forall t \geq 1$$

degree of v at time t

! Stop process when no edges can be added.

$$\tau = \min\{t \geq 0 : \text{no edge can be added to } G_t\}$$

If $T = \frac{1}{2}\sum m(v)$ then $\tau \leq T$

To implement process, keep track of deficit of vertices:

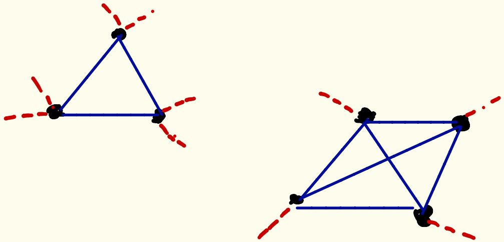
$$m(v) - d_t(v) \begin{cases} > & \text{unsaturated} \\ = & \text{saturated} \end{cases}$$

The process saturates if final G_τ has ≤ 1 unsat. vertices

Probability that $\{G_t\}$ saturates?

Bad case $m(v) \equiv d$:

At time τ, graph contains a clique of size $k \in \{2, 3, \ldots d\}$ of unsaturated vertices



Problem:

Keep track of forbidden edges

Generalized d - process G_t:

Include a subgraph Φ of bounded degree such that

Step (0) G_0 starts with n isolated vertices

Step (t) edge chosen at random such that

is missing from G_{t-1}

don't violate degree restriction

does not belong to Φ

Let $M = \max_v m(v)$. If $m(v) - d_t(v) = M$ then v is full isolate

Recall $T = \frac{1}{2} \sum m(v)$

Theorem: If by time $T - I$ there are $o(I)$ full isolates and $I = I(n) \to \infty$ then $\exists\ \sigma(n) \to \infty$ such that a.a.s.

there are no full isolates by $T - \sigma(n)$

For d-process: $d \geq 2$ Divide into stages

① $G_t^{(1)}$ with $\Phi^{(1)} = \phi$ and $m_1(v) \equiv d$

until $\tau_1 = \inf\{t : \text{mindeg in } G_t^{(1)} \geq 1\}$

② $G_t^{(2)}$ with $\Phi = E(G_{\tau_1}^{(1)})$ and $m_2(v) = d - d_{\tau_1}^{(1)}(v)$

until $\tau_2 = \inf\{t : \text{mindeg in } G_t^{(2)} \geq 1\}$

$\vdots$

ⓓ $G_t^{(d)}$ with $\Phi = E(G_{\tau_{d-1}}^{(d-1)})$ and $m_d(v) = d - d_{\tau_{d-1}}^{(d-1)}(v)$

until $\tau_d = \inf\{t : \text{mindeg in } G_t^{(d)} \geq 1\}$

<u>Obs</u>: Concatenate the choices of edges in $G_t^{(i)}$ to get the original d-process

We are using: Can go from one stage to the other

Grönwall's inequality discrete version

The bound: Suppose for $j \le m$ $\quad x_j < c + \sum_{i=1}^{j-1} a x_i + b$

then $\forall$ all j $\quad x_j < b\tilde{c}(1+a)^j$ $\quad$ where $\tilde{c} = \min\{a^{-1}, m\}$

$$x_j + \left(\frac{b}{a}\right) < \left(c + \frac{b}{a}\right) + \sum_{0 \le i < j} a\left(x_i + \frac{b}{a}\right)$$

There are two relaxations for the assumptions

So it suffices to consider $b = 0$

$$x_j < (c + bm) + \sum_{0 \le i < j} a x_i$$

Now, observe that

$$(1+a)^j = (1+a)^{j-1} + a(1+a)^{j-1}$$

$$(1+a) = 1 + a(1+a)^0$$

By induction $\quad x_j < c(1+a)^j \quad$ since

$$x_j < c + \sum_{0 \le i < j} a x_i \le c\left(1 + \sum_{0 \le i < j} a(1+a)^i\right) = c(1+a)^j$$

Finally, we use the fact $(1+a)^j \le e^{aj}$

The Differential Equation theorem

Suppose: $Y(i)$ is $\mathcal{F}_i$-measurable for $i \geq 0$

and $\exists \begin{cases} \text{bounded domain } D \subseteq \mathbb{R}^2 \\ L\text{-Lipschitz cont } F : D \to \mathbb{R} \end{cases}$

Whenever $\left(\frac{i}{n}, \frac{Y(i)}{n}\right) \in D$ we have:

i) Trend Hypothesis:

$$\left| \mathbb{E}[Y(i+1) - Y(i) \mid \mathcal{F}_i] - F\left(\frac{i}{n}, \frac{Y(i)}{n}\right) \right| \leq \delta$$

ii) Boundedness:

$$|Y(i+1) - Y(i)| \leq \beta$$

iii) Initial Condition:

$$|Y(0) - \hat{y}n| \leq \lambda n$$

Then: $\exists!\ y(t)$ with

$$y'(t) = F(t, y(t))$$
$$y(0) = \hat{y}$$

$\forall \lambda \geq \delta \min\{T, L^{-1}\} + \frac{R}{n}$

with prob $\geq 1 - 2e^{-n\lambda^2/8T\beta^2}$

$$\max_{0 \leq i \leq \sigma n} \left| \frac{Y(i)}{n} - y\left(\frac{i}{n}\right) \right| < 3e^{LT}\lambda$$

Details

$\exists\ R = R(D, F, L) \geq 1$ $\quad |y'(t)| \leq R$

$T = T(D) > 0$

$\sigma = \sigma(\hat{y}) \in [0, T]$ $\quad \forall t \leq \sigma \quad (t, y(t))$ is $3e^{LT}\lambda$ away from boundary

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$\forall \lambda \geq \delta \min\{T, L^{-1}\} + \frac{R}{n}$ with prob $\geq 1 - 2e^{-n\lambda^2 / 8T\beta^2}$

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Back to balls and bins example

Y_i = #empty bins when i balls placed

i) $\mathbb{E}[\Delta Y_i \mid \mathcal{F}_i] = -\frac{Y_i}{n} + \underbrace{\text{lower order terms}}_{\delta = 0}$

ii) $|Y_{i+1} - Y_i| \leq \beta = 1$

iii) $y(0) = 1 \Rightarrow \lambda > 0$ has no restriction

Domain: $\begin{array}{l} i \leq cn = Tn \\ Y_i \leq n \end{array}$ $\quad D = \left\{ (t, x) : \begin{array}{l} t \in [0, c] \\ x \in [0, 1] \end{array} \right\}$

$F(t, x) = -x$ then $y(t) = e^{-t}$ $\quad |y'(t)| \leq e^{-c} < 1 = R$

1-Liptchitz $L = 1$

Select $\lambda = n^{-1/3}$ then w.p. $1 - e^{-\frac{2n^{1/3}}{8}}$

$$\max_{0 \leq i \leq \sigma n} \left| \frac{Y(i)}{n} - y\left(\frac{i}{n}\right) \right| \leq 3e^{c} n^{-1/3}$$

Let $\sigma = T$ by extending the domain a little bit

A second and third analysis of d-process

The emergence of a giant component

$\iff$ Suceptibility blows up

$$\sum_{j\in J} |C_j|^2 \longleftarrow \text{step changes depend on products}$$

of $\sum_{j\in J} |C_{j,a}||C_{j,b}| = Y_{a,b}$ $\quad a,b \in [d]^2$

vertices of deg b in C_j

The growth of the giant component

$\iff$ Control on moments of a Branching Process...

$$\sum_{j\in J} |C_j|^k \longleftarrow \text{step changes depend on products}$$

of $\sum_{j\in J} \prod_{i=1}^{4} |C_{j,a_i}| = Y_{\underline{a}}$ $\quad \underline{a} \in [d]^2$

References:

- The cook-book approach to the DEM — Díaz, Mitsche
- On Wormald's DEM — Warnke
- The DEM for graph processes and greedy alg. — Wormald
- Concentration — McDiarmid
- On the method of bounded differences — McDiarmid

On d-processes

- Growth of the giant component — Eslava, Warnke
- Emergence of the giant component — Warnke, Wormald
- Random graph processes with maximum degree 2 — Ruciński, Wormald
- Diff. Equations for random processes and random graphs — Wormald
- Random Graph Processes with degree restrictions — Ruciński, Wormald