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Martingale convergence and the stopped branching random walk

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Abstract. We discuss the construction of stopping lines in the branching random walk and thus the existence of a class of supermartingales indexed by sequences of stopping lines. Applying a method of Lyons (1997) and Lyons, Pemantle and Peres (1995) concerning size biased branching trees, we establish a relationship between stopping lines and certain stopping times. Consequently we develop conditions under which these supermartingales are also martingales. Further we prove a generalization of Biggins' Martingale Convergence Theorem, Biggins (1977a) within this context.

1. Introduction

The branching random walk begins with an initial ancestor that we shall label $\mho$. After one unit of time this individual gives birth to a random number of offspring, the first generation, scattered at random points in $\mathbb{R}^d$ according to the point process Z. Attached to each child is an independent copy of Z and so on. We assume that the average number of children born to each individual is greater than one ($\mathbb{E}Z\left(\mathbb{R}^d\right) > 1$) thus guaranteeing the survival of the process with positive probability. We shall also assume throughout that each individual has an almost surely finite family size. Any individual that appears in the process may be identified through its ancestry using the Ulam-Harris notation. So for example an individual $u = (i_1, \ldots, i_{n-1}, i_n)$ is the i_nth child of the i_{n-1}th child of....of the i_1th child of $\mho$. In this way, we understand $|u|$ to mean the generation in which u resides, $u < v$ to indicate that u is a strict ancestor of v and uv to refer to the individual who, from u's perspective, has line of descent expressed as v.

We understand ζ_u to be the position in $\mathbb{R}^d$ of each realized individual u in the branching random walk. For $\theta \in \mathbb{R}^d$, define $W_n(\theta) = \sum_{|u|=n} \exp\{-\theta \cdot \zeta_u\}/m^n(\theta)$ whenever $m(\theta) := \mathbb{E}\left(\sum_{|u|=1} \exp\{-\theta \cdot \zeta_u\}\right)$ is finite. (We have written $\theta \cdot \zeta_u$ meaning the inner product in the usual sense). For the case that $d = 1$ and reproduction

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is restricted to $\mathbb{R}^+$, Kingman (1975) observed that $W_n(\theta)$ is a positive martingale with mean 1, almost surely convergent to some limit $W(\theta)$. Biggins (1977a) later demonstrated that the spatial restriction is not necessary and further provided conditions equivalent to the convergence in mean of $W_n(\theta)$ (Biggins' martingale convergence theorem). Later, Lyons (1997) reproved this result for $d = 1$ with a stronger probabilistic argument, weakening slightly the conditions of the theorem. Lyons' version of the Theorem is easily upgraded to d dimensions by simply considering a one dimensional branching random walk with offspring distribution $\{\theta \cdot \zeta_u : |u| = 1\}$. In its enhanced form the theorem reads as follows.

Theorem 1 (Biggins' Martingale Convergence Theorem). *Suppose that* $m(\theta) < \infty$ *and* $\theta \cdot \nabla m(\theta) := -\mathbb{E}\left(\sum_{|u|=1} \theta \cdot \zeta_u e^{-\theta \cdot \zeta_u}\right)$ *exists and is finite, then* $W_n(\theta)$ *converges in expectation to* $W(\theta)$ *if and only if* $\mathbb{E}\left[W_1(\theta) \log^+ W_1(\theta)\right] < \infty$ *and* $\log m(\theta) - (\theta \cdot \nabla m(\theta))/m(\theta) > 0$. *If either of these two conditions fail then* $W(\theta)$ *is identically zero.*

Since its first appearance, this result has been identified as being of relevance within the study of the growth and spread of spatial branching processes [see for example Biggins (1977b, 1979, 1992), Biggins and Kyprianou (1997), Chauvin and Rouault (1988), McDiarmid (1995), Neveu (1988) and Uchiyama (1978, 1982)] and other related topics such as statistical physics, random cascades and the theory of smoothing transforms [see for example Chauvin and Rouault (1997), Liu and Rouault (1997), Koukiou (1997), Waymire and Williams (1996) and Liu (1997, 1998)]. From these references we should mention in particular that when the limit $W(\theta)$ is non-trivial, its Laplace transform $\mathbb{E}(\exp\{-xW(\theta)\})$ is a also a non-trivial solution to the functional equation

$$\Phi(x) = \mathbb{E}\left[\prod_{|u|=1} \Phi\left(x\frac{\exp\{-\theta \cdot \zeta_u\}}{m(\theta)}\right)\right] . \tag{1}$$

This identity offers travelling wave solutions to a discrete time version of the K-P-P equation which have an intimate relation with the asymptotic behaviour of the right most individual in the branching random walk when $d = 1$. See for example the discussion in Dekking and Host (1991) or Kyprianou (1998).

We can look upon the martingale $W_n(\theta)$ as a Laplace-Stieltjes transformation of the point process $\{\zeta_u : |u| = n\}$ (the n-th generation) weighted by its mean. Close examination of the proof that $W_n(\theta)$ is a martingale (Kingman (1975)) reveals that the branching property on this point process plays an inevitable and indispensable role. It is therefore natural to ask if there are any other sequences of 'subpopulations' like $\{\zeta_u : |u| = n\}$, on which the branching property can be established and from which we could show the existence of other martingales like $W_n(\theta)$. Further, if such martingales exist, what is the relationship between their limit and the limiting variable $W(\theta)$?

Similar questions to these have already been addressed for almost surely convergent multiplicative martingales that are built from solutions to (1) that typically look like

$$M_n(\theta) := \prod_{|u|=n} \Phi\left(x \frac{\exp\{-\theta \cdot \zeta_u\}}{m(\theta)^n}\right) . \tag{2}$$

Between them, Lalley and Sellke (1997), Neveu (1988), Chauvin (1991), Biggins and Kyprianou (1997), Harris (1999) and Kyprianou (1999) have shown that for both the branching random walk and certain branching diffusions, multiplicative martingales like (2) may be generalized to products over sequences of stopping lines. Biggins and Kyprianou (1997) show that there is a close relationship between the limit, as n tends to infinity, of $-\log M_n(\theta)$ and $W_n(\theta)$ and the uniqueness and asymptotic behaviour of solutions to the functional equation (1). Similar relationships are also exposed for branching diffusions in Neveu (1988), Chauvin (1991) and Harris (1999). The work that is presented in this paper thus also alludes to alternative ways of characterizing solutions to (1) through martingales built from stopping lines.

In the following sections we shall discuss the idea of stopping lines, a class of which provide suitable 'subpopulations' from which we are able to construct additive supermartingales. By applying a method of probabilistic analysis used by Lyons (1997) and Lyons, Pemantle and Peres (1995), concerning size biased trees, we are able to distinguish circumstances under which these supermartingales are also martingales. Following this we prove what may be considered a generalization of Biggins' martingale convergence theorem; demonstrating necessary and sufficient conditions under which our identified additive supermartingales are L^1-convergent martingales. In this case their limit is precisely W. Lastly we conclude this paper by discussing some examples of stopping lines.

For the multi-type general branching processes, Jagers (1989) has also studied stopping lines and shown the existence of an intrinsic additive martingale. The stopping lines we shall stick to here are less general in definition however than those used by Jagers. Some of the results here overlap with those of Jagers in certain circumstances, although the method of analysis is considerably different. The application of measures on size biased trees that we use here provides an attractive link between stopping lines and stopping times that is not apparent in Jagers' presentation (at a price however of a more restrictive definition of stopping line). We shall indicate where appropriate in the text the points at which our results meet those of Jagers.

2. Stopping lines

The branching random walk may also be considered as a marked tree $\mathscr{T}$ with vertices realized from the space of possible nodes

$$\mathscr{I} = \mho \cup \bigcup_{n=1}^{\infty} \mathbb{N}^n .$$

Suppose now that Ω is the space of possible events constituting the life of the initial ancestor. Each individual u in the process is thus endowed with an independent copy of this life space; referred to as Ω_u. Define a *line* to be a set of individuals $\ell \subset \mathscr{I}$ with the property that for two distinct members of ℓ, neither one is a descendent of the other. Associated with a line ℓ is its sigma algebra $\mathscr{F}_\ell := \sigma\left(\Omega_u : u \notin \mathscr{D}_\ell\right)$, where $\mathscr{D}_\ell := \{u : \exists\, v \leq u \text{ such that } v \in \ell\}$, the set of descendants from individuals in the line. Thus $\mathscr{F}_\ell$ gives information about all individuals who are neither members of ℓ nor descendent of members of ℓ. Let us call $\mathscr{G}_u = \sigma\left(\Omega_v : v \leq u\right)$ the sigma algebra describing the life histories of u and all its ancestry. Suppose that $\tau_u : [\Omega_v : v \in \mathscr{T}] \to \{0, 1\}, u \in \mathscr{T}$ is an ensemble of maps.

Definition 1. *The set of individuals $\mathscr{L}(\tau) := \{u \in \mathscr{T} : \tau_u = 1\}$ is a stopping line if (i) it is a line (that is to say $\tau_u = 1 \Longrightarrow \tau_v = 0\ \forall\ v < u$) and (ii) τ_u is $\mathscr{G}_u$-measurable for all $u \in \mathscr{T}$. Associated with the stopping line $\mathscr{L}$ is the sigma algebra $\mathscr{F}_{\mathscr{L}} := \sigma\left(\Omega_u : u \notin \mathscr{D}_{\mathscr{L}}\right)$. (Note that from now we will often refrain from indicating the dependence of $\mathscr{L}$ on τ).*

This definition is consistent with that of Chauvin (1991) for branching Brownian motion. Note there is a slight difference in terminology to Chauvin's definition though. In her case, it is the suite of maps τ that is referred to as the stopping line. Jagers (1989) however gives a definition of stopping lines which is less restrictive. He defines a stopping line $\mathscr{L} \subset \mathscr{T}$ to be a random line which is optional in the sense that $\{\mathscr{L} \leq \ell\} \in \mathscr{F}_\ell$ for all lines ℓ (here we understand $\mathscr{L} \leq \ell$ to mean $\forall\ u \in \ell, \exists\ v \in \mathscr{L}$ such that $u \geq v$). The following example, due to Peter Jagers and communicated to me by John Biggins, shows nicely a set of nodes that is a stopping line by Jagers' definition, but not by Definition 1. Let $\mathscr{N}^{(1)}$ be the first generation in which there is a family of 27 children. For any individual u it cannot be established whether it is a member of the $\mathscr{N}^{(1)}$th generation simply by looking back at the history of its ancestry. This is because it may be another group of siblings distinct from u's but in the same generation which is first of size 27 and therefore responsible for the inclusion of u in $\mathscr{N}^{(1)}$. Therefore, in the context of Definition 1 we can see that Jagers' stopping lines can accommodate for greater correlation between the individual mappings of τ.

Let us now introduce some more notation, some of which is not necessarily new to this presentation and can be seen used in the papers of Chauvin (1991) and Biggins and Kyprianou (1997). For any two stopping lines $\mathscr{L}$ and $\mathscr{Q}$, we say that $\mathscr{L}$ dominates $\mathscr{Q}$ if for each $u \in \mathscr{L}$ there exists a $v \in \mathscr{Q}$ such that $v \leq u$ almost surely. A sequence of stopping lines $\{\mathscr{L}_t\}_{t \geq 0}$ is said to be increasing to infinity if $\mathscr{L}_t$ dominates $\mathscr{L}_s$ for all $t \geq s$ and $\lim_{t \uparrow \infty} \inf\{|u| : u \in \mathscr{L}\} = \infty$ almost surely. Let

$$\mathscr{A}_{\mathscr{L}}(\mathscr{Q}) := \{u \in \mathscr{Q} : \nexists\ v \leq u,\ v \in \mathscr{L}\}$$

be those individuals in $\mathscr{Q}$ who have no ancestor, including themselves, in the stopping line $\mathscr{L}$. We can think of $\mathscr{A}_{\mathscr{L}}(\mathscr{Q})$ as being the part of $\mathscr{Q}$ that is 'below' $\mathscr{L}$. We will reserve the special notation $\mathscr{A}_{\mathscr{L}}(n)$ for the case that $\mathscr{Q}$ is the nth generation (the simplest example of a stopping line, induced by setting $\tau_u = I(|u| = n)$, $u \in \mathscr{T}$).

Lemma 2. *For any two stopping lines $\mathscr{L}$ and $\mathscr{Q}$, $\mathscr{A}_{\mathscr{L}}(\mathscr{Q})$ and $\mathscr{L} \cap \mathscr{Q}$ are stopping lines which are both measurable with respect to $\mathscr{F}_{\mathscr{Q}}$ and $\mathscr{F}_{\mathscr{L}}$.*

Proof. Suppose that $\mathscr{L} = \mathscr{L}(\tau)$ and $\mathscr{Q} = \mathscr{Q}(\rho)$, then

$$\mathscr{A}_{\mathscr{L}}(\mathscr{Q}) = \{u : \rho_u = 1 \text{ and } \tau_v = 0 \;\forall\; v \leq u\} \text{ and } \mathscr{L} \cap \mathscr{Q} = \{u : \tau_u = 1 \text{ and } \rho_u = 1\} \;\; .$$

Define η and χ such that $\eta_u = \rho_u I(\tau_v = 0 \;\forall\; v \leq u)$ and $\chi_u = \tau_u \rho_u$. It is easily checked that $\mathscr{R} := \mathscr{R}(\eta)$ and $\mathscr{P} := \mathscr{P}(\chi)$ are stopping lines that are both measurable with respect to $\mathscr{F}_{\mathscr{Q}}$ and $\mathscr{F}_{\mathscr{L}}$. Finally we note that $\mathscr{A}_{\mathscr{L}}(\mathscr{Q}) = \mathscr{R}$ and $\mathscr{L} \cap \mathscr{Q} = \mathscr{P}$ almost surely. □

Definition 2. *A stopping line $\mathscr{L}$ is called almost surely dissecting if*

$$\sup\{n : \mathscr{A}_{\mathscr{L}}(n) = \emptyset\} < \infty \; a.s.$$

This definition refers to the fact that under the given condition, there exists an almost surely finite N such that all members of the N-th generation are descendent from members of $\mathscr{L}$.

Already established from previous works in this field, we have the following two important Lemmas which are valid for the definition of stopping lines we have given here.

Lemma 3. *(Jagers (1989), Theorem 4.14) Let $\mathscr{L}$ be a stopping line and T_u be the shift operator that renders u the initial ancestor. Given $\mathscr{F}_{\mathscr{L}}$, the trees $\mathscr{T} \circ T_u$ emanating from each $u \in \mathscr{L}$ are independent stochastic copies of $\mathscr{T}$.*

Lemma 4. *(Chauvin (1991), Lemma 2.4) Suppose that $\mathscr{L}$ and $\mathscr{Q}$ are two almost surely dissecting stopping lines such that $\mathscr{L}$ dominates $\mathscr{Q}$. Then $\mathscr{L}$ can be partitioned exhaustively and uniquely into mutually exclusive subsets*

$$\mathscr{L}^{(v)} := \{u \in \mathscr{L} : u \geq v\}, \, v \in \mathscr{Q}$$

such that conditional on $\mathscr{F}_{\mathscr{Q}}$, $\mathscr{L}^{(v)}$ is an almost surely dissecting stopping line on the tree $\mathscr{T} \circ T_v$.

The second of these two Lemmas is not necessarily true for stopping lines with Jagers' definition. The earlier example, $\mathscr{N}^{(1)}$, can be used to show this is the case. It suffices to consider the two stopping lines $\mathscr{N}^{(1)}$ and $\mathscr{N}^{(2)}$ (the generation in which there is a family of size 27 appearing for the second time). See also the discussion in Kyprianou (1999).

For convenience, set $y_u(\theta) = e^{-\theta \cdot \zeta_u}/m^{|u|}(\theta)$. Where it is not necessary we will not include the dependence on θ of these terms and functions of these terms.

We need a further classification of our stopping lines.

Definition 3. *A stopping line $\mathscr{L}$ is L^1-dissecting if*

$$\lim_{n \uparrow \infty} \mathbb{E}\left(\sum_{u \in \mathscr{A}_{\mathscr{L}}(n)} y_u \right) = 0 \;\; .$$

As we shall later see almost sure dissection does not imply L^1-dissection nor vice versa. The above condition guarantees that lines of descent (essentially random walks) are not able to drift too far from the origin in one direction creating unusually large values of y_u before (if at all) hitting the stopping line $\mathscr{L}$. Note that L^1-dissection also implies that if $v < u \in \mathscr{L}$ almost surely then

$$\lim_{n\uparrow\infty} \mathbb{E}\left(\sum_{u\in\mathscr{A}_{\mathscr{L}(v)}(n)} y_u \middle| \mathscr{F}_v \right) = 0 \ .$$

That is to say that conditional on $\mathscr{F}_v$, $\mathscr{L}^{(v)}$ is an L^1-dissecting stopping line on $\mathscr{T} \circ T_v$. If this were not true then, since $v \in \mathscr{A}_{\mathscr{L}}(|v|)$

$$\begin{aligned}
\lim_{n\uparrow\infty} \mathbb{E}\left(\sum_{u\in\mathscr{A}_{\mathscr{L}}(n)} y_u \right) &= \lim_{n\uparrow\infty} \mathbb{E}\left[\sum_{u\in\mathscr{A}_{\mathscr{L}}(|v|)} y_u \sum_{uw\in\mathscr{A}_{\mathscr{L}}(n)} y_w \circ T_u \right] \\
&= \lim_{n\uparrow\infty} \mathbb{E}\left[\sum_{u\in\mathscr{A}_{\mathscr{L}}(|v|)} y_u \mathbb{E}\left(\sum_{w\in\mathscr{A}_{\mathscr{L}(v)}(n-|v|)} y_w \circ T_u \middle| \mathscr{F}_{|v|} \right) \right] \\
&\geq \mathbb{E}\left[y_v \lim_{n\uparrow\infty} \mathbb{E}\left(\sum_{w\in\mathscr{A}_{\mathscr{L}(v)}(n-|v|)} y_w \circ T_v \middle| \mathscr{F}_v \right) \right] > 0
\end{aligned}$$

which is a contradiction.

3. Martingales

For any stopping line $\mathscr{L}$, define $W_{\mathscr{L}} = \sum_{u\in\mathscr{L}} y_u$. Clearly as the n-th generation is a stopping line, W_n is also of this form.

Lemma 5. *Suppose that $\mathscr{L}$ and $\mathscr{Q}$ are two stopping lines, then*

$$\mathbb{E}(W_{\mathscr{Q}} | \mathscr{F}_{\mathscr{L}}) \leq \sum_{u\in\mathscr{A}_{\mathscr{L}}(\mathscr{Q})} y_u + \sum_{u\in\mathscr{Q}\cap\mathscr{L}} y_u + \sum_{u\in\mathscr{A}_{\mathscr{Q}}(\mathscr{L})} y_u \ . \tag{3}$$

It is sufficient that $\mathscr{Q}$ is almost surely and L^1-dissecting and that $\mathscr{L}$ is almost surely dissecting for equality to hold.

Proof. It is not difficult to show (a rough sketch may help) that $\mathscr{Q}$ can be decomposed according to that part which is 'below' $\mathscr{L}$, the part that intersects with $\mathscr{L}$ and the descendents of the part of $\mathscr{L}$ that sits 'below' $\mathscr{Q}$ (although some of these three may be empty sets with positive probability). Taking advantage of the fact that we can write y_{uv} as $y_u \times (y_v \circ T_u)$ (this decomposition is discussed in more detail in Biggins and Kyprianou (1997)) we have

$$W_{\mathscr{Q}} = \sum_{u\in\mathscr{A}_{\mathscr{L}}(\mathscr{Q})} y_u + \sum_{u\in\mathscr{Q}\cap\mathscr{L}} y_u + \sum_{u\in\mathscr{A}_{\mathscr{Q}}(\mathscr{L})} y_u \sum_{v\in\mathscr{Q}^{(u)}} y_v \circ T_u \tag{4}$$

where $\mathscr{Q}^{(u)} = \{v \in \mathscr{Q} : v > u\}$ is the (possibly empty) subset of $\mathscr{Q}$ which descends from u. Taking conditional expectations we have as a consequence of Lemma 2

$$\mathbb{E}\left(W_{\mathscr{Q}} \mid \mathscr{F}_{\mathscr{L}}\right) = \sum_{u\in\mathscr{A}_{\mathscr{L}}(\mathscr{Q})} y_u + \sum_{u\in\mathscr{Q}\cap\mathscr{L}} y_u + \sum_{u\in\mathscr{A}_{\mathscr{Q}}(\mathscr{L})} y_u \mathbb{E}\left(\sum_{v\in\mathscr{Q}^{(u)}} y_v \circ T_u \middle| \mathscr{F}_{\mathscr{L}}\right) . \tag{5}$$

Lemmas 3 and 4 imply that each of the conditional expectations on the right hand side of (5) are independent and each is a version of $\mathbb{E}W_{\mathscr{R}}$ for some appropriately defined stopping line $\mathscr{R}$; which may, with probability one, be degenerately the empty set. If both $\mathscr{L}$ and $\mathscr{Q}$ are almost surely dissecting, then each of the $\mathscr{Q}^{(u)}$ are non-empty with positive probability. If further $\mathscr{Q}$ is L^1-dissecting, then in view of the comments following Definition 3, we see (3) holds with equality as a consequence of the next Theorem. Otherwise, without these restrictions on $\mathscr{L}$ and $\mathscr{Q}$, the best we can say, again with the help of the next Theorem, is the inequality (3). □

Theorem 6. *Suppose $\mathscr{R}$ is a stopping line, then $\mathbb{E}W_{\mathscr{R}} \leq 1$. Further, equality holds if and only if $\mathscr{R}$ is L^1-dissecting.*

In order to prove this theorem we must import some ideas from Lyons (1997) and Lyons, Pemantle and Peres (1995) concerning size biased trees. The marked size biased tree associated with the branching random walk is constructed as follows. From the initial ancestor, $\mho =: \xi_0$, we begin with a reproduction process $\widetilde{Z}$ defined by a random number of offspring and spatial positions whose distribution has Radon-Nikodym derivative W_1 with respect to the joint probability distribution of offspring numbers and spacings of Z. From this first generation we select an individual ξ_1 with probability $y_{\xi_1} / \left(\sum_{|u|=1} y_u\right)$. From those individuals not selected we generate independent branching random walks with reproduction process Z. The selected individual ξ_1 reproduces according to an independent copy of $\widetilde{Z}$. From the offspring of ξ_1 we select at random an individual $\xi_2 = \xi_1 v$ with probability $\left(y_v \circ T_{\xi_1}\right) / \left(\sum_{|u|=1} y_u \circ T_{\xi_1}\right)$ and so on. Lyons (1997) shows that there exists a joint probability measure, $\widetilde{\mathbb{Q}}$, of the marked tree $\widetilde{\mathscr{T}}$ with random ray or spine $\xi = \left(\xi_0, \xi_1, \xi_2, \xi_3, ...\right)$. Let $\mathbb{Q}$ be the probability measure that is the projection of $\widetilde{\mathbb{Q}}$ to the space generated by the sequence of point processes $\left\{\widetilde{Z}^{(\xi_n)}\right\}_{n\geq 0}$ on the spine, where $\widetilde{Z}^{(\xi_n)}$ is an independent copy of $\widetilde{Z}$, for each $n \geq 0$.

Proof of Theorem 6. We can prove Theorem 6 by finding an expression for $\mathbb{E}W_{\mathscr{R}}$ in terms of an event measured under $\mathbb{Q}$. Let $\mathscr{R} = \mathscr{R}(\tau)$ (not necessarily dissecting in either sense) and define $U_n(\mathscr{R}) = \mathbb{E}\left(\sum_{|u|=n} I(\tau_u = 1)\, y_u\right)$ so that

$$\mathbb{E}W_{\mathscr{R}} = \mathbb{E}\left[\sum_{u\in\mathscr{T}} I(\tau_u = 1)\, y_u\right] = \sum_{n\geq 1} U_n(\mathscr{R}) \quad .$$

Let $N(\xi, \mathscr{R})$ be a stopping time such that $\{N(\xi, \mathscr{R}) = n\}$ if and only if $\left\{\tau_{\xi_n} = 1\right\}$. For the first part of the theorem we shall demonstrate that $U_n(\mathscr{R}) = \mathbb{Q}(N = n)$

and hence $\mathbb{E}W_{\mathscr{R}} = \mathbb{Q}(N < \infty) \leq 1$ (we have abused our notation and written N as shorthand for $N(\xi, \mathscr{R})$). We argue by induction, so for $n = 1$,

$$\begin{aligned}
U_1(\mathscr{R}) &= \mathbb{E}\left(\sum_{|u|=1} I(\tau_u = 1)\, y_u\right) \\
&= \mathbb{E}\left(\sum_{|u|=1} I(\tau_u = 1) \frac{y_u}{\sum_{|v|=1} y_v} \sum_{|v|=1} y_v\right) \\
&= \mathbb{E}\left(\sum_{|u|=1} I\left(u = \xi_1, \tau_u = 1\right) \frac{y_{\xi_1}}{\sum_{|v|=1} y_v} W_1\right) \\
&= \int I(N = 1)\, d\mathbb{Q} \\
&= \mathbb{Q}(N = 1) \quad .
\end{aligned}$$

Assume now that $U_n(\mathscr{L}) = \mathbb{Q}(N(\xi, \mathscr{L}) = n)$ for all stopping lines $\mathscr{L}$, then

$$\begin{aligned}
U_{n+1}(\mathscr{R}) &= \mathbb{E}\left(\sum_{|u|=n+1} I(\tau_u = 1)\, y_u\right) \\
&= \mathbb{E}\left(\sum_{|u|=n+1} I(\tau_u = 1,\ \tau_v = 0\ \forall\ v < u)\, y_u\right) \\
&= \mathbb{E}\left[\sum_{|u|=1} I(u = \xi_1, \tau_u = 0) \frac{y_{\xi_1}}{\sum_{|v|=1} y_v} W_1\right. \\
&\qquad\qquad \left. \times \mathbb{E}\left(\sum_{|w|=n} I\left(\tau_w \circ T_{\xi_1} = 1\right) y_w \circ T_{\xi_1} \middle| \mathscr{F}_1\right)\right]
\end{aligned}$$

where the inner conditional expectation in the second equality is a version of $U_n\left(\mathscr{R}^{(\xi_1)}\right)$ on the tree $\widetilde{\mathscr{T}} \circ T_{\xi_1}$. It is important to note at this stage that the previous computation is not necessarily possible had we been working with stopping lines defined in the sense of Jagers (1989).

Define $\mathbb{Q}_1$ the restriction of $\mathbb{Q}$ to $\mathscr{F}_1 := \sigma(\Omega)$, $\xi' = (\xi_1, \xi_2, \xi_3, ...)$, $\mathbb{Q}^{(\xi_1)}$ to be the version of $\mathbb{Q}$ associated with the spine ξ' on $\widetilde{\mathscr{T}} \circ T_{\xi_1}$ and finally $N_{\xi_1} = N\left(\xi', \mathscr{R}^{(\xi_1)}\right)$. We have

$$\begin{aligned}
U_{n+1}(\mathscr{R}) &= \mathbb{E}\left[\sum_{|u|=1} I\left(u = \xi_1, \tau_u = 0\right) \mathbb{Q}^{(\xi_1)}\left(N_{\xi_1} = n\right) \frac{y_{\xi_1}}{\sum_{|v|=1} y_v} W_1\right] \\
&= \int I(N > 1)\, \mathbb{Q}^{(\xi_1)}\left(N_{\xi_1} = n\right) d\mathbb{Q}_1 \\
&= \mathbb{Q}(N = n + 1)
\end{aligned}$$

thus proving $\mathbb{E}W_{\mathscr{R}} = \mathbb{Q}(N < \infty) \leq 1$. As we shall see from Example 1 in the final section, it is not necessarily true that N is $\mathbb{Q}$-almost surely finite even when we impose that $\mathscr{R}$ is almost surely dissecting.

To prove the second part of the Theorem, define

$$\rho_u(n) = I(|u| = n \text{ and } \tau_v = 0\ \forall\, v \leq u) \quad ,$$

so that $\mathscr{A}_{\mathscr{R}}(n) = \mathscr{L}(\rho(n))$. We now have

$$\{N(\xi, \mathscr{A}_{\mathscr{R}}(n)) = m\} \text{ if and only if } \left\{\rho_{\xi_m}(n) = 1\right\}$$

and

$$\{N(\xi, \mathscr{A}_{\mathscr{R}}(n)) = m\} = \begin{cases} \emptyset & \text{if } n \neq m \\ \{N(\xi, \mathscr{R}) > n\} & \text{if } n = m \end{cases} \quad .$$

The previous part of the Theorem shows that

$$\begin{aligned} \mathbb{E}\left(\sum_{u \in \mathscr{A}_{\mathscr{R}}(n)} y_u\right) &= \sum_{m \geq 1} \mathbb{Q}\left[N(\xi, \mathscr{A}_{\mathscr{R}}(n)) = m\right] \\ &= \mathbb{Q}\left[N(\xi, \mathscr{R}) > n\right] \quad . \end{aligned} \tag{6}$$

Since $\lim_{n\uparrow\infty} \mathbb{Q}[N(\xi, \mathscr{R}) > n] = 0$ if and only if $\mathbb{Q}[N(\xi, \mathscr{R}) < \infty] = 1$, equality (6) completes the proof of the second part of the Theorem. □

Clearly the size biased tree provides us with a mechanism of looking at stopping lines in terms of stopping times. The condition of L^1-dissection has significance in terms of stopping the random ray ξ on the size biased tree. The equality (6) gives us the following useful corollary in this respect.

Corollary 7. *A stopping line $\mathscr{R}$ is L^1-dissecting if and only if $N(\xi, \mathscr{R})$ is $\mathbb{Q}$-almost surely finite.*

We also now have the following Corollary to Lemma 5 and Theorem 6 which identifies for us sufficient conditions for the existence of additive supermartingales and martingales built from stopping lines.

Corollary 8. *Let $\{\mathscr{L}_t\}_{t\geq 0}$ be an increasing sequence of almost surely dissecting stopping lines tending to infinity, then $W_{\mathscr{L}_t}$ is an almost surely convergent supermartingale with respect to $\mathscr{F}_{\mathscr{L}_t}$. If further $\mathscr{L}_t$ is also L^1-dissecting for each $t \geq 0$, then $W_{\mathscr{L}_t}$ is a mean 1 martingale.*

Proof. This result follows simply from the fact that for $s < t$, $\mathscr{L}_s = \mathscr{A}_{\mathscr{L}_t}(\mathscr{L}_s) \cup (\mathscr{L}_s \cap \mathscr{L}_t)$ and $\mathscr{A}_{\mathscr{L}_s}(\mathscr{L}_t) = \emptyset$ almost surely. □

So far we have concerned ourselves with conditions under which we can produce martingales. The following theorem, which we may consider as a generalization of Biggins' Martingale Convergence Theorem, gives necessary and sufficient conditions under which $W_{\mathscr{L}_t}$ is an L^1-convergent martingale with a specific limit.

Theorem 9. *Suppose that* $m(\theta) < \infty$ *and* $\theta \cdot \nabla m(\theta) := -\mathbb{E}\left(\sum_{|u|=1} \theta \cdot \zeta_u e^{-\theta \cdot \zeta_u}\right)$ *exists and is finite and* $\{\mathscr{L}_t\}_{t \geq 0}$ *is as in Corollary 8. Then* $W_{\mathscr{L}_t}(\theta)$ *is an* L^1*-convergent martingale with limit* $W(\theta)$ *(the limit of* $W_n(\theta)$*) if and only if*

$$\mathbb{E}\left[W_1(\theta) \log^+ W_1(\theta)\right] < \infty \quad \textit{and} \quad \log m(\theta) - (\theta \cdot \nabla m(\theta))/m(\theta) > 0.$$

Note that the combined effect of the two necessary and sufficient conditions overrides the need for an L^1-dissecting sequence of stopping lines in order that $W_{\mathscr{L}_t}$ may enjoy martingale status.

Proof. Assume first that $\mathbb{E}\left[W_1(\theta) \log^+ W_1(\theta)\right] < \infty$ and $\log m(\theta) - (\theta \cdot \nabla m(\theta)) / m(\theta) > 0$. From the decomposition (4) we have that for any $t \geq 0$,

$$W_n = \sum_{u \in \mathscr{A}_{\mathscr{L}_t}(n)} y_u + \sum_{\substack{|u| \leq n \\ u \in \mathscr{L}_t}} y_u \sum_{|v|=n-|u|} y_u \circ T_u \tag{7}$$

where conditional on $\mathscr{F}_{\mathscr{L}_t}$, $\sum_{|v|=n-|u|} y_u \circ T_u$ is an independent version of $W_{n-|u|}$ for each $u \in \mathscr{L}_t$, $|u| \leq n$. As $\mathscr{L}_t$ is almost surely dissecting for all $t \geq 0$, taking expectations of (7) conditional on $\mathscr{F}_{\mathscr{L}_t}$ and then the limit as $n \uparrow \infty$ we have

$$\begin{aligned}
\lim_{n\uparrow\infty} \mathbb{E}_{\mathscr{F}_{\mathscr{L}_t}}(W_n) &= \lim_{n\uparrow\infty} \left[\sum_{u \in \mathscr{A}_{\mathscr{L}_t}(n)} y_u + \sum_{\substack{|u| \leq n \\ u \in \mathscr{L}_t}} y_u \times \mathbb{E}_{\mathscr{F}_{\mathscr{L}_t}}\left(\sum_{|v|=n-|u|} y_u \circ T_u \right) \right] \\
&= \lim_{n\uparrow\infty} \left[\sum_{u \in \mathscr{A}_{\mathscr{L}_t}(n)} y_u + \sum_{\substack{|u| \leq n \\ u \in \mathscr{L}_t}} y_u \right] \\
&= W_{\mathscr{L}_t} \ .
\end{aligned} \tag{8}$$

Since W_n converges in mean,

$$\lim_{n\uparrow\infty} \mathbb{E}\left|\mathbb{E}_{\mathscr{F}_{\mathscr{L}_t}}(W_n) - \mathbb{E}_{\mathscr{F}_{\mathscr{L}_t}}(W)\right| \leq \lim_{n\uparrow\infty} \mathbb{E}|W_n - W| = 0$$

and hence $W_{\mathscr{L}_t} = \mathbb{E}_{\mathscr{F}_{\mathscr{L}_t}}(W)$ thus revealing that $W_{\mathscr{L}_t}$ is a martingale with mean 1. Now let $\mathscr{F}_\infty = \sigma\left(\cup_{t \geq 0} \mathscr{F}_{\mathscr{L}_t}\right)$. Since $\mathscr{L}_t$ is tending to infinity with t we have that $\lim_{t\uparrow\infty} \mathscr{A}_{\mathscr{L}_t}(n) = \{u : |u| = n\}$ and $\lim_{t\uparrow\infty} \{u \in \mathscr{L}_t : |u| \leq n\} = \emptyset$ almost surely. Therefore taking limits in (8) with respect to t instead of n yields

$$\begin{aligned}
\mathbb{E}_{\mathscr{F}_\infty}(W_n) &= \lim_{t\uparrow\infty} \mathbb{E}_{\mathscr{F}_{\mathscr{L}_t}}(W_n) \\
&= \lim_{t\uparrow\infty} \left[\sum_{u \in \mathscr{A}_{\mathscr{L}_t}(n)} y_u + \sum_{\substack{|u| \leq n \\ u \in \mathscr{L}_t}} y_u \right] \\
&= W_n
\end{aligned}$$

thus implying that W_n is $\mathscr{F}_\infty$-measurable for all n which in turn implies that so is the limit W. We can now easily identify the martingale limit as W.

Suppose now that we assume that $W_{\mathscr{L}_t}$ is a martingale convergent almost surely and in mean to W. Then as $W_{\mathscr{L}_t}$ is a positive martingale we must have $W > 0$ with positive probability and thus by Biggins' martingale convergence theorem the two conditions $\mathbb{E}\left[W_1(\theta)\log^+ W_1(\theta)\right] < \infty$ and $\log m(\theta) - (\theta \cdot \nabla m(\theta))/m(\theta) > 0$ are implied. □

The sufficient conditions of this theorem reflect those of Corollary 6.6 of Jagers (1989) in which the existence of additive martingales for the multi-type general branching process is also demonstrated using a less restrictive definition of stopping lines. In fact, aside from the difference in definition of stopping lines, the 'sufficient' direction above and Jagers' Corollary 6.6 say the same thing when we consider a single type general branching process.

4. Examples

We shall finish off by giving some examples of stopping lines, demonstrating more clearly the difference between almost surely dissecting and L^1-dissecting and further, how to take advantage of the interpretation in Corollary 7.

Example 1. *Suppose* $m(\theta) < \infty$ *and* $\log m(\theta) - (\theta \cdot \nabla m(\theta))/m(\theta) < 0$. *Let*

$$\begin{aligned}\tau_u(t) &= I\left(\theta \cdot \zeta_u + |u| \log m(\theta) > t,\ \theta \cdot \zeta_v + |v| \log m(\theta) \le t\ \forall\, v < u\right)\\ &= I\left(y_u < e^{-t},\ y_v \ge e^{-t}\ \forall\, v < u\right)\ .\end{aligned}$$

It is easily checked that $\mathscr{L}_t := \mathscr{L}(\tau(t))$ *is a stopping line and constitutes the set of individuals who are first in their line of descent to cross the hyperplane* $\theta \cdot y + s \log m(\theta) = t$ *where* $y \in \mathbb{R}^d$, $s \in \mathbb{R}^+$ *and* $t > 0$ *is a fixed constant. For the case that* $d = 1$ *it was proved in Biggins and Kyprianou (1997) that* $\{\mathscr{L}_t\}_{t\ge0}$ *is a sequence of almost surely dissecting stopping lines tending to infinity, albeit under the slightly more restrictive condition that* $\theta \in int\{\phi : m(\phi) < \infty\} \neq \emptyset$. *The proof of this result (which we shall refrain from reproducing here) is centred around the fact that* $\lim_{n\uparrow\infty} \sup\{y_u : |u| = n\} = 0$ *and also works also for the d-dimensional case. By considering Theorem 3 of Biggins (1998) it is easy to see that this argument holds even when we relax the condition on* θ *to simply* $m(\theta) < \infty$. *We shall now consider to what extent* $\{\mathscr{L}_t\}_{t\ge0}$ *is an* L^1*-dissecting sequence. As was shown in Lyons (1997) the positions of the ray* ξ *are a d-dimensional* $\mathbb{Q}$*-random walk with mean step size*

$$\mathbb{E}_{\mathbb{Q}}\left(\zeta_{\xi_1}\right) = \mathbb{E}_{\mathbb{Q}}\left(\sum_{|u|=1} \zeta_u \frac{e^{-\theta\cdot\zeta_u}}{\sum_{|v|=1} e^{-\theta\cdot\zeta_v}} W_1\right) = -\frac{\nabla m(\theta)}{m(\theta)}\ .$$

Let $S_n - n\log m(\theta) = \theta \cdot \zeta_{\xi_n}$ *be the random walk which is the projection of* $\{\zeta_{\xi_n}\}_{n\ge1}$ *onto the line passing through the origin and parallel to* θ. *Then*

$$\mathbb{Q}\left[N(\xi, \mathscr{L}_t) < \infty\right] = \sum_{n\ge1} \mathbb{Q}\left(S_n > t,\ S_k \le t\ \forall\, k < n\right) = \mathbb{Q}\left(S_n > t \text{ eventually}\right)\ .$$

Since the average step size of S_n is equal to

$$\mathbb{E}_{\mathbb{Q}}(S_1) = \log m(\theta) - \frac{\theta \cdot \nabla m(\theta)}{m(\theta)} \ ,$$

which is assumed negative, we see that $\mathbb{Q}(S_n > t$ eventually$) < 1$ (for any $t > 0$) and thus $\mathscr{L}_t$ is an example of an almost surely dissecting stopping line that is not L^1-dissecting. In this instance $W_{\mathscr{L}_t}$ is a supermartingale. On the other hand, had we insisted that

$$\log m(\theta) - \frac{\theta \cdot \nabla m(\theta)}{m(\theta)} \geq 0$$

then $\mathscr{L}_t$ would be L^1-dissecting thus making $W_{\mathscr{L}_t}$ a martingale.

Example 2. *Suppose in the previous example we assume that*

$$\log m(\theta) - \frac{\theta \cdot \nabla m(\theta)}{m(\theta)} = 0 \ .$$

Define for some fixed t,

$$\rho_u(t) = \sum_{\mho < v \leq u} I\left(y_v < e^{-t},\ y_{m(v)} \geq e^{-t}\right)$$

and

$$\tau_u(k,t) = I\left(\rho_u(t) = k,\ \rho_{m(u)}(t) = k-1\right) \ .$$

where $m(v)$ is v's mother, $v \in \mathscr{T}$. We now have the stopping line $\mathscr{L}_{t,k} := \mathscr{L}(\tau(k,t))$ consisting of those individuals who are k-th in their line of descent to cross the hyperplane $\theta \cdot y + s \log m(\theta) = t$ ($y \in \mathbb{R}^d$, $s \in \mathbb{R}^+$) from below. Note in particular that $\mathscr{L}_{t,1} = \mathscr{L}_t$ as before. In light of the discussion in the previous example we have that

$$\mathbb{Q}\left[N\left(\xi, \mathscr{L}_{t,k}\right) < \infty\right] = \mathbb{Q}\left(S_n \text{ eventually crosses } t \text{ from left to right } k \text{ times}\right) \ .$$

This last probability is one since by our initial assumption the random walk S_n is oscillating (a little thought concerning ladder heights justifies this statement, see for example the chapter on random walks in Feller (1971)). Therefore $\mathscr{L}_{t,k}$ is an L^1-dissecting stopping line. We can show that $\mathscr{L}_{t,k}$ is tending to infinity as $k \uparrow \infty$. This follows as a simple consequence of the fact that $\inf\{|u| : u \in \mathscr{L}_{t,k}\} \geq 2k-1$ *as the random walk S_n can, at maximum efficiency, cross t (in alternating directions) at every step. It is not clear whether $\mathscr{L}_{t,k}$ is almost surely dissecting in view of the fact that* $\lim_{n\uparrow\infty} \sup\{y_u : |u| = n\} = 0$. *It may indeed be possible to find a counter example in this respect. Until then we shall not pass judgement as to whether $W_{\mathscr{L}_{t,k}}$ is a martingale.*

Example 3. *Suppose now we consider a branching random walk in which each individual u has position $\zeta_u = (p_u, \sigma_u)$, where p_u is the its point of birth in $\mathbb{R}^d$ and σ_u is its moment of birth in $\mathbb{R}^+$. This process, is an example of what Biggins (1995) calls a general branching random walk. It is in effect a general branching process in which we have assigned spatial positions in $\mathbb{R}^d$ to each individual. Let*

$$\tau_u(t) = I(\sigma_u > t,\ \sigma_v \leq t\ \forall\ v < u)$$

then $\mathscr{C}_t := \mathscr{C}(\tau(t))$ is precisely the coming generation (defined by Nerman (1981)). That is to say, those who are born after time t but whose mother is born before time t. It is not difficult to reason that $\mathscr{C}_t$ is both almost surely dissecting and also tending to infinity as $t \uparrow \infty$. As with Example 1, $\left\{\zeta_{\xi_n}\right\}_{n\geq 1}$ is a $(d+1)$-dimensional $\mathbb{Q}$-random walk and hence so is its projection onto the real time axis $\Sigma_n := \sigma_{\xi_n}$. Also as before

$$\mathbb{Q}\left[N(\xi, \mathscr{C}_t) < \infty\right] = \mathbb{Q}(\Sigma_n > t \textit{ eventually})$$

which equals one since the mean step size

$$\mathbb{E}_{\mathbb{Q}}(\Sigma_1) = \mathbb{E}\left(\sum_{|u|=1} \sigma_u \exp\{-\theta\cdot\zeta_u\}/m(\theta)\right) > 0$$

provided $m(\theta) < \infty$. Consequently, under this last condition $\mathscr{C}_t$ is both almost surely and L^1-dissecting and $W_{\mathscr{C}_t}$ is always a martingale.

Example 4. *In the two type branching random walk we colour individuals either red or blue independently of one another such that the probability of being coloured red is p. Let c_u be valued 1 if u is red and zero otherwise. Define for $n \geq 1$*

$$\tau_u(n) = I\left(c_u = 1,\ \sum_{v<u} c_v = n-1\right)$$

then $\mathscr{R}_n := \mathscr{R}(\tau(n))$ is a stopping line consisting of those individuals who are n-th in their line of descent to be coloured red. This stopping line has been used in the past by Doney (1976) and Ryan (1968) (although not presented at the time as a stopping line). Note that

$$\mathscr{R}_{n+1} = \bigcup_{u\in\mathscr{R}_n} \mathscr{R}_1^{(u)} \textit{ where } \mathscr{R}_1^{(u)} \overset{i.i.d}{\sim} \mathscr{R}_1 \textit{ on } \mathscr{T}\circ T_u,$$

so that $|\mathscr{R}_n|$ is the number of offspring in the n-th generation of an embedded Galton-Watson process. Therefore to guarantee that $\mathscr{R}_n$ is almost surely dissecting, it suffices to prove that $\mathscr{R}_1$ is almost surely dissecting. Consider $\mathscr{B}_n$, the stopping line consisting of individuals n-th in their line of descent to be coloured blue. As before $|\mathscr{B}_n|$ is also a Galton-Watson Process and further, it becomes extinct with probability 1 if and only if $\mathscr{R}_1$ is almost surely dissecting. Hence $\mathscr{R}_1$ is almost surely dissecting if and only if $(1-p)\mathbb{E}\left(Z\left(\mathbb{R}^d\right)\right) \leq 1$. Under the probability measure $\widetilde{\mathbb{Q}}$, *nodes in* $\widetilde{\mathscr{T}}$ *are coloured with the same probability and thus* $N(\xi, \mathscr{R}_n)$ *are renewal times with*

$$\mathbb{Q}\left[N(\xi, \mathscr{R}_1) < \infty\right] = \sum_{n \geq 1} p(1-p)^{n-1} = 1$$

implying that $\mathbb{E}W_{\mathscr{R}_n} = 1$ *for all* $n \in \mathbb{Z}^+$ *and that* $W_{\mathscr{R}_n}$ *is a martingale. Note that the above equalities also hold even when* $(1-p)\mathbb{E}\left(Z\left(\mathbb{R}^d\right)\right) > 1$*, thus giving us an example of when* L^1*-dissection does not imply almost sure dissection. In this case it is still true that* $W_{\mathscr{R}_n}$ *is a martingale showing that the martingale conditions in Corollary 8 are not necessary.*

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