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CALLABLE PUTS AS COMPOSITE EXOTIC OPTIONS

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Introduced by Kifer (2000), game options function in the same way as American options with the added feature that the writer may also choose to exercise, at which time they must pay out the intrinsic option value of that moment plus a penalty. In Kyprianou (2004) an explicit formula was obtained for the value function of the perpetual put option of this type. Crucial to the calculations which lead to the aforementioned formula was the perpetual nature of the option. In this paper we address how to characterize the value function of the *finite expiry* version of this option via mixtures of other exotic options by using mainly martingale arguments.

Key Words: game options, Dynkin games, fluctuation theory, local time

1. INTRODUCTION

Consider the Black–Scholes market. That is, a market with a risky asset S and a riskless bond, B. The bond evolves according to the dynamic

$$dB_t = rB_t\, dt \text{ where } r, t \geq 0.$$

The risky asset is written as the process $S = \{S_t : t \geq 0\}$ where

$$S_t = x \exp\{\sigma W_t + \mu t\} \text{ where } x > 0$$

is the initial value of S and $W = \{W_t : t \geq 0\}$ is a Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions and $T \in (0, \infty)$ is the time horizon. A callable put is an American put with the additional feature that the seller can recall the option prematurely paying besides the intrinsic value a constant penalty δ.

If the holder exercises first, (s)he may claim the value $(K - S_t)^+$ at the exercise date and if the writer exercises prematurely, (s)he is obliged to pay to the holder the value $(K - S_t)^+ + \delta$ at the time of exercise. If neither have exercised at time T then the writer pays the holder the value $(K - S_T)^+$. If both decide to claim at the same time then the lesser of the two claims is paid (however, it turns out that the agreement in this marginal case has no impact on the resulting option price). Our objective in this paper is to characterize the

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value and rational behavior of writer and holder that lead to this value. However, before getting involved with technicalities, let us address the following fundamental question.

Who should buy and who should sell a callable put?

Assume that the option starts far out of the money, that is, $x \gg K$. If S_t hits K (goes in the money) quite late, that is, near to expiry, the risk for the writer is comparatively small. By contrast, when S_t hits K quite early in the option lifetime it becomes a long term at-the-money-option which has a large time value. Against this risky situation, the writer is insured by the right to recall the option. Put differently, by the callable feature the writer has an upper bound on the time value conceded to the put holder.

In the context of an illiquid market where the writer cannot compensate her/his short position by buying an American residual put option recalling is possibly the only way to close the position. Also, in view of model risk or violation of the hypothesis of market completeness, the cheapest superhedging strategy for the writer of an American put option can be the trivial one—consisting of an investment of K units in the riskless bank account. In this situation, strategic recalling can be an efficient instrument to limit risk—especially when the writer expects falling stock prices.

On the other hand for the buyer, the incentive is the lower price (as it is for callable bonds). This comes at the price that extreme gains become less likely.

A deposit insurance can be viewed as a callable perpetual put option on the market value of the assets issued by the insured bank. The put writer—usually a federal deposit insurer—agrees to purchase the bank's insured deposits for the market value of the bank's assets (if the bank closes itself). On the other hand, the deposit insurer can enfore premature exercise of the option by recalling the put option and closing the bank. For details, see Allen and Saunders (1993).

Returning now to the technical description of the callable put, let $\mathcal{T}_{t,T}$ be the class of $\mathbb{F}$-stopping times valued in $[t, T]$ and let $\mathbb{P}_x$ be the risk-neutral measure for S under the assumption that $S_0 = x$. (Note that standard Black–Scholes theory dictates that this measure exists and is uniquely defined via a Girsanov change of measure.) We shall denote $\mathbb{E}_x$ to be expectation under $\mathbb{P}_x$. From Kifer (2000) it follows that there is a unique no-arbitrage price process of the callable put under the Black–Scholes framework which can be represented by the right continuous process $V = \{V_t : t \in [0, T]\}$ where

$$\begin{aligned} V_t &= \operatorname*{ess\text{-}inf}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\text{-}sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}_x\big(\mathbf{1}_{(\sigma\le\tau)}e^{-r(\sigma-t)}(K-S_\sigma)^+ + \mathbf{1}_{(\tau<\sigma)}e^{-r(\tau-t)}\{(K-S_\tau)+\delta\}\,\big|\,\mathcal{F}_t\big) \\ &= \operatorname*{ess\text{-}sup}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\text{-}inf}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_x\big(\mathbf{1}_{(\sigma\le\tau)}e^{-r(\sigma-t)}(K-S_\sigma)^+ + \mathbf{1}_{(\tau<\sigma)}e^{-r(\tau-t)}\{(K-S_\tau)+\delta\}\,\big|\,\mathcal{F}_t\big). \end{aligned}$$

Further, for all $t \in [0, T]$ there exist stopping strategies

$$\begin{aligned} \sigma^*_{T-t} &= \inf\big\{s \in [t, T] : V_s = (K-S_s)^+\big\} \text{ and} \\ \tau^*_{T-t} &= \inf\big\{s \in [t, T] : V_s = (K-S_s)^+ + \mathbf{1}_{(s<T)}\delta\big\}. \end{aligned} \tag{1.1}$$

such that

(1.2)
$$V_t = \mathbb{E}_x\big(\mathbf{1}_{(\sigma^*_{T-t}\le\tau^*_{T-t})}e^{-r(\sigma^*_{T-t}-t)}\big(K-S_{\sigma^*_{T-t}}\big) + \mathbf{1}_{(\tau^*_{T-t}<\sigma^*_{T-t})}e^{-r(\tau^*_{T-t}-t)}\big\{\delta+\big(K-S_{\tau^*_{T-t}}\big)\big\}\,\big|\,\mathcal{F}_t\big).$$

By the Markov property we can write $V_t = v^{CP}(S_t, T-t)$ where

$$(1.3)\quad v^{CP}(x,u) = \inf_{\tau\in\mathcal{T}_{0,u}} \sup_{\sigma\in\mathcal{T}_{0,u}} \mathbb{E}_x\big(\mathbf{1}_{(\sigma\le\tau)}e^{-r\sigma}(K-S_\sigma) + \mathbf{1}_{(\tau<\sigma)}e^{-r\tau}\{(K-S_\tau)^+ + \delta\}\big)$$
$$= \sup_{\sigma\in\mathcal{T}_{0,u}} \inf_{\tau\in\mathcal{T}_{0,u}} \mathbb{E}_x\big(\mathbf{1}_{(\sigma\le\tau)}e^{-r\sigma}(K-S_\sigma)^+ + \mathbf{1}_{(\tau<\sigma)}e^{-r\tau}\{(K-S_\tau)^+ + \delta\}\big)$$

defined on $(x,u)\in(0,\infty)\times[0,T]$. Note that by considering strategies $\sigma=0$ and $\tau=0$ it can be seen that

$$(1.4)\qquad (K-x)^+ \le v^{CP}(x,u) \le (K-x)^+ + \delta.$$

Our interest is in showing how the value function $v^{CP}(x,u)$ can be characterized in terms of the value functions of other more familiar exotic options. However, it is first necessary to understand whether the writer's rights really makes a significant difference to the case of the American put.

In Kyprianou (2004) for $T=\infty$ explicit formulae expressions are achieved for V_t in terms of the process S. The calculations are greatly eased by the perpetual nature of the option. For the finite expiry version, no explicit formulae are possible for the same reason that there are no explicit formulae for the value function of an American put. In this paper we establish representations of finite expiry versions of the callable put via mixtures of other familiar exotic options. The method of proof relies on the classical technique of "guess and verify."

We close this section with an overview of the paper. Section 2 reviews the American put for later reflection. For a suitably large value of δ, that is, exceeding a specified threshold, it turns out that the value of the callable put is nothing more than the value of the American put. That is to say the writer will never exercise. This is dealt with in Section 3. In Section 4, the more interesting and complicated case of when δ is smaller than the aforementioned threshold is considered. The paper presents its conclusions in Section 5.

2. REVIEWING THE AMERICAN PUT

It will be of help to review some facts concerning the pricing of a regular American put option (cf. Karatzas and Shreve (1998), Lamberton (1998) and Myneni (1992)). That is to say, a contract with finite expiry date T which rewards the holder with $(K-S_t)^+$ at the moment they decide to exercise and forces a payment of $(K-S_t)^+$ if they have not exercised by the time the contract expires. Classical analysis of the American put tells us that

$$V_t = \operatorname*{ess\text{-}sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}_x\big(e^{-r(\sigma-t)}(K-S_\sigma)^+ \,\big|\, \mathcal{F}_t\big)$$
$$= v^A(S_t, T-t)$$

where

$$v^A(x,u) = \sup_{\sigma\in\mathcal{T}_{0,u}} \mathbb{E}_x\big(e^{-r\sigma}(K-S_\sigma)^+\big)$$

defined on $(x,u)\in(0,\infty)\times[0,T]$ is jointly continuous, convex, and non-increasing in x and non-decreasing in u. Further, the optimal stopping strategy is given by the stopping time

$$(2.1)\qquad \sigma_T^A := \inf\{t\ge 0 : V_t \le (K-S_t)^+\}$$

so that on the event $t < \sigma_T^A$

$$V_t = \mathbb{E}_x\big(e^{-r\sigma_T^A}\big(K - S_{\sigma_T^A}\big)^+ \,\big|\, \mathcal{F}_t\big) = \mathbb{E}_{x'}\big(e^{-r\sigma_{T-t}^A}\big(K - S_{\sigma_{T-t}^A}\big)^+\big),$$

where $x' = S_t$ and σ_{T-t}^A has the same definition as (2.1), but with T replaced by $T - t$. Note that we shall use here and throughout the standard definition $\inf\emptyset = \infty$. Based on the facts above, one can show that there exists a continuous monotone decreasing curve $\varphi^A : [0, T] \to (0, \mathrm{K}]$ with $\varphi^A(0) = K$ such that the optimal stopping strategy can otherwise to be defined as

$$\sigma_T^A = \inf\big\{t \geq 0 : S_t \leq \varphi^A(T - t)\big\} \wedge T.$$

Finally, from the theory of optimal stopping which drives the rational behind the pricing of American options, we have that

$$\big\{e^{-r(t\wedge\sigma_T^A)} v^A\big(S_{t\wedge\sigma_T^A}, T - \big(t \wedge \sigma_T^A\big) : t \in [0, T]\big\}$$

and

$$\big\{e^{-rt} v^A(S_t, T - t) : t \in [0, T]\big\}$$

are a $\mathbb{P}_x$-martingale and a $\mathbb{P}_x$-supermartingale, respectively, for each $x > 0$.

3. REPRESENTATION OF v^{CP} FOR LARGE δ

Suppose that δ is very large, for example when

$$\delta > \sup_{(x,u)\in(0,\infty)\times[0,T]} v^A(x, u).$$

With such a large value of δ it would not make sense for the writer to exercise at all. For then they would be left with the responsibility of a compensation which far exceeds any amount the holder themselves would ever have claimed. We should therefore expect that in this case, the saddle point in Kifer's theorem simply requires the writer to leave the decision making to the holder. That is to say, in this case, the callable put option becomes nothing more than the standard American put. For smaller values of δ however, as indicated in the introduction, one should expect that there can be rational in the writer exercising before the holder. Suppose that δ is very small. The writer can force the holder to exercise the option untimely by paying in addition to the intrinsic value $(K - S_t)^+$ a small penalty. The following lemma gives an upper bound for the smallest δ beyond which the callable δ-penalty put is nothing more than an American put. We shall later see that it is the smallest upper bound.

LEMMA 3.1. *If $\delta \geq v^A(K, T)$ then $v^{CP} = v^A$, $\sigma^* = \sigma_T^A$ and $\tau^* = T$.*

Proof. For two different game options the difference between their option values is bounded by the maximal deviation between the two exercising and between the two recall processes. Therefore, v^{CP} is varying with δ in a continuous way and it is sufficient to show the assertion for $\delta > v^A(K, T)$. Then, we have that for all $x \in (0, \infty)$, $u \in [0, T]$

$$v^{CP}(x, u) \leq v^A(x, u) \leq (K - x)^+ + v^A(K, u) < (K - x)^+ + \delta. \tag{3.1}$$

Note that, the first inequality is justified by considering $\tau = u$ in the definition of $v^{CP}(x, u)$. Since $V_t = v^{CP}(S_t, T - t)$, (3.1) implies that the optimal recall time for the seller, given by (1.1), is $\tau_T^* = T$. This implies $v^{CP}(x, u) = v^A(x, u)$ and $\sigma_T^* = \sigma_T^A$. □

4. REPRESENTATION OF v^{CP} FOR SMALL δ

Suppose now that $0 < \delta < v^A(K, T)$. That is to say at the beginning of the contract, for certain paths of S the American option is worth strictly more than the callable δ-penalty put; recall the bounds (1.4). Despite this fact, since the value function v^A is continuous and increasing in the time to expiry with $v^A(x, 0) = (K - x)^+$, for all times sufficiently close to expiry the value of the American option will become uniformly in x less than the writers obligation should they decide to exercise; $v^A(x, T - t) \leq (K - x)^+ + \delta$, uniformly in x, for all sufficiently large $t < T$. Suppose that a callable δ-penalty put has survived to almost the expiry date, say a time t'. By the Markov property, the option has the same value of a fresh callable δ-penalty put initiated at t' with the same strike but with duration $T - t'$. Since $\delta \geq v^A(K, T - t')$, Lemma 3.1 tells us that the writer has no interest in exercising and the option proceeds as the tail end of an American put with strike K and expiry $T - t'$.

In this light, we shall proceed by investigating the following heuristic for the exercising strategy of the option writer.

Writer's Perspective. As long as $S_t > K$ it is not rewarding to cancel the contract by paying the penalty δ. Namely, as the interest rate r is positive, it is better to wait and not to cancel the contract, if at all, until S hits K. On the other hand, if $S_t < K$ we have that $e^{-rt}\{(K - S_t)^+ + \mathbf{1}_{(t<T)}\delta\} = e^{-rt}\{K - S_t + \mathbf{1}_{(t<T)}\delta\}$. This payoff, considered as a process stopped when S hits K, is a strict $\mathbb{P}_x$-supermartingale (as the process $e^{-rt}S_t$ is a $\mathbb{P}_x$-martingale). Thus the writer is doing well to wait—independent of the stopping strategy of the holder. Summing up, acting optimally the writer *can* (if at all) only stop when $S_t = K$.

Let t^* be the time for which $v^A(K, T - t^*) = \delta$. Note that continuity and strict monotonicity of the function $v^A(K, \cdot)$ guarantees that this value is uniquely defined. Assume now that (s)he does not recall at $S_t = K$ for some remaining lifetime $u > T - t^*$. Then, (s)he will nor recall sometime in the future. This would imply that

$$v^{CP}(x, u) = v^A(x, u), \quad \forall x \in (0, \infty). \tag{4.1}$$

However, for $x = K$, (4.1) implies that

$$\delta = v^A(K, T - t^*) < v^A(K, u) = v^{CP}(K, u)$$

which is a contradiction to the fact that $v^{CP}(x, u)$ has to lie in the interval $[(K - x)^+, (K - x)^+ + \delta]$. Thus, we might guess that the writer should exercise according to the strategy

$$\hat{\tau} = \inf\{t \in [0, t^*] : S_t = K\} \wedge T. \tag{4.2}$$

The strategy in (4.2) has another interpretation. It is well known that the time value of an American put, defined as $v^A(x, u) - (K - x)^+$, is maximal at $x = K$ and increasing in u. (4.2) suggests that also the time value of the callable put, that is, $v^{CP}(x, u) - (K - x)^+$, is maximal at $x = K$ taking the value δ in case of $u \geq T - t^*$ (and a smaller value when $u < T - t^*$).

Holder's Perspective. The holder on the other hand will reason in the same way as they would for the associated American put. That is to make a compromise between S reaching a prescribed low value and not waiting too long. Following these strategies, if neither holder nor writer takes action by time t^* the option should go on, as we have already seen, as a regular American put.

In order to turn these heuristics into rigor, it will be helpful to consider the following related exotic option.

4.1. The American Knock-out Option

THEOREM 4.1. *Consider an American-type exotic option of duration t^* which offers the holder the right to exercise at any time claiming $(K - S_t)^+$, however, if the value of S hits K then the option is "knocked-out" with a rebate of δ and further, if at expiry the option is still active then the holder is rewarded with an American put option with strike K and duration $T - t^*$.*

(i) The holder of this option behaves rationally by exercising according to the stopping time

$$\hat{\sigma} = \inf\{t \geq 0 : \hat{v}(S_t, t^* - t) = (K - S_t)^+\} \wedge t^*, \tag{4.3}$$

where

$$\hat{v}(x, u) = \sup_{\sigma \in \mathcal{T}_{0,u}} \mathbb{E}_x\big(e^{-r\hat{\tau}}\delta \mathbf{1}_{(\hat{\tau} \leq \sigma)} + \mathbf{1}_{(\sigma < \hat{\tau} \wedge u)} e^{-r\sigma}(K - S_\sigma) + \mathbf{1}_{(\sigma = u < \hat{\tau})} e^{-ru} v^A(S_u, T - t^*)\big). \tag{4.4}$$

(ii) The discounted value of the option is given by

$$\{e^{-r(t\wedge\hat{\tau})}\hat{v}\,(S_{t\wedge\hat{\tau}}, t^* - (t \wedge \hat{\tau})) : t \in [0, t^*]\}.$$

(iii) The process

$$\{e^{-r(t\wedge\hat{\tau})}\hat{v}\,(S_{t\wedge\hat{\tau}}, t^* - (t \wedge \hat{\tau})) : t \in [0, t^*]\}$$

is a $\mathbb{P}_x$-supermartingale and the process

$$\{e^{-r(t\wedge\hat{\tau}\wedge\hat{\sigma})}\hat{v}\,(S_{t\wedge\hat{\tau}\wedge\hat{\sigma}}, t^* - (t \wedge \hat{\tau} \wedge \hat{\sigma})) : t \in [0, t^*]\}$$

is a $\mathbb{P}_x$-martingale.

Proof.

(*i*) First note that the discounted claim process $\{\pi_t : t \in [0, t^*]\}$ where

$$\pi_t = \mathbf{1}_{(t<\hat{\tau}\wedge t^*)} e^{-rt}(K - S_t)^+ + e^{-r\hat{\tau}}\delta \mathbf{1}_{(\hat{\tau}\leq t^* \text{ and } t \geq \hat{\tau})} + \mathbf{1}_{(t^* < \hat{\tau} \text{ and } t = t^*)} e^{-rt^*} v^A(S_{t^*}, T - t^*).$$

is an $\mathbb{F}$-adapted process with càdlàg paths that have no *negative* jumps and satisfies $\mathbb{E}_x(\sup_{t\in[0,t^*]} \pi_t) < \infty$. Now consider the optimal stopping problem

$$\sup_{\sigma \in \mathcal{T}_{0,t^*}} \mathbb{E}_x(\pi_\sigma). \tag{4.5}$$

Standard theory of American-type option pricing (cf. Shiryaev *et al.* (1995)) now tells us that this problem charaterizes the value of this option. In particular, optimal stopping strategy occurs at

$$\widetilde{\sigma} = \inf\{t \geq 0 : v_t^\pi = \pi_t\}$$

where $v^\pi = \{v_t^\pi : t \in [0, t^*]\}$ is the Snell envelope of $\{\pi_t : t \in [0, t^*]\}$. By the Strong Markov Property of S we have that on the set $\{t \leq \hat{\tau}\}$

$$\begin{aligned} v_t^\pi &= \operatorname*{ess-sup}_{\sigma \in \mathcal{T}_{t,t^*}} \mathbb{E}_x(\pi_\sigma \mid \mathcal{F}_t) \\ &= e^{-rt} \sup_{\sigma \in \mathcal{T}_{0,t^*-t}} \mathbb{E}_{x'}(\pi_\sigma) \text{ where } x' = S_t \\ &= e^{-rt} \hat{v}\big(S_t, t^* - t\big). \end{aligned}$$

Therefore, on the set $\{\hat{\sigma} < \hat{\tau}\}$ we have that $\tilde{\sigma} = \hat{\sigma}$ and on the set $\{\hat{\sigma} \geq \hat{\tau}\}$ we have that $\tilde{\sigma} = \hat{\tau}$. Thus, $\tilde{\sigma} = \hat{\sigma} \wedge \hat{\tau}$. As $v_t^\pi = \delta$, for $\hat{\tau} \leq t^*$ and $t \geq \hat{\tau}$, it follows that $\mathbb{E}_x(\pi_{\hat{\sigma}}) = \mathbb{E}_x(\pi_{\tilde{\sigma}})$. Thus, also $\hat{\sigma}$ is optimal for the stopping problem (4.5).

(*ii–iii*) From standard theory of optimal stopping, the Snell envelope v^π is a supermartinagle and further when stopped at an opimal stopping time it forms a martingale. $e^{-r(t\wedge\hat{\tau})}\hat{v}(S_{t\wedge\hat{\tau}}, t^* - t \wedge \hat{\tau}) = v^\pi_{t\wedge\hat{\tau}}$ implies the assertion. □

REMARK 4.1. The option described in the previous theorem has two different interpretations depending on the initial stock price x.

If $x < K$ then we can understand this option to be an American "up-and-out" put option with reimbursement δ at the point of "knock-out." Further the holder is rewarded with an American put option of further duration $T - t^*$ and strike K if the option reaches its natural maturity.

If on the other hand, $x > K$ the option cannot come into the money before t^* without knocking out. This gives the interpretation of our option being a European "down-and-out" option with contingent claim $v^A(S_{t^*}, T - t^*)$ and rebate δ when the option "knocks out."

Without specifying on which side of K the initial value of the risky asset lies, we can say that the option in the previous theorem is the sum of the above compound American up-and-out with rebate and European down-and-out with rebate. For future reference, we shall refer to this combined derivative as "the American knock-out."

4.2. Analytical Properties of the American Knock-out Option

Let us progress to look at some of the analytical properties of the American knock-out option, presented as a series of lemmas, which will be of later use.

The lemmas are partial steps to establishing convexity of $\hat{v}$ which in turn is crucial in establishing a submartingale associated with $\hat{v}$. This submartingale serves to justify the heuristic at the beginning of this section.

LEMMA 4.1. *There exists a function $f : (0, \infty) \times [0, t^*] \to \mathbb{R}$ which is convex in its first variable such that*

$$f(x, u) \leq \hat{v}(x, u) \text{ and } f(K, u) = \hat{v}(K, u) = \delta. \tag{4.6}$$

Proof. Recall that for $u \in [0, t^*]$, $v^A(K, T - t^* + u) \geq \delta$, where v^A is the value function of the corresponding standard American put. Define

$$f(x, u) = v^A(x, T - t^* + u) + \delta - v^A(K, T - t^* + u).$$

We have of course that $f(K,u)=\hat{v}(K,u)$ and convexity follows from the fact that v^A is known to be convex. It remains to show that

$$f(x,u)\le\hat{v}(x,u). \tag{4.7}$$

As the rebate δ is always smaller than the value of the American put at the strike, we have that $\hat{v}\le v^A(\cdot,T-t^*+\cdot)$. Thus (4.7) states that the maximal distance between $\hat{v}(x,u)$ and $v^A(x,u)$ is attained at $x=K$. This seems to be plausible as the payoffs of the underlying options only differ when S_t hits K. But, if the process S_t starts away from K it hits K only with some probability and after some time. By this, the strong Markov property and the fact that $v^A(K,\cdot)$ is increasing, one can verify (4.7). For a formal proof write σ^A as short hand for $\sigma^A_{T-t^*+u}$, the optimal exercising time for an American put with maturity $T-t^*+u$, and recall that $\hat{\tau}$ is the first hitting time of K. We have

$$\begin{aligned}
\hat{v}(x,u) &\ge \mathbb{E}_x\big(e^{-r(\sigma^A\wedge\hat{\tau}\wedge u)}\hat{v}\big(S_{\sigma^A\wedge\hat{\tau}\wedge u},u-\sigma^A\wedge\hat{\tau}\wedge u\big)\big)\\
&\ge \mathbb{E}_x e^{-r(\sigma^A\wedge\hat{\tau}\wedge u)}\big(\mathbf{1}_{(\hat{\tau}<\sigma^A\wedge u)}\delta+\mathbf{1}_{(\hat{\tau}\ge\sigma^A\wedge u)}v^A\big(S_{\sigma^A\wedge u},T-t^*+u-\sigma^A\wedge u\big)\big)\\
&= \mathbb{E}_x\big(e^{-r(\sigma^A\wedge\hat{\tau}\wedge u)}v^A\big(S_{\sigma^A\wedge\hat{\tau}\wedge u},T-t^*+u-\sigma^A\wedge\hat{\tau}\wedge u\big)\big)\\
&\quad-\mathbb{E}_x\big(e^{-r\hat{\tau}}\mathbf{1}_{(\hat{\tau}<\sigma^A\wedge u)}(v^A(K,T-t^*+u-\hat{\tau})-\delta)\big)\\
&\ge \mathbb{E}_x\big(e^{-r(\sigma^A\wedge\hat{\tau}\wedge u)}v^A\big(S_{\sigma^A\wedge\hat{\tau}\wedge u},T-t^*+u-\sigma^A\wedge\hat{\tau}\wedge u\big)\big)\\
&\quad-\mathbb{E}_x\big(\mathbf{1}_{(\hat{\tau}<\sigma^A\wedge u)}(v^A(K,T-t^*+u)-\delta)\big)\\
&\ge \mathbb{E}_x\big(e^{-r(\sigma^A\wedge\hat{\tau}\wedge u)}v^A\big(S_{\sigma^A\wedge\hat{\tau}\wedge u},T-t^*+u-\sigma^A\wedge\hat{\tau}\wedge u\big)\big)\\
&\quad-(v^A(K,T-t^*+u)-\delta)\\
&= v^A(x,T-t^*+u)+\delta-v^A(K,T-t^*+u)\\
&= f(x,u).
\end{aligned}$$

The first inequality is due to the supermartingale property stated in Theorem 4.1. The second inequality can be seen by a case differentiation. In case of $\hat{\tau}<\sigma^A\wedge u$ it is obvious thereas for $\sigma^A\wedge u\le\hat{\tau}$ we use that $\hat{v}(S_{\sigma^A},u-\sigma^A)\ge(K-S_{\sigma^A})^+=v^A(S_{\sigma^A},T-t^*+u-\sigma^A)$ and $\hat{v}(S_u,0)=v^A(S_u,T-t^*)$, resp. Then, the first equality is just rewriting and the third and fourth inequality use that the value of the American put is increasing in the time to maturity and the difference $v^A(K,T-t^*+u)-\delta$ is non-negative. Finally, the second equality comes from the martingale property of the American put. □

LEMMA 4.2. *We have that for all $u\ge 0$ and $x>0$, $\hat{v}(x,u)>0$ and*

$$(K-x)^+\le\hat{v}(x,u)\le(K-x)^++\delta. \tag{4.8}$$

Further, for each $x>0$ the function $\hat{v}(x,\cdot)$ is monotone increasing and continuous and for each $u\in[0,t^*]$ the function $\hat{v}(\cdot,u)$ is monotone decreasing and continuous; and hence $\hat{v}$ is jointly continuous.

Proof. The lower bounds follow by considering the stopping times $\sigma=u$ and $\sigma=0$ in the expression given for $\hat{v}$ in (4.4).

For the upper bound we make a case differentiation. For initial price $x\ge K$ the assertion is trivial as the (discounted) payoff by the American knock-out cannot exceed δ. For $x<K$ we bring to mind that the stopped process $(e^{-rt}(K-S_t)^+)^{\hat{\tau}}_{t\in[0,u]}=(e^{-rt}(K-S_t))^{\hat{\tau}}_{t\in[0,u]}$, where $\hat{\tau}$ was defined as the first time S_t hits K, is a supermartingale. Therefore from the definition of $\hat{v}(x,u)$ we have

$$\hat{v}(x,u) \le \sup_{\sigma\in\mathcal{T}_{0,u}} \mathbb{E}_x\big(e^{-r(\sigma\wedge\hat{\tau})}(K-S_{\sigma\wedge\hat{\tau}})^+ + \delta\big)$$
$$= (K-x)^+ + \delta.$$

For the inequality above, recall that at $\sigma = u$ the option is switched to an American put with duration $T-t^*$. Its value becomes $v^A(S_u, T-t^*) \le (K-S_u)^+ + v^A(K, T-t^*) = (K-S_u)^+ + \delta$.

Let us now show the monotonicity of $\hat{v}(x,\cdot)$. As a partial step we shall show that

$$v^A(x, T-t^*) \le \hat{v}(x,u) \quad \text{for all } x>0 \text{ and } u\in[0,t^*]. \tag{4.9}$$

By the dynamic programming principle, the value of the standard American put option coincides with the value of an American put which is knocked out when S_t hits K, paying then the amout $v^A(K, T-t^*-\hat{\tau})$. We thus obtain for $u\in[0,t^*]$ that

$$\begin{aligned} (4.10)\quad & v^A(x, T-t^*) \\ &= \sup_{\sigma\in\mathcal{T}_{0,T-t^*}} \mathbb{E}_x\big(e^{-r\sigma}(K-S_\sigma)^+\big) \\ &= \sup_{\sigma\in\mathcal{T}_{0,T-t^*}} \mathbb{E}_x\big(\mathbf{1}_{(\hat{\tau}\le\sigma\wedge u)}e^{-r\hat{\tau}}v^A(K, T-t^*-\hat{\tau}) + \mathbf{1}_{(\hat{\tau}>\sigma\wedge u)}e^{-r\sigma}(K-S_\sigma)^+\big). \end{aligned}$$

Further, since by the definition of t^*, an American option with remaining term less than $T-t^*$ is less than δ, it follows from the right-hand side above that

$$\begin{aligned} (4.11)\quad & v^A(x, T-t^*) \\ &\le \sup_{\sigma\in\mathcal{T}_{0,T-t^*}} \mathbb{E}_x\big(\mathbf{1}_{(\hat{\tau}\le\sigma\wedge u)}e^{-r\hat{\tau}}\delta + \mathbf{1}_{(\hat{\tau}>\sigma\wedge u)}e^{-r\sigma}(K-S_\sigma)^+\big) \\ &\le \sup_{\sigma\in\mathcal{T}_{0,T-t^*+u}} \mathbb{E}_x\big(\mathbf{1}_{(\hat{\tau}\le\sigma\wedge u)}e^{-r\hat{\tau}}\delta + \mathbf{1}_{(\hat{\tau}>\sigma\wedge u)}e^{-r\sigma}(K-S_\sigma)^+\big). \end{aligned}$$

Finally, we note that by considering the American knock-out option with expiry u as having ultimate expiry time $T-t^*+u$ by taking into account the rebated American option of length $T-t^*$ issued at time t^*, we may simply identify the right-hand side above as $\hat{v}(x,u)$.

Now, suppose that $0\le u_1\le u_2\le t^*$. We compare American knock-out options with remaining times u_1 and u_2, respectively. Until u_1, the payoffs coincide. At time u_1 the first option is switched to an American put and the second is still a knock-out option with remaining time u_2-u_1. We use (4.9) with $u = u_2-u_1$ and obtain

$$\begin{aligned} (4.12)\quad \hat{v}(x,u_1) &= \sup_{\sigma\in\mathcal{T}_{0,u_1}} \mathbb{E}_x\big(\mathbf{1}_{(\hat{\tau}\le\sigma)}e^{-r\hat{\tau}}\delta + \mathbf{1}_{(\sigma<\hat{\tau}\wedge u_1)}e^{-r\sigma}(K-S_\sigma) \\ &\qquad + \mathbf{1}_{(\sigma=u_1<\hat{\tau})}e^{-ru_1}v^A\big(S_{u_1}, T-t^*\big)\big) \\ &\le \sup_{\sigma\in\mathcal{T}_{0,u_1}} \mathbb{E}_x\big(\mathbf{1}_{(\hat{\tau}\le\sigma)}e^{-r\hat{\tau}}\delta + \mathbf{1}_{(\sigma<\hat{\tau}\wedge u_1)}e^{-r\sigma}(K-S_\sigma)^+ \\ &\qquad + \mathbf{1}_{(\sigma=u_1<\hat{\tau})}e^{-ru_1}\hat{v}(S_{u_1}, u_2-u_1)\big) \\ &= \hat{v}(x,u_2). \end{aligned}$$

This is the required monotonicity in u.

For continuity in u, have again a look at (4.11). As we have $v^A(K, T-t^*-u)\to\delta$ for $u\to 0$, the second line in (4.11) becomes an approximation for the last line of (4.10) as $u\to 0$. For $u<T-t^*$ the last line in (4.11) coincides with

$$\sup_{\sigma\in\mathcal{T}_{0,T-t^*}} \mathbb{E}_x\big(\mathbf{1}_{(\hat\tau\le\sigma\wedge u)}e^{-r\hat\tau}\delta+\mathbf{1}_{(\hat\tau\wedge T-t^*>\sigma\wedge u)}e^{-r\sigma}(K-S_\sigma)^+$$
$$+\mathbf{1}_{(\sigma=T-t^*>\hat\tau)}e^{-r(T-t^*)}v^A(S_{T-t^*},u)\big).$$

As $v^A(x,u)\to(K-x)^+$ for $u\to0$, uniformly in $x\in(0,\infty)$, also the third line in (4.11) becomes an approximation for the line before as $u\to0$. We obtain

$$\hat v(x,u)\to v^A(x,T-t^*),\quad u\to0,\ \text{uniformly in } x\in\mathbb{R}_+. \tag{4.13}$$

The asymptotic (4.13) together with an inspection of (4.12) reveals that $\hat v(x,u_1)-\hat v(x,u_2)\to0$ for $u_2-u_1\to0$.

For monotonicity in x, let $0\le x_1<x_2\le K$ and write

$$\hat\tau_{x_2}=\inf\{t\ge0: x_2S_t=K\}.$$

By (4.6) and the monotonicity of v^A we have that for every $u\in[0,t^*]$

$$\hat v\left(\frac{x_1}{x_2}K,u\right)\ge v^A\left(\frac{x_1}{x_2}K,u\right)+\delta-v^A(K,u)\ge\delta. \tag{4.14}$$

It now follows that (4.14) and the monotonicity of $x\mapsto K-x$ and $x\mapsto v^A(x,u)$ imply that

$$\begin{aligned}\hat v(x_2,u)&=\sup_{\sigma\in\mathcal{T}_{0,u}}\mathbb{E}_1\big(\mathbf{1}_{(\hat\tau_{x_2}\le\sigma)}e^{-r\hat\tau_{x_2}}\delta+\mathbf{1}_{(\sigma<\hat\tau_{x_2}\wedge u)}e^{-r\sigma}(K-x_2S_\sigma)\\&\qquad+\mathbf{1}_{(\sigma=u<\hat\tau_{x_2})}e^{-ru}v^A\big(x_2S_u,T-t^*\big)\big)\\&\le\sup_{\sigma\in\mathcal{T}_{0,u}}\mathbb{E}_1\left(\mathbf{1}_{(\hat\tau_{x_2}\le\sigma)}e^{-r\hat\tau_{x_2}}\hat v\left(\frac{x_1}{x_2}K,u-\hat\tau_{x_2}\right)+\mathbf{1}_{(\sigma<\hat\tau_{x_2}\wedge u)}e^{-r\sigma}(K-x_1S_\sigma)\right.\\&\qquad\left.+\mathbf{1}_{(\sigma=u<\hat\tau_{x_2})}e^{-ru}v^A\big(x_1S_u,T-t^*\big)\right)\\&=\hat v(x_1,u).\end{aligned} \tag{4.15}$$

The last equality follows from the dynamic programming principle (Note that $\hat\tau_{x_2}\le\inf\{t\ge0:x_1S_t=K\}$). By (4.8) we have $\hat v(Kx_1/x_2,u)\le(K-Kx_1/x_2)^++\delta$ and therefore

$$\hat v\left(\frac{x_1}{x_2}K,u\right)\to\delta,\quad x_1\to x_2>0. \tag{4.16}$$

The limiting relation (4.16) and an inspection of (4.15) reveals continuity in x.

On $[K,\infty)$ the proof of monotonicity and continuity is similar, but easier. It makes again use of the fact that $\hat v(x,u)\ge v^A(x,u)+\delta-v^A(K,u)\to\delta$, $x\to K$, uniformly in $u\in[0,t^*]$. The complete proof is left to the reader. □

The monotonicity properties of $\hat v$ and its lower bounds together with the fact that $\hat v(K,u)=\delta$, this implies that there exists an open set $\mathcal{C}$ taking the form

$$\mathcal{C}=(K,\infty)\times(0,t^*)\cup\{(x,u)\in(0,K)\times(0,t^*):x>b(u)\}$$

where $b:(0,t^*]\to[0,\mathrm{K})$, given by

$$b(u)=\sup\{x\ge0:\hat v(x,u)=K-x\}$$

(with the convention that $\sup\emptyset=0$) is monotone decreasing, satisfying $\lim_{u\downarrow0}b(u)\le\varphi^A(T-t^*)$ such that the optimal stopping time $\hat\tau\wedge\hat\sigma$ corresponds to

$$\tau^{\mathcal{C}}:=\inf\{t>0:(S_t,t^*-t)\notin\mathcal{C}\}.$$

LEMMA 4.3. *The value function $\hat{v}(x, u)$ is twice continuously differentiable in x and once continuously differentiable in u on the continuation region $\mathcal{C}$ with*

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \hat{v}}{\partial x^2} + rx\frac{\partial \hat{v}}{\partial x} - r\hat{v} - \frac{\partial \hat{v}}{\partial u} = 0 \text{ in } \mathcal{C}.$$

Proof. We recall a technique used in Karatzas and Shreve (1991), p. 243. That is to say, construct the parabolic Dirichlet problem

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} - rV - \frac{\partial V}{\partial u} = 0 \text{ in } \mathcal{R}$$
$$V = \hat{v} \text{ on } \partial^0\mathcal{R},$$

where $\mathcal{R}$ is the open rectangle $(x_1, x_2) \times (u_1, u_2) \subset \mathcal{C}$ with parabolic boundary

$$\partial^0\mathcal{R} = \partial\mathcal{R} - ((x_1, x_2) \times \{u_2\}).$$

On account of the fact that $\hat{v}$ is joinly continuous in u and x, classical theory of boundary value problems dictates that the above Dirichlet problem has a unique solution which is $C^{2,1}$ in $\mathcal{R}$ (cf. Friedman 1976). By part (iii) of Theorem 4.1 we have that

$$\left\{e^{-rt}\hat{v}\left(S_t, t^* - t\right) : t \in \left[t^* - u_2, \tau^{\mathcal{R}}\right]\right\}$$

is a uniformly integrable martingale where $\tau^{\mathcal{R}} = \inf\{t \geq t^* - u_2 : (S_t, t^* - t) \notin \mathcal{R}\}$. On the other hand, stochastic representation tells us also that

$$\left\{e^{-rt}V\left(S_t, t^* - t\right) : t \in \left[t^* - u_2, \tau^{\mathcal{R}}\right]\right\}$$

is also a uniformly integrable martingale. Since both have the same terminal value, we are forced to conclude they are the same martingale and hence $V = \hat{v}$ in $\mathcal{R}$. Since $\mathcal{R}$ may be placed anywhere in $\mathcal{C}$ the theorem is proved. □

LEMMA 4.4. *For each $u \in [0, t^*]$ the function $\hat{v}(\cdot, u)$ is convex on $(0, \infty)$*

Proof. Let

$$L = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} - r - \frac{\partial}{\partial u}$$

and recall that $L\hat{v} = 0$ on $\mathcal{C}$ (in particular $\hat{v}$ is smooth on $\mathcal{C}$). From Lemma 4.2, we have that $\hat{v}$ is decreasing in its first variable and increasing in its second variable. Hence, it follows that $\partial\hat{v}/\partial x \leq 0$ and $\partial\hat{v}/\partial u \geq 0$ on $\mathcal{C}$. These latter two observations together with the fact that $\hat{v} \geq 0$ and $L\hat{v} = 0$ leads to the conclusion that $\partial^2\hat{v}/\partial x^2 \geq 0$ on $\mathcal{C}$.

Since $\hat{v}$ is jointly continuous and bounded below by a convex function f (cf. Lemma 4.1) having the property that $f(K, u) = \hat{v}(K, u)$ it follows from the conclusion of the previous paragraph that $\hat{v}(\cdot, u)$ is convex on $(b(u), \infty)$. As $\hat{v}(x, u) \geq (K - x)^+$, when $\hat{v}$ joins the function $(K - x)^+$ it does so with an increasing gradient in x. It now follows that $\hat{v}(\cdot, u)$ is convex on $(0, \infty)$. □

LEMMA 4.5. *Let $t' \in [0, t^*]$ and*

$$\hat{\sigma}_{t'} := \inf\left\{t \geq t' : \hat{v}\left(S_t, t^* - t\right) = (K - S_t)^+\right\} \wedge t^*.$$

We have that

$$\left\{e^{-r(t\wedge\hat{\sigma}_{t'})}\hat{v}\left(S_{t\wedge\hat{\sigma}_{t'}}, t^* - (t \wedge \hat{\sigma}_{t'})\right) : t \in [t', t^*]\right\}$$

is a $\mathbb{P}_x$-submartingale.

Proof. Recall again that $L\hat{v} = 0$ on $\mathcal{C}$. Using a modern version of Itô's formula as given in Theorem 3.1 of Peskir (2005) we may deduce that on $t \in [t', \hat{\sigma}_{t'}]$

$$e^{rt} d\big[e^{-rt}\hat{v}\big(S_t, t^* - t\big)\big] = L\hat{v}\big(S_t, t^* - t\big)\, dt + dM_t + \left\{\frac{\partial \hat{v}}{\partial x}(K^+, t^* - t) - \frac{\partial \hat{v}}{\partial x}(K^-, t^* - t)\right\} dL_t^K \tag{4.17}$$

where L^K is local time of S at level K and M is a martingale. Note that Theorem 3.1 of Peskir (2005) has three conditions which need checking. It can easily be confirmed that these conditions are automatically satisfied here.

Since $\hat{v}$ is convex in x, we know that the local time term in (4.17) is monotone increasing and hence the result follows. □

4.3. The Callable δ-Penalty Put Is a Composite Exotic Option

Now we are ready to show what we have already alluded to. Namely that the callable δ-penalty put option of length T is nothing more than the American knock-out option with expiry t^* followed through to the expiration of the rebated American put of length $T - t^*$ if appropriate.

THEOREM 4.2. *Suppose that $\delta < v^A(K, T)$ and define*

$$t^* = \sup\{t \geq 0 : v^A(K, T - t) = \delta\}.$$

The δ − penalty Israeli put value function v^{CP} is given by

$$v^{CP}(x, u) = \begin{cases} v^A(x, u) & \text{for } (x, u) \in (0, \infty) \times [0, T - t^*] \\ \hat{v}(x, u - T + t^*) & \text{for } (x, u) \in (0, \infty) \times [T - t^*, T]. \end{cases} \tag{4.18}$$

Further, with

$$\mathcal{S}^{CP} = \{(x, u) : x \geq \varphi^A(u), \quad u \in [0, T - t^*]\}$$
$$\cup\{(x, u) : x \geq b(u + t^* - T), \quad u \in (T - t^*, T]\}$$

the optimal stopping strategy of the holder is given by

$$\sigma^{CP} = \inf\big\{t \geq 0 : (S_t, T - t) \in \mathcal{S}^{CP}\big\} \wedge T$$

and the optimal stopping strategy of the writer is

$$\tau^{CP} = \inf\big\{t \in [0, t^*] : S_t = K\big\} \wedge T.$$

Proof. Let us define a new function $v(x, u)$ which will be equal to the right-hand side of (4.18). Already from the definitions of v^{CP} and $\hat{v}$ in (1.3) and (4.4), respectively, it becomes evident that $v \geq v^{CP}$ as v corresponds to the value in case of a certain recall strategy of the seller, namely τ^{CP}, whereas for v^{CP} we take the infimum over all stopping times τ. All that we need is to prove that $v \leq v^{CP}$; then v is the solution to the saddle point problem (1.3).

It turns out that the submartingale properties associated with $\hat{v}$ will be crucial for the proof. As the value of an American put is a martingale up to the optimal exercise time, it follows from Lemma 4.5 that

$$\big\{e^{-r(t\wedge\sigma^{CP})} v\big(S_{t\wedge\sigma^{CP}}, T - (t \wedge \sigma^{CP})\big) : t \geq 0\big\} \text{ is a } \mathbb{P}_x\text{-submartingale.}$$

We can perform a calculation similar in nature to the calculations in Kyprianou (2004). To this end, define

$$\sigma_t^{CP} = \inf\{q \geq 0 : (S_q, T - t - q) \in \mathcal{S}^{CP}\} \wedge (T - t)$$

and

$$\tau_t^{CP} = \begin{cases} \inf\{q \in [0, t^* - t] : S_q = K\} \wedge (T - t) & \text{if } t \leq t^* \\ T - t & \text{if } t > t^*. \end{cases}$$

That is when $x' = S_t$

$$\begin{aligned} v(x', T - t) &= \inf_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}_{x'}\left(e^{-r(\tau \wedge \sigma_t^{CP})} v\left(S_{\tau \wedge \sigma_t^{CP}}, T - t - \left(\tau \wedge \sigma_t^{CP}\right)\right)\right) \\ &\leq \inf_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}_{x'}\left(e^{-r(\tau \wedge \sigma_t^{CP})}\left\{\mathbf{1}_{(\sigma_t^{CP} \leq \tau)}\left(K - S_{\sigma_t^{CP}}\right)^+ + \mathbf{1}_{(\sigma_t^{CP} > \tau)}\left[(K - S_\tau)^+ + \delta\right]\right\}\right) \\ &\leq \sup_{\sigma \in \mathcal{T}_{0,T-t}} \inf_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}_{x'}\left(e^{-r(\tau \wedge \sigma)}\left\{\mathbf{1}_{(\sigma \leq \tau)}(K - S_\sigma)^+ + \mathbf{1}_{(\sigma > \tau)}\left[(K - S_\tau)^+ + \delta\right]\right\}\right) \\ &= v^{CP}(x', T - t), \end{aligned}$$

where the first equality holds by Lemma 4.5 and the corresponding martingale property for the American put. In the first inequality we have used that $v(S_{\sigma_t^{CP}}, T - t - \sigma^{CP}) = (K - S_{\sigma_t^{CP}})^+$ and the fact that $(K - x)^+ + \delta$ is an upper bound for v. The latter follows from (4.8) and the estimation $v^A(x, u) \leq (K - x)^+ + v^A(K, u) \leq (K - x)^+ + \delta$ for American puts with duration u less than $T - t^*$. □

REMARK 4.2. From this proof and Theorem 4.1 (iii), we saw that

$$\left\{e^{-r(t \wedge \sigma^{CP} \wedge \tau^{CP})} v^{CP}\left(S_{t \wedge \sigma^{CP} \wedge \tau^{CP}}, T - t \wedge \sigma^{CP} \wedge \tau^{CP}\right) : t \in [0, T]\right\}$$

is a martingale. This is the martingale which the writer should hedge in order to replicate the option.

REMARK 4.3. In the proof of Lemma 4.5, and hence Theorem 4.2, it is not clear that the discounted value process is a genuine submartingale (as opposed to just a martingale) as there may be smooth pasting of $\hat{v}$ at $x = K$ which would knock out the integral with respect to local time. The following proposition excludes this possibility and thus τ^{CP} is the unique optimal strategy for the option writer.

PROPOSITION 4.1. *For $u > T - t^*$ we have*

$$\frac{\partial v^{CP}}{\partial x}(K^+, u) > \frac{\partial v^{CP}}{\partial x}(K^-, u).$$

Proof. As, by Lemma 4.2, $v^{CP}(x, \cdot)$ is monotone increasing and $v^{CP}(K, u) = \delta$ for all $u \in [T - t^*, T]$, we have that on $[T - t^*, T]$ the difference

$$\frac{v^{CP}(K + \Delta x, \cdot) - v^{CP}(K, \cdot)}{\Delta x} - \frac{v^{CP}(K, \cdot) - v^{CP}(K - \Delta x, \cdot)}{\Delta x}$$

is monotone increasing for all $\Delta x > 0$ and therewith also its limit

$$\frac{\partial v^{CP}}{\partial x}(K^+, \cdot) - \frac{\partial v^{CP}}{\partial x}(K^-, \cdot) \tag{4.19}$$

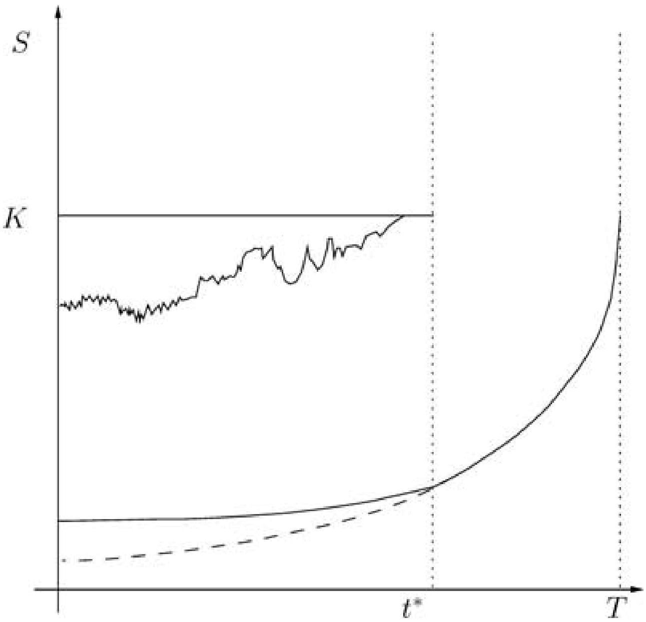

FIGURE 4.1. A sketch of the boundaries which characterize the optimal strategies of the writer and holder. The dotted line is where one would expect to see the optimal stopping barrier of a regular American put with the same parameters.

(letting Δx tend to zero). By convexity, the difference in (4.19) is non-negative. Assume that it vanishes for a fixed $u \in (T - t^*, T]$. Then, it has to vanish for all $u' \in (T - t^*, u]$ and the local time part in (4.17) disappears after $T - u$. Consequently, the process $t \mapsto e^{-rt} v^{CP}(S_t, T - t)$ started at $t = T - u$ is a supermartingale which impies that $v^{CP}(\cdot, u) = v^A(\cdot, u)$ (as the price process of the American put is the smallest supermartingale dominating the intrinsic option value). This contradicts to $v^{CP}(K, u) = \delta = v^A(K, T - t^*) < v^A(K, u)$ for $u \in (T - t^*, T]$. □

Let us conclude this section with some sketches of aspects of the function v^{CP}. Figure 4.1 gives an impression of the two barriers which form the saddle point strategy of the stochastic game behind the callable put. The upper barrier at $S = K$ represents the stopping region of the writer and the domain below the lower curved line represents the stopping domain of the writer. The dotted line represents the continuation of the barrier in the case of a regular American put with the same parameters.

In Figures 4.2 and 4.3 depict time slices of the function $v^{CP}(x, u)$. Figure 4.2 is a time slice from the region where $T \geq u > T - t^*$. The profile of $v^{CP}(\cdot, u)$ is constrained by the upper and lower gain functions $(K - x)^+ + \delta$ and $(K - x)^+$ respectively. Further, the value function pastes smoothly onto the lower gain function and fits under the corner of the upper gain function with a discontinuity in its first derivative as indicated in the previous proposition.

In Figure 4.3, we see $v^{CP}(\cdot, u)$ closer to the expiry of the option when $0 < u < T - t^*$. In this case, the callable put has the same value as an American put with the same parameters close to expiry. One sees in the figure that the upper gain function is everywhere strictly greater than the value curve which is consistent with the logic that the writer of the callable put prefers never to exercise close to expiry.

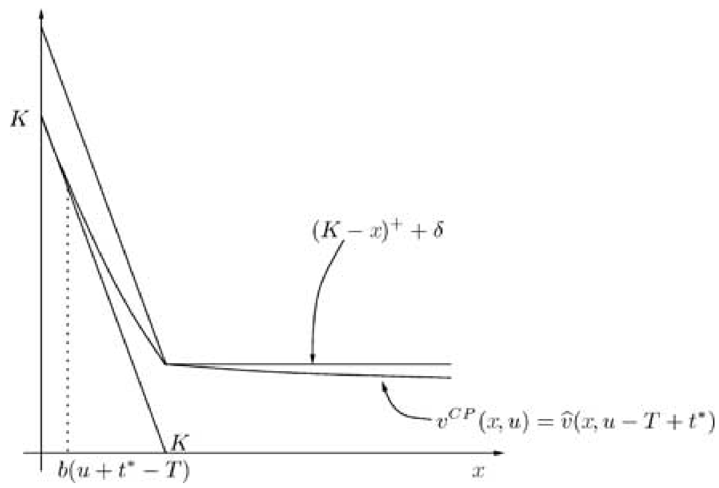

FIGURE 4.2. A profile of the function $v^{CP}(\cdot, u)$ for $u \in (T - t^*, T]$.

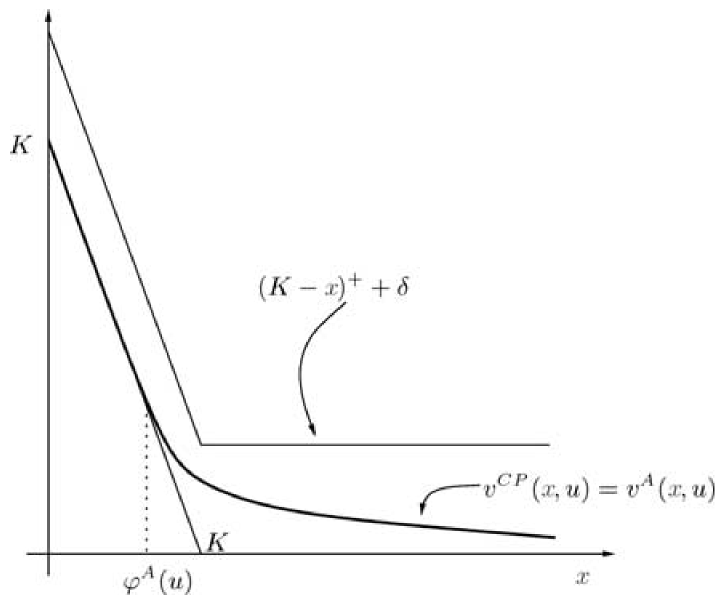

FIGURE 4.3. A profile of the function $v^{CP}(\cdot, u)$ for $u \in (0, T - t^*)$.

5. CONCLUSION

We have shown that the callable put is equivalent to the composition of other known exotic options. That is to say the stochastic saddle point in Kifer's pricing formula of game contingent claims is semi-explicitly identifiable thus giving a basis for further research of these options. Indeed with further work, one should be able to show that the given composite exotic options characterize uniquely the solution to a free boundary problem as one sees for American put and Russian options. See the preprint preceding this paper, Kühn and Kyprianou (2003a).

In related work, the reader is also referred to Kühn, Kyprianou, and van Schaik (2007) where the value of a more general class of finite expiry game contingent claims are characterized via a pathwise pricing formula.

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