Skeletal stochastic differential equations for continuous-state branching process

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Definition of ψ-CSBP.

A CSBP $(X, \mathbb{P}_x)$ is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x, x \geq 0)$ such that for any $x, y \geq 0$, $\mathbb{P}_{x+y} = \mathbb{P}_x * \mathbb{P}_y$.

In particular

$$\mathbb{E}_x(\mathrm{e}^{-\theta X_t}) = \mathrm{e}^{-x u_t(\theta)}, \qquad x, \theta, t \geq 0,$$

where $u_t(\theta)$ uniquely solves the evolution equation

$$u_t(\theta) + \int_0^t \psi(u_s(\theta))\mathrm{d}s = \theta, \qquad t \geq 0.$$

Here, we assume that the so-called branching mechanism ψ takes the form

$$\psi(\theta) = -\alpha\theta + \beta\theta^2 + \int_{(0,\infty)} (\mathrm{e}^{-\theta x} - 1 + \theta x)\Pi(\mathrm{d}x), \ \ \theta \geq 0,$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and Π is a measure concentrated on $(0, \infty)$ which satisfies $\int_{(0,\infty)} (x \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

PROPERTIES.

We assume that the process is conservative, i.e.

$$\int_{0+} \frac{1}{|\psi(\xi)|} \mathrm{d}\xi = \infty.$$

It is easily verified that

$$\mathbb{E}_x[X_t] = x\mathrm{e}^{-\psi'(0+)t}, \qquad t, x \geq 0.$$

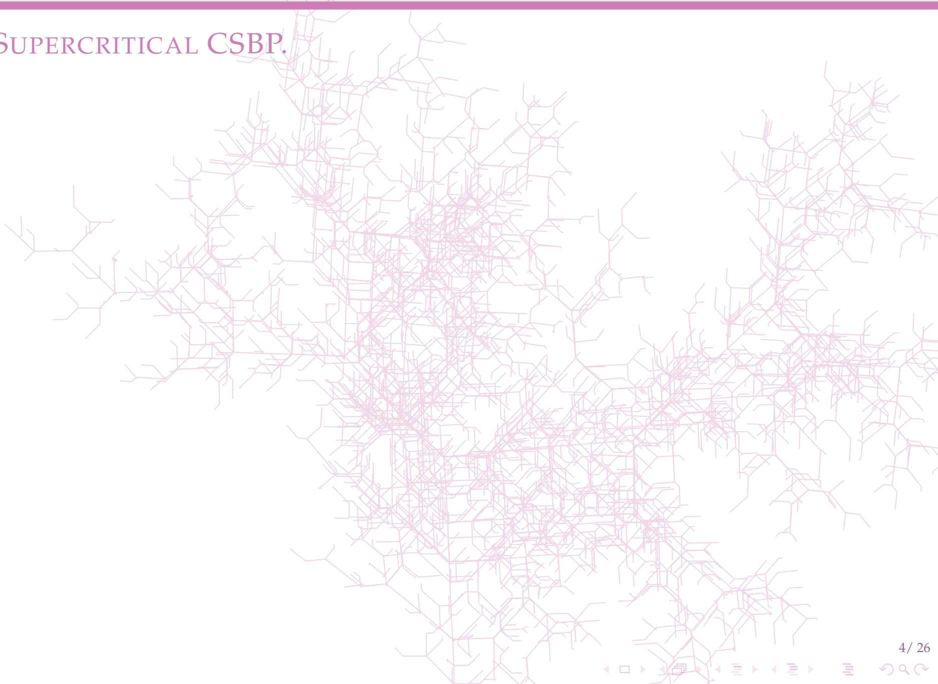
We say that the CSBP is supercritical, critical or subcritical accordingly as $-\psi'(0+) = \alpha$ is strictly positive, equal to zero or strictly negative.

For a **supercritical** ψ-CSBP the probability of extinction is

$$\mathbb{P}_x(\lim_{t\uparrow\infty} X_t = 0) = \mathrm{e}^{-\lambda^* x},$$

where λ^* is the unique root on $(0, \infty)$ of the equation $\psi(\theta) = 0$.

SUPERCRITICAL CSBP.



Prolific skeleton I.

The supercritical ψ-CSBP is equal in law to the total mass process obtained by the following construction.

- Initiate $\text{Po}(\lambda^* x)$ independent Galton-Watson processes with branching generator

$$q\left(\sum_{k\geq 0} p_k r^k - r\right) = \frac{1}{\lambda^*}\psi(\lambda^*(1-r)), \qquad r \in [0,1],$$

where $q = \psi'(\lambda^*)$, $p_0 = p_1 = 0$ and for $k \geq 2$

$$p_k = \frac{1}{\lambda^* \psi'(\lambda^*)}\left\{\beta(\lambda^*)^2 \mathbf{1}_{\{k=2\}} + (\lambda^*)^k \int_{(0,\infty)} \frac{r^k}{k!} \mathrm{e}^{-\lambda^* r} \Pi(\mathrm{d}r)\right\}.$$

- Along the edges immigrate CSBPs at rate

$$2\beta \mathrm{d}\mathbb{Q}^* + \int_0^\infty y \mathrm{e}^{-\lambda^* y} \Pi(\mathrm{d}y) \mathrm{d}\mathbb{P}^*_y,$$

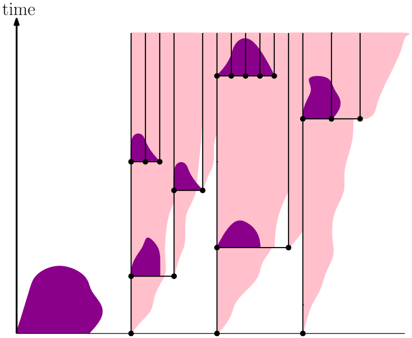
where $\mathbb{P}^*_x,\ x \geq 0$ is the law of the CSBP with branching mechanism $\psi^*(\lambda) = \psi(\lambda + \lambda^*)$ and $\mathbb{Q}^*$ is the associated excursion measure.

Prolific skeleton II.

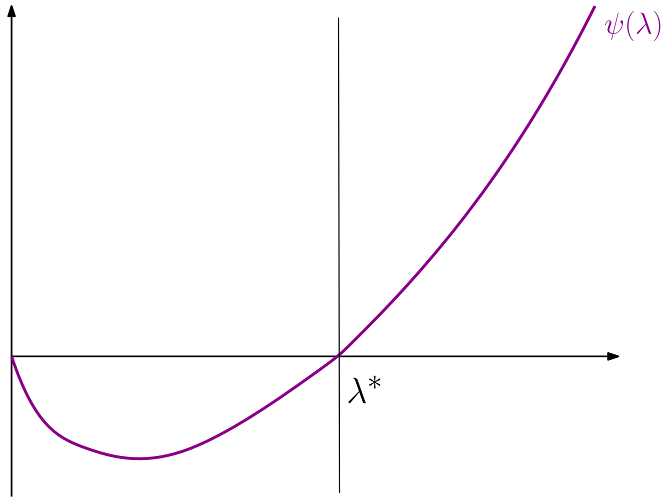
- Given that an individual dies and branches into $k \geq 2$ offspring, an independent ψ^*-CSBP is immigrated with initial mass r with probability

$$\eta_k(\mathrm{d}r) = \frac{1}{p_k \lambda^* \psi'(\lambda^*)} \left\{ \beta(\lambda^*)^2 \delta_0(\mathrm{d}r) \mathbf{1}_{\{k=2\}} + (\lambda^*)^k \frac{r^k}{k!} \mathrm{e}^{-\lambda^* r} \Pi(\mathrm{d}r) \right\}.$$

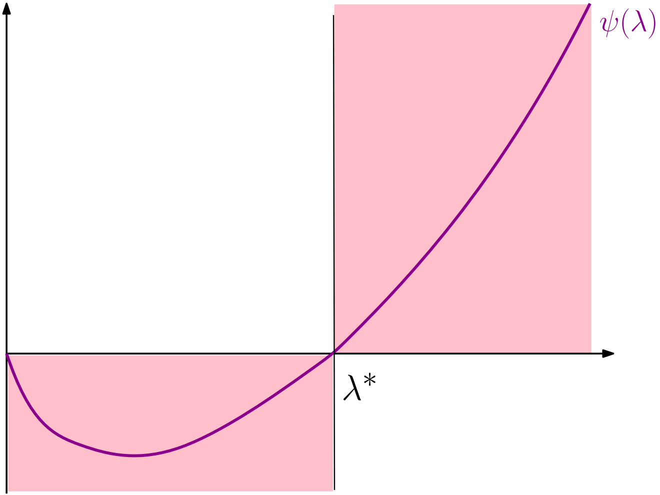
- Finally an independent ψ^*-CSBP is issued at time zero with initial mass x.



BRANCHING MECHANISM.



Branching mechanism.



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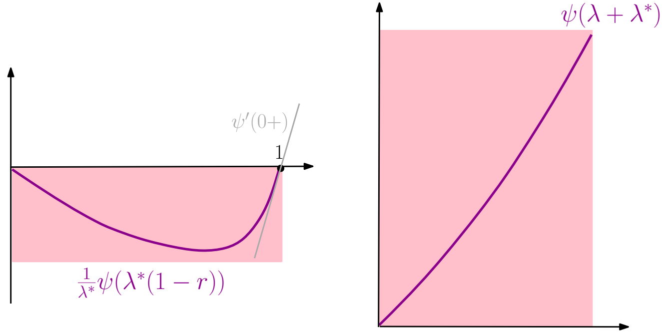

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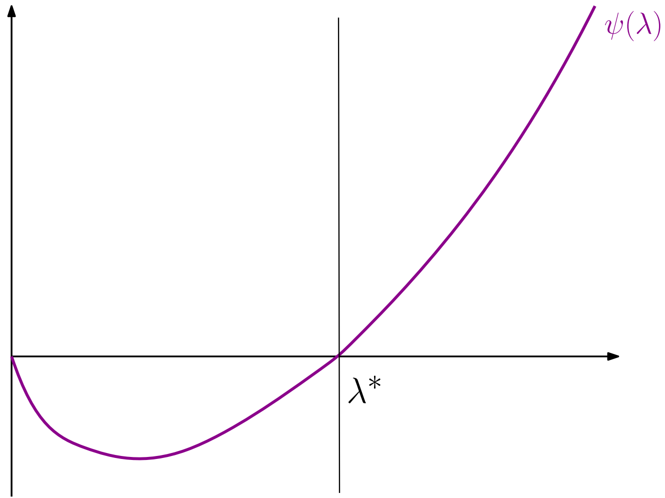

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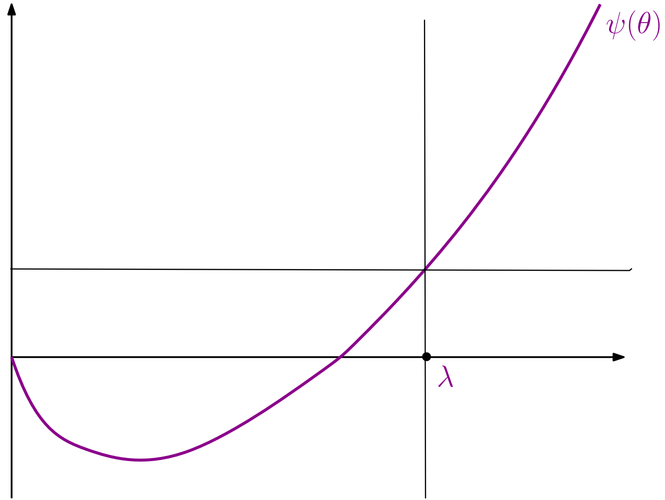

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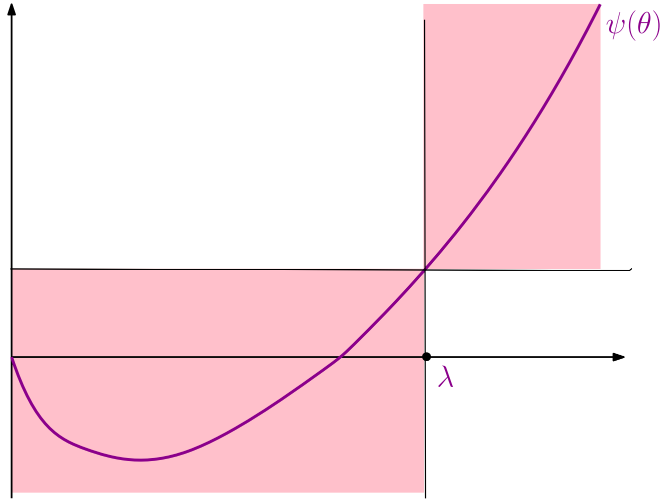

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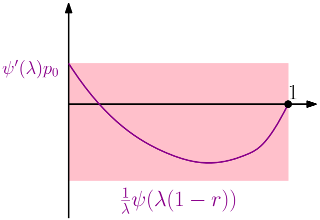
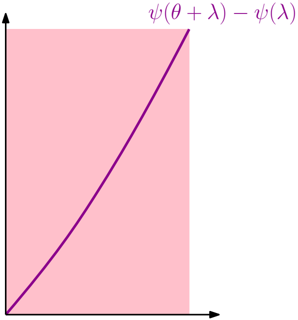


λ-skeleton



ψ_λ-CSBP

λ-SKELETON I.

Let $\lambda \geq \lambda^*$.
Define the Esscher transformed branching mechanism $\psi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ for $\theta \geq -\lambda$ and $\lambda \geq \lambda^*$ by $\psi_\lambda(\theta) = \psi(\theta + \lambda) - \psi(\lambda)$.

The supercritical ψ-CSBP is equal in law to the total mass process obtained by the following construction.

- Initiate $\mathrm{Po}(\lambda x)$ independent Galton-Watson processes with branching generator

$$q\left(\sum_{k\geq 0} p_k r^k - r\right) = \frac{1}{\lambda}\psi(\lambda(1-r)), \qquad r \in [0,1],$$

where $q = \psi'(\lambda)$, $\boldsymbol{p_0 = \psi(\lambda)/\lambda\psi'(\lambda)}$, $p_1 = 0$ and for $k \geq 2$

$$p_k = \frac{1}{\lambda\psi'(\lambda)}\left\{\beta\lambda^2 \mathbf{1}_{\{k=2\}} + \int_{(0,\infty)} \frac{(\lambda r)^k}{k!} \mathrm{e}^{-\lambda r} \Pi(\mathrm{d}r)\right\}.$$

λ-SKELETON II.

- Along the edges immigrate CSBPs at rate

$$2\beta \mathrm{d}\mathbb{Q}^{(\lambda)} + \int_0^\infty y e^{-\lambda y} \Pi(\mathrm{d}y) \mathrm{d}\mathbb{P}_y^{(\lambda)},$$

where $\mathbb{P}_x^{(\lambda)}$, $x \geq 0$ is the law of the CSBP with branching mechanism ψ_λ and $\mathbb{Q}^{(\lambda)}$ is the associated excursion measure.

- Given that an individual dies and branches into $k \in \mathbb{N}_0 \setminus \{1\}$ offspring, an independent ψ_λ-CSBP is immigrated with initial mass r with probability

$$\eta_k(\mathrm{d}r) = \frac{1}{p_k \lambda \psi'(\lambda)} \Big\{ \psi(\lambda) \mathbf{1}_{\{k=0\}} \delta_0(\mathrm{d}r) + \beta \lambda^2 \mathbf{1}_{\{k=2\}} \delta_0(\mathrm{d}r) + \mathbf{1}_{\{k\geq 2\}} \frac{(\lambda r)^k}{k!} e^{-\lambda r} \Pi(\mathrm{d}r) \Big\},$$

- Finally an independent ψ_λ-CSBP is issued at time zero with initial mass x.

SDE.

The process $(X, \mathbb{P}_x)$, $x > 0$, can be represented as the unique strong solution to the stochastic differential equation (SDE)

$$X_t = x + \alpha \int_0^t X_{s-} \mathrm{d}s + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r\tilde{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu), \tag{1}$$

for $x > 0, t \geq 0$, where

- $W(\mathrm{d}s, \mathrm{d}u)$ is a white noise process on $(0, \infty)^2$ based on the Lebesgue measure $\mathrm{d}s \otimes \mathrm{d}u$,
- $N(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$ is a Poisson point process on $(0, \infty)^3$ with intensity $\mathrm{d}s \otimes \Pi(\mathrm{d}r) \otimes \mathrm{d}\nu$, and $\tilde{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$ the compensated measure of $N(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$.

Thinning of the SDE I.

We can introduce an additional mark to atoms of N, resulting in an 'extended' Poisson random measure, $\mathcal{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu, \mathrm{d}k)$ on $(0,\infty)^3 \times \mathbb{N}_0$ with intensity

$$\mathrm{d}s \otimes \Pi(\mathrm{d}r) \otimes \mathrm{d}\nu \otimes \frac{(\lambda r)^k}{k!}\mathrm{e}^{-\lambda r}\sharp(\mathrm{d}k).$$

Define three random measures by

$$N^0(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) = \mathcal{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu, \{k=0\}),$$

$$N^1(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) = \mathcal{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu, \{k=1\})$$

and

$$N^2(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) = \mathcal{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu, \{k\geq 2\}).$$

We have that N^0, N^1 and N^2 are independent Poisson point processes on $(0,\infty)^3$ with respective intensities $\mathrm{d}s \otimes \mathrm{e}^{-\lambda r}\Pi(\mathrm{d}r) \otimes \mathrm{d}\nu$, $\mathrm{d}s \otimes (\lambda r)\mathrm{e}^{-\lambda r}\Pi(\mathrm{d}r) \otimes \mathrm{d}\nu$ and $\mathrm{d}s \otimes \sum_{k=2}^{\infty}(\lambda r)^k\mathrm{e}^{-\lambda r}\Pi(\mathrm{d}r)/k! \otimes \mathrm{d}\nu$.

Thinning of the SDE II.

$$
\begin{aligned}
X_t &= x + \alpha \int_0^t X_{s-} \mathrm{d}s + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^{\infty} \int_0^{X_{s-}} r\tilde{N}^0(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) \\
&\quad + \int_0^t \int_0^{\infty} \int_0^{X_{s-}} rN^1(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) + \int_0^t \int_0^{\infty} \int_0^{X_{s-}} rN^2(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) \\
&\quad - \int_0^t \int_0^{\infty} X_{s-} \sum_{n=1}^{\infty} \frac{(\lambda r)^n}{n!} \mathrm{e}^{-\lambda r} r\Pi(\mathrm{d}r)\mathrm{d}s \\
&= x - \psi'(\lambda) \int_0^t X_s \mathrm{d}s + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^{\infty} \int_0^{X_{s-}} r\tilde{N}^0(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) \\
&\quad + \int_0^t \int_0^{\infty} \int_0^{X_{s-}} rN^1(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) + 2\beta\lambda \int_0^t X_{s-} \mathrm{d}s \\
&\quad + \int_0^t \int_0^{\infty} \int_0^{X_{s-}} rN^2(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu),
\end{aligned}
$$

(In the last equality we have used that $-\int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda r}) r\Pi(\mathrm{d}r) = -\alpha + 2\beta\lambda - \psi'(\lambda)$).

Thinning of the SDE II.

$$X_t = x + \alpha \int_0^t X_{s-} \mathrm{d}s + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r\tilde{N}^0(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$$

$$+ \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^1(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) + \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^2(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$$

$$- \int_0^t \int_0^\infty X_{s-} \sum_{n=1}^\infty \frac{(\lambda r)^n}{n!} \mathrm{e}^{-\lambda r} r\Pi(\mathrm{d}r)\mathrm{d}s$$

$$= x - \psi'(\lambda) \int_0^t X_s \mathrm{d}s + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r\tilde{N}^0(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$$

$$+ \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^1(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) + 2\beta\lambda \int_0^t X_{s-} \mathrm{d}s$$

$$+ \int_0^t \int_0^\infty \int_0^{X_{s-}} rN^2(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu),$$

(In the last equality we have used that $-\int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda r}) r\Pi(\mathrm{d}r) = -\alpha + 2\beta\lambda - \psi'(\lambda)$).

Theorem

Suppose that ψ corresponds to a supercritical branching mechanism (i.e. $\alpha > 0$) and $\lambda \geq \lambda^*$. Consider the coupled system of SDEs

$$\begin{aligned}
\begin{pmatrix} \Lambda_t \\ Z_t \end{pmatrix} = \begin{pmatrix} \Lambda_0 \\ Z_0 \end{pmatrix} - \psi'(\lambda)\int_0^t \begin{pmatrix} \Lambda_{s-} \\ 0 \end{pmatrix} \mathrm{d}s + \sqrt{2\beta}\int_0^t\int_0^{\Lambda_{s-}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} W(\mathrm{d}s, \mathrm{d}u) \\
+ \int_0^t\int_0^\infty\int_0^{\Lambda_{s-}} \begin{pmatrix} r \\ 0 \end{pmatrix} \tilde{\mathrm{N}}^0(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) \\
+ \int_0^t\int_0^\infty\int_1^{Z_{s-}} \begin{pmatrix} r \\ 0 \end{pmatrix} \mathrm{N}^1(\mathrm{d}s, \mathrm{d}r, \mathrm{d}j) \\
+ \int_0^t\int_0^\infty\int_0^\infty\int_1^{Z_{s-}} \begin{pmatrix} r \\ k-1 \end{pmatrix} \mathrm{N}^2(\mathrm{d}s, \mathrm{d}r, \mathrm{d}k, \mathrm{d}j) \\
+ 2\beta\int_0^t \begin{pmatrix} Z_{s-} \\ 0 \end{pmatrix} \mathrm{d}s, \qquad t \geq 0,
\end{aligned} \tag{2}$$

with $\Lambda_0 \geq 0$ given and fixed. Under the assumption that Z_0 is an independent random variable which is Poisson distributed with intensity $\lambda\Lambda_0$ the system (2) has a unique strong solution such that:

- *(i) For $t \geq 0$, $Z_t|\mathcal{F}_t^\Lambda$ is Poisson distributed with intensity $\lambda\Lambda_t$, where $\mathcal{F}_t^\Lambda := \sigma(\Lambda_s : s \leq t)$;*
- *(ii) The process $(\Lambda_t, t \geq 0)$ is a weak solution to (1).*

(Driving sources of randomness I.)

Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\sharp(\mathrm{d}\ell) = \sum_{i\in\mathbb{N}_0} \delta_i(\mathrm{d}\ell)$, $\ell \geq 0$.

Then in the previous theorem

- $\mathtt{N}^0$ is a Poisson random measure on $(0,\infty)^3$ with intensity measure $\mathrm{d}s \otimes \mathrm{e}^{-\lambda r}\Pi(\mathrm{d}r) \otimes \mathrm{d}\nu$, $\tilde{\mathtt{N}}^0$ is the associated compensated version of $\mathtt{N}^0$,
- $\mathtt{N}^1(\mathrm{d}s, \mathrm{d}r, \mathrm{d}j)$ is a Poisson point process on $(0,\infty)^2 \times \mathbb{N}$ with intensity $\mathrm{d}s \otimes r\mathrm{e}^{-\lambda r}\Pi(\mathrm{d}r) \otimes \sharp(\mathrm{d}j)$,
- $\mathtt{N}^2(\mathrm{d}s, \mathrm{d}r, \mathrm{d}k, \mathrm{d}j)$ is a Poisson point process on $(0,\infty)^2 \times \mathbb{N}_0 \times \mathbb{N}$ with intensity $\psi'(\lambda)\mathrm{d}s \otimes \eta_k(\mathrm{d}r) \otimes p_k\sharp(\mathrm{d}k) \otimes \sharp(\mathrm{d}j)$, and
- $W(\mathrm{d}s, \mathrm{d}u)$ is the white noise process on $(0,\infty)^2$ based on the Lebesgue measure $\mathrm{d}s \otimes \mathrm{d}u$.

SUBCRITICAL CSBP.

Ray-Knight representation.

Assume that Grey's condition is satisfied, ie.

$$\int^{\infty} \frac{1}{\psi(\theta)} \mathrm{d}u < \infty.$$

Let

- $(\xi_t, t \geq 0)$ be a spectrally positive Lévy process with Laplace exponent ψ,
- $(\hat{\xi}_r^{(t)}, 0 \leq r \leq t)$, where $\hat{\xi}_r^{(t)} := \xi_t - \xi_{(t-r)-}$, the time reversed process at time t,
- $\hat{S}_r^{(t)} := \sup_{s \leq r} \hat{\xi}_s^{(t)}$.

The process $(H_t, t \geq 0)$ is called the height process if H_t is the local time at level 0, at time t of $\hat{S}^{(t)} - \hat{\xi}^{(t)}$.
Denote by L_t^a the local time up to time t of H at level $a \geq 0$, and let
$T_x := \inf\{t \geq 0 : \xi_t = -x\}$.

Then the generalised Ray-Knight theorem for the ψ-CSBP process states that $(L_{T_x}^a, a \geq 0)$ has a càdàg modification for which

$$(L_{T_x}^t, t \geq 0) \stackrel{d}{=} (X, \mathbb{P}_x),$$

that is, the two processes are equal in law.

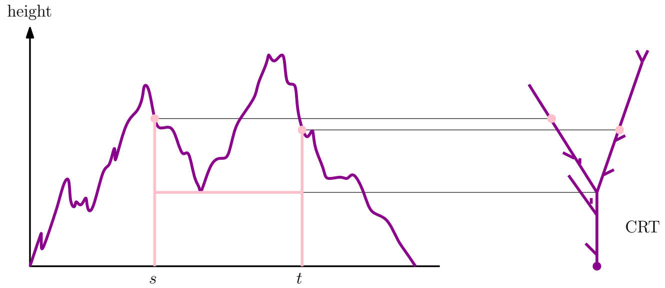
Genealogy of subcritical CSBP.

Excursions of H away from 0 form a PPP, denote by $\mathtt{n}$ its intensity, and let ϵ be a canonical excursion under $\mathtt{n}$.
Let $\zeta = \inf\{s > 0, \epsilon_s = 0\}$, and define

$$d_\epsilon(s,t) = \epsilon_s + \epsilon_t - \inf_{s \wedge t \leq r \leq s \vee t} \epsilon_r, \quad (s,t) \in [0,\zeta]^2.$$

Then we can define the equivalence relation $\sim_\epsilon$, such that $(s \sim_\epsilon t)$ is and only if $d_\epsilon(s,t) = 0$, and $\mathcal{T}_\epsilon = [0,\zeta] \backslash \sim_\epsilon$.

The compact metric space $(\mathcal{T}_\epsilon, d_\epsilon)$ is called a Lévy random tree.



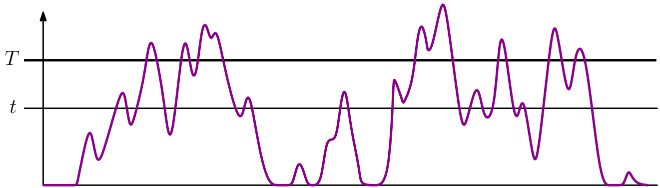
T-SKELETON.

Fix $T > 0$.
Define $(Z_t^T, 0 \leq t < T)$ as the process that counts the number of excursions above level t that hit level T.
Then Z^T is a time-dependent continuous-time Galton-Watson process which at time t branching at rate

$$q^{T-t} = \frac{u_{T-t}(\infty)\psi'(u_{T-t}(\infty)) - \psi(u_{T-t}(\infty))}{u_{T-t}(\infty)}, \qquad t \in [0, T),$$

and its offspring distribution $(p_k^{T-t}, k \geq 0)$ is given by $p_0^{T-t} = p_1^{T-t} = 0$,

$$p_k^{T-t} = \frac{1}{u_{T-t}(\infty)q^{T-t}} \times \left\{ \beta u_{T-t}^2(\infty)\mathbf{1}_{\{k=2\}} + \int_0^\infty \frac{(u_{T-t}(\infty)x)^k}{k!} \mathrm{e}^{-u_{T-t}(\infty)x}\Pi(dx) \right\}.$$



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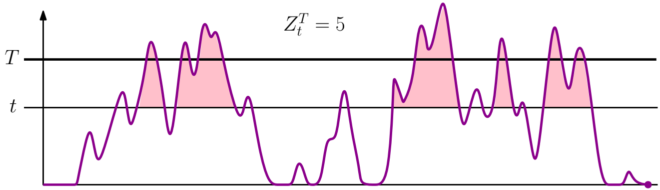

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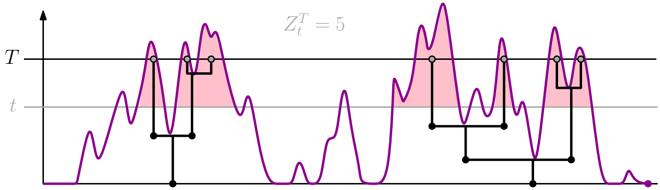

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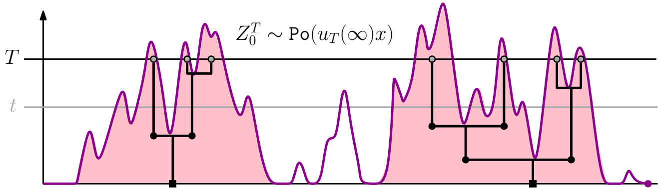


IMMIGRATION.

As

$$\mathbb{P}\left[X_T = 0 | \mathcal{F}_t\right] = \mathrm{e}^{-X_t u_{T-t}(\infty)},$$

the law of X conditioned to die out by time T can be obtained by the following change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}_x^T}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{\mathrm{e}^{-X_t u_{T-t}(\infty)}}{\mathrm{e}^{-x u_T(\infty)}}, \qquad t \geq 0, x > 0.$$

We get that $(X, \mathbb{P}_x^T)$ is a time-dependent CSBP with Laplace transform

$$\mathbb{E}_x^T[\mathrm{e}^{-\theta X_t}] = \mathrm{e}^{-x V_t^T(\theta)}, \quad 0 \leq t < T,\ x, \theta \geq 0,$$

where

$$V_t^T(\theta) = u_t(\theta + u_{T-t}(\infty)) - u_T(\infty).$$

Note that

$$\lim_{t \to T} u_{T-t}(\infty) = \infty, \quad \text{and} \quad \lim_{T \to \infty} u_{T-t}(\infty) = 0.$$

Theorem

Suppose that ψ corresponds to a (sub)critical branching mechanism (i.e. $\alpha \le 0$) which satisfies Grey's condition. Fix a time horizon $T > 0$ and consider the coupled system of SDEs

$$
\begin{aligned}
\begin{pmatrix} \Lambda_t^T \\ Z_t^T \end{pmatrix} = \begin{pmatrix} \Lambda_0^T \\ Z_0^T \end{pmatrix} - \int_0^t \psi'(u_{T-s}(\infty)) \begin{pmatrix} \Lambda_{s-}^T \\ 0 \end{pmatrix} \mathrm{d}s + \sqrt{2\beta} \int_0^t \int_0^{\Lambda_{s-}^T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} W(\mathrm{d}s, \mathrm{d}u) \\
+ \int_0^t \int_0^\infty \int_0^{\Lambda_{s-}^T} \begin{pmatrix} r \\ 0 \end{pmatrix} \tilde{\mathrm{N}}_T^0(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu) \\
+ \int_0^t \int_0^\infty \int_1^{Z_{s-}^T} \begin{pmatrix} r \\ 0 \end{pmatrix} \mathrm{N}_T^1(\mathrm{d}s, \mathrm{d}r, \mathrm{d}j) \\
+ \int_0^t \int_0^\infty \int_0^\infty \int_1^{Z_{s-}^T} \begin{pmatrix} r \\ k-1 \end{pmatrix} \mathrm{N}_T^2(\mathrm{d}s, \mathrm{d}r, \mathrm{d}k, \mathrm{d}j) \\
+ 2\beta \int_0^t \begin{pmatrix} Z_{s-}^T \\ 0 \end{pmatrix} \mathrm{d}s, \qquad 0 \le t < T. \qquad (3)
\end{aligned}
$$

with $\Lambda_0^T \ge 0$ given and fixed. Under the assumption that Z_0^T is an independent random variable which is Poisson distributed with intensity $u_T(\infty)\Lambda_0^T$ the system (3) has a unique strong solution such that:

(i) For $T > t \ge 0$, $Z_t^T | \mathcal{F}_t^{\Lambda^T}$ is Poisson distributed with intensity $u_{T-t}(\infty)\Lambda_t^T$, where $\mathcal{F}_t^{\Lambda^T} := \sigma(\Lambda_s^T : s \le t)$;

(ii) Conditional on $(\mathcal{F}^{\Lambda_t^T}, 0 \le t < T)$, the process $(\Lambda_t^T, 0 \le t < T)$ is a weak solution to (1)

(Driving forces of randomness II.)

In the previous theorem

- N_T^0 is a Poisson random measure on $[0,\infty)^3$ with intensity $\mathrm{d}s \otimes \mathrm{e}^{-u_{T-s}(\infty)r}\Pi(\mathrm{d}r) \otimes \mathrm{d}\nu$.

- N_T^1 is a Poisson process on $[0,\infty)^2 \times \mathbb{N}_0$ with intensity $\mathrm{d}s \otimes r\mathrm{e}^{-u_{T-s}(\infty)r}\Pi(\mathrm{d}r) \otimes \sharp(\mathrm{d}j)$,

- $\mathrm{N}_T^2(\mathrm{d}s, \mathrm{d}r, \mathrm{d}k, \mathrm{d}j)$ is a Poisson process on $[0,\infty)^2 \times \mathbb{N}_0 \times \mathbb{N}$ with intensity

$$\left\{\frac{u_{T-s}(\infty)\psi'(u_{T-s}(\infty)) - \psi(u_{T-s}(\infty))}{u_{T-s}(\infty)}\right\}\mathrm{d}s \otimes \eta_k^{T-s}(\mathrm{d}r) \otimes p_k^{T-s}\sharp(\mathrm{d}k) \otimes \sharp(\mathrm{d}j),$$

where, for $k \geq 2$,

$$\eta_k^{T-s}(\mathrm{d}r) = \frac{\beta u_{T-s}^2(\infty)\mathbf{1}_{\{k=2\}}\delta_0(\mathrm{d}r) + (u_{T-s}(\infty)r)^k\,\mathrm{e}^{-u_{T-s}(\infty)r}\Pi(\mathrm{d}r)/k!}{p_k^{T-s}\left(u_{T-s}(\infty)\psi'(u_{T-s}(\infty)) - \psi(u_{T-s}(\infty))\right)}, \qquad r \geq 0,$$

- $W(\mathrm{d}s, \mathrm{d}u)$ is the white noise process on $(0,\infty)^2$ based on the Lebesgue measure $\mathrm{d}s \otimes \mathrm{d}u$.

Conditioning on survival.

The law of $(\Lambda_t^T, 0 \leq t < T)$ conditional on $(\mathcal{F}^{\Lambda_t^T} \cap \{Z_0^T \geq 1\}, 0 \leq t < T)$ is that of the law of the ψ-CSBP, X, conditioned to survive until time T.

- This law is can be obtained by the following change of measure for $t \geq 0, x > 0$

$$\left.\frac{d\widetilde{\mathbb{P}}_x^T}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{1 - e^{-X_t u_{T-t}(\infty)}}{1 - e^{-x u_T(\infty)}}.$$

- We have for $k \geq 1$

$$\mathbf{P}_x^T[Z_0 = k | Z_0 \geq 1] = \frac{(u_T(\infty)x)^k}{k!} \frac{e^{-u_T(\infty)x}}{1 - e^{-u_T(\infty)x}}.$$

- If $\mathtt{n}_T$ denotes the conditional probability $\mathtt{n}(\cdot | \sup_{s \geq 0} \epsilon_s \geq T)$, then the first branch time γ_T of the individual corresponding to the excursion ϵ is given by

$$\mathtt{n}_T(\gamma_T > t) = \frac{\psi(u_T(\infty))}{u_T(\infty)} \frac{u_{T-t}(\infty)}{\psi(u_{T-t}(\infty))},$$

for $t \in [0, T)$.

CONDITIONING ON SURVIVAL.

The law of $(\Lambda_t^T, 0 \le t < T)$ conditional on $(\mathcal{F}^{\Lambda_t^T} \cap \{Z_0^T \ge 1\}, 0 \le t < T)$ is that of the law of the ψ-CSBP, X, conditioned to survive until time T.

Take $T \to \infty$.

- This law is can be obtained by the following change of measure for $t \ge 0, x > 0$

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}_x^T}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{1 - \mathrm{e}^{-X_t u_{T-t}(\infty)}}{1 - \mathrm{e}^{-x u_T(\infty)}} \longrightarrow \mathrm{e}^{-\alpha t}\frac{X_t}{x}.$$

- We have for $k \ge 1$

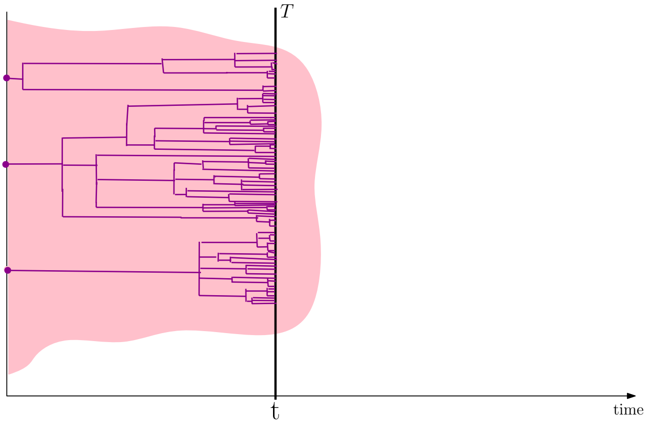
$$\mathbf{P}_x^T[Z_0 = k | Z_0 \ge 1] = \frac{(u_T(\infty)x)^k}{k!} \frac{\mathrm{e}^{-u_T(\infty)x}}{1 - \mathrm{e}^{-u_T(\infty)x}} \longrightarrow 0, \text{ unless } k = 1.$$

- If $\mathtt{n}_T$ denotes the conditional probability $\mathtt{n}(\cdot | \sup_{s \ge 0} \epsilon_s \ge T)$, then the first branch time γ_T of the individual corresponding to the excursion ϵ is given by

$$\mathtt{n}_T(\gamma_T > t) = \frac{\psi(u_T(\infty))}{u_T(\infty)} \frac{u_{T-t}(\infty)}{\psi(u_{T-t}(\infty))} \longrightarrow 1,$$

for $t \in [0, T)$.

Emergence of the spine.



Note that the convergence is in a weak sense.

SPINE.

Theorem

Suppose that ψ is a critical or subcritical branching mechanism such that Grey's condition holds. Suppose, moreover, that $((\Lambda_t^T, Z_t^T), 0 \leq t < T)$ is a weak solution to (3) and that Z_0^T is an independent random variable which is Poisson distributed with intensity $u_T(\infty)\Lambda_0^T$. Then, conditional on the event $Z_0^T > 0$, in the sense of weak convergence with respect to the Skorokhod topology on $\mathbb{D}([0,\infty), \mathbb{R}^2)$, for all $t > 0$,

$$((\Lambda_s^T, Z_s^T), 0 \leq s \leq t) \to ((X_s^\uparrow, 1), 0 \leq s \leq t),$$

where $X^\uparrow$ is a weak solution to

$$X_t = x + \alpha \int_0^t X_{s-} \mathrm{d}s + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r\tilde{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u) + \int_0^t \int_0^\infty rN^*(\mathrm{d}s, \mathrm{d}r) + 2\beta t, \qquad t \geq 0.$$

Spine.

Theorem

Suppose that ψ is a critical or subcritical branching mechanism such that Grey's condition holds. Suppose, moreover, that $((\Lambda_t^T, Z_t^T), 0 \le t < T)$ is a weak solution to (3) and that Z_0^T is an independent random variable which is Poisson distributed with intensity $u_T(\infty)\Lambda_0^T$. Then, conditional on the event $Z_0^T > 0$, in the sense of weak convergence with respect to the Skorokhod topology on $\mathbb{D}([0,\infty), \mathbb{R}^2)$, for all $t > 0$,

$$((\Lambda_s^T, Z_s^T), 0 \le s \le t) \to ((X_s^{\uparrow}, 1), 0 \le s \le t),$$

where $X^{\uparrow}$ is a weak solution to

$$X_t = x + \alpha \int_0^t X_{s-} \mathrm{d}s + \sqrt{2\beta} \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^\infty \int_0^{X_{s-}} r\tilde{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u)$$
$$+ \int_0^t \int_0^\infty rN^*(\mathrm{d}s, \mathrm{d}r) + 2\beta t, \qquad t \ge 0.$$

(Driving sources of randomness III.)

In the previous theorem

- $W(\mathrm{d}s, \mathrm{d}u)$ is a white noise process on $(0,\infty)^2$ based on the Lebesgue measure $\mathrm{d}s \otimes \mathrm{d}u$,
- $N(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$ is a Poisson point process on $(0,\infty)^3$ with intensity $\mathrm{d}s \otimes \Pi(\mathrm{d}r) \otimes \mathrm{d}\nu$, and $\tilde{N}(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$ is the compensated measure of $N(\mathrm{d}s, \mathrm{d}r, \mathrm{d}\nu)$,
- N^* is a Poisson random measure on $[0,\infty) \times (0,\infty)$ with intensity measure $\mathrm{d}s \otimes r\Pi(\mathrm{d}r)$.

References.

[1] D. A. Dawson and Z. Li. Stochastic equations, flows and measure-valued processes. *Ann. Probab.*, 40(2):813-857, 2012.

[2] T. Duquesne and J.-F. Le Gall. Random Trees, Lévy Processes and Spatial Branching Processes. *ArXiv Mathematics e-prints*, September 2005.

[3] T. Duquesne and M. Winkel. Growth of Lévy trees. *Probab. Theory Related Fields*, 139(3-4):313Ð371, 2007.

[4] D. Fekete, J. Fontbona, A. E. Kyprianou. Skeletal stochastic differential equations for continuous state branching process. *https://arxiv.org/pdf/1702.03533.pdf*, 2017.

[5] M. C. Fittipaldi and J. Fontbona. On SDE associated with continuous-state branching processes conditioned to never be extinct. *Electron. Commun. Probab.*, 17:no. 49, 13, 2012.

[6] Z. Li. *Measure Valued Branching Markov Processes.* Probability and its Applications (New York). Springer, Heidelberg, 2011.