Moments of branching Markov processes and related problems

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Summary

A fundamental question concerning general spatial branching processes, both superprocesses and branching Markov processes, pertains to their moments. Whilst the setting of first and second moments has received quite some attention, limited information seems to be known about higher moments, in particular, their asymptotic behaviour with time. Relevant references that touch upon this topic include [18, 14, 23, 33, 20]. In this work, we provide general results that pertains to both superprocesses and spatial branching Markov processes and which provides a very precise result for moment growth.

We show that, under the assumption that the first moment semigroup of the process exhibits a natural Perron Frobenious type behaviour, the k-th moment functional of either a superprocess or branching Markov process, when appropriately normalised, limits to a precise constant. The setting in which we work is remarkably general, even allowing for the setting of nonlocal branching; that is, where mass is created at a different point in space to the position of the parent. Moreover, the methodology we use appears to be extremely robust and we show that the asymptotic k-th moments of the running occupation measure are equally accessible using essentially the same approach. Our results will thus expand on what is known for branching diffusions and superdiffusions e.g. in [10], [31], as well as giving precise growth rates for the moments of occupations.

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Chapter 1

Introduction

A branching process is a mathematical object used to describe and study the development of an asexual population, with members called individuals or particles, which independently reproduce and behave the same way as their parents.

One of the simplest examples is the Galton-Watson process, named after Francis Galton and Henry William Watson who studied the process in [47]. This process describes the evolution of a population of individuals in discrete-time who reproduce themselves according to a certain offspring distribution. The associated Galton Watson process is a Markov chain $(X_n, n \geq 0)$ with values in $\mathbb{N}$ such that

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i, \quad n \geq 0, \tag{1.1}$$

where the random variables ξ_i are independent and identically distributed. The continuous analogue of the Galton-Watson branching process are known as continuous-state branching processes (CSBP), which describe the evolution in continuous time of a population taking values in a continuous space. For example, in a continuous time Galton Watson process $(X_t, t \geq 0)$, the initial particle waits an exponential time e with some parameter $\beta > 0$ at which it splits and produces a random number of offspring, each of which behaves the same way independently, and then, for $s \leq t$,

$$X_t = \sum_{i=1}^{N_s} X^{(i)}_{t-s},$$

where N_s is the number of particles alive at time s, and the $X^{(i)}_\cdot$ are independent copies of the original process. If we look at the split times, the discrete GW process is embedded.

In this thesis we will consider a more general setting in which spatial dependence and movement are considered, with some assumptions regarding the linear semigroup and the offspring distribution, which we detail in the following sections. Let us begin providing the details of the setting in which we wish to work. Let E be a Lusin [1] space. Throughout, will write $B(E)$ for the Banach space of bounded measurable functions on E with norm $\|\cdot\|$, $B^+(E)$ for non-negative bounded measurable functions on E and $B_1^+(E)$ for the subset of functions in $B^+(E)$ which are uniformly bounded by unity. We are interested in spatial branching processes that are defined in terms of a Markov process and a branching operator. The former can be characterised by a semigroup on E, denoted by $\mathtt{P} = (\mathtt{P}_t, t \geq 0)$. We do not need $\mathtt{P}$ to possess the Feller property, and it is not necessary that $\mathtt{P}$ is conservative. That said, if so desired, we can append a cemetery state $\{\dagger\}$ to E, which is to be treated as an absorbing state, and regard $\mathtt{P}$ as conservative on the extended space $E \cup \{\dagger\}$, which can also be treated as a Lusin space. Equally, we can extend the branching operator to $E \cup \{\dagger\}$ by defining it to be zero on $\{\dagger\}$, i.e. no branching activity on the cemetery state. Examples of this setting include Polish spaces, bounded Euclidean spaces with cemetery state $\{\dagger\}$ on the boundary, discrete spaces $\{1, \ldots, n\} \cup \{\dagger\}$, etc. Note that the event of killing does not happen only at the boundary, the cemetery state $\{\dagger\}$ is also reached, for example, if the process dies in the middle of the domain.

1.1 Branching Markov processes

Branching Markov processes enjoy a very long history in the literature, dating back as far as the late 50's early 60's, cf. [42, 41, 43], with a broad base of literature that is arguably too voluminous to give a fair summary of here. In this thesis we consider a spatial branching process in which, given their point of creation, particles evolve independently according to a $\mathtt{P}$-Markov process. In an event which we refer to as 'branching', particles positioned at x die at rate $\beta \in B^+(E)$ and instantaneously, new particles are created in E according to a point process. The configurations of these offspring are described by the random counting measure

$$\mathcal{Z}(A) = \sum_{i=1}^{N} \delta_{x_i}(A), \tag{1.2}$$

[1] A metrizable space is Lusin if it is the image of a Polish space under a bijective continuous map.

for Borel A in E. The law of the aforementioned point process depends on x, the point of death of the parent, and we denote it by $\mathcal{P}_x$, $x \in E$, with associated expectation operator given by $\mathcal{E}_x$, $x \in E$. This information is captured in the so-called branching mechanism

$$\mathtt{G}[f](x) := \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N} f(x_i) - f(x)\right], \qquad x \in E, \tag{1.3}$$

where we recall $f \in B_1^+(E) := \{f \in B^+(E) : \sup_{x\in E} f(x) \leq 1\}$. Without loss of generality we can assume that $\mathcal{P}_x(N = 1) = 0$ for all $x \in E$ by viewing a branching event with one offspring as an extra jump in the motion. On the other hand, we do allow for the possibility that $\mathcal{P}_x(N = 0) > 0$ for some or all $x \in E$, that is, we allow deaths of particles with no offspring in addition to branching.

Henceforth we refer to this spatial branching process as a $(\mathtt{P}, \mathtt{G})$-branching Markov process. It is well known that if the configuration of particles at time t is denoted by $\{x_1(t), \ldots, x_{N_t}(t)\}$, then, on the event that the process has not become extinct or exploded, the branching Markov process can be described as the co-ordinate process $X = (X_t, t \geq 0)$ in the space of finite counting measures on E, denoted by $N(E)$, where

$$X_t(\cdot) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(\cdot), \qquad t \geq 0,$$

and N_t is the number of particles alive at time T. In particular, X is Markovian in $N(E)$. Its probabilities will be denoted $\mathbb{P} := (\mathbb{P}_\mu, \mu \in N(E))$.

1.2 Semigroups

Branching Markov processes are strongly related to the theory of semigroups, and its study conforms an important tool to understand these type of processes as well as giving a probabilistic construction through their linear and non linear semigroups, see for example the Branching Markov processes series [28, 29, 30] from Ikeda, Nagasawa and Watanabe. Some of the evolution equations presented in this thesis can be found in the literature e.g. in [27].

1.2.1 Linear and Non linear semigroups

Throughout this work we will use the notation $\langle f, \mu\rangle$ for a function f and a measure μ to denote $\int_E f \mathrm{d}\mu$. With this notation we define the linear semigroup of the process $(X_t, t \geq 0)$

as

$$\mathtt{T}_t\left[f\right](x) := \mathbb{E}_{\delta_x}\left[\langle f, X_t\rangle\right] = \mathbb{E}_{\delta_x}\left[\sum_{i=1}^{N_t} f(x_i(t))\right], \tag{1.4}$$

for $f \in B_1^+(E)$, $x \in E$ and $t \geq 0$, and the non linear semigroup as

$$\mathtt{v}_t[f](x) = \mathbb{E}_{\delta_x}\left[\prod_{i=1}^{N_t} f(x_i(t))\right], \qquad f \in B_1^+(E), t \geq 0, \tag{1.5}$$

For $f \in B^+(E)$, it is well known that the mean semigroup evolution satisfies

$$\mathtt{T}_t[f](x) = \mathtt{P}_t[f](x) + \int_0^t \mathtt{P}_s\left[\mathtt{F}\mathtt{T}_{t-s}[f]\right](x)\mathrm{d}s, \qquad t \geq 0, x \in E, \tag{1.6}$$

where

$$\mathtt{F}[f](x) = \beta(x)\mathcal{E}_x\left[\sum_{i=1}^{N} f(x_i) - f(x)\right] =: \beta(x)(\mathtt{m}[f](x) - f(x)), \qquad x \in E.$$

To see this, we use a standard branching decomposition, conditioning the right-hand side of (1.4) on the time of the first branching event to get that

$$\begin{aligned}
\mathtt{T}\left[f\right](x) &= \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s} f(\xi_t)\right] + \int_0^t \mathbf{E}_x\left[\mathrm{e}^{-\int_0^s \beta(\xi_u)\mathrm{d}u}\beta(\xi_s)\mathtt{m}\left[\mathtt{T}_{t-s}\left[f\right](\xi_s)\right]\right]\mathrm{d}s \\
&= \mathbf{E}_x\left[f(\xi_t)\right] + \int_0^t \mathbf{E}_x\Big[\beta(\xi_s)\left(\mathtt{m}\left[\mathtt{T}_{t-s}\left[f\right](\xi_s)\right] - \mathtt{T}_{t-s}\left[f\right](\xi_s)\right)\Big]\mathrm{d}s \\
&= \mathtt{P}_t\left[f\right](x) + \int_0^t \mathtt{P}_s\left[\mathtt{F}\mathtt{T}_{t-s}\left[f\right]\right](x)\mathrm{d}s,
\end{aligned} \tag{1.7}$$

where we have used Lemma A.1 of the Appendix to get the first equality in (1.7). For similar calculations, see for example the calculations in [26] or Section 4 in [45].

It is worth noting that the independence that is manifest in the definition of branching events and movement implies that, for $\mu \in N(E)$ given by $\mu = \sum_{i=1}^n \delta_{y_i}$, we have

$$\mathbb{E}_\mu\left[\prod_{i=1}^{N_t} f(x_i(t))\right] = \prod_{i=1}^{n} \mathtt{v}_t[f](y_i), \qquad t \geq 0. \tag{1.8}$$

Moreover, for $f \in B^+(E)$ and $x \in E$,

$$\mathtt{v}_t[f](x) = \hat{\mathtt{P}}_t[f](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathtt{v}_{t-s}[f]]\right](x)\mathrm{d}s, \qquad t \geq 0, \tag{1.9}$$

where $\hat{\mathtt{P}}_t$ is a slight adjustment of $\mathtt{P}_t$, which returns a value of 1 on the event of killing, that is $\hat{\mathtt{P}}_t[0] = 1$.This is needed as in this thesis we will consider the product over an empty set to be equal to 1, that is $\prod_\emptyset = 1$, and use the definition of the non linear semigroup similar to the one found for example in [28]. To see that (1.9) holds we define

$$\mathtt{H}[f](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N} f(x_i)\right]$$

then we write

$$\mathtt{v}[f](x) = \mathbf{E}_{\delta_x}\left[\prod_{i=1}^{N_t} f(x_i(t))\mathbb{1}(N_t > 0) + \mathbb{1}(N_t = 0)\right]$$

and use the usual branching decomposition along with Lemma A.1 to get that

$$\begin{aligned}
\mathtt{v}[f](x) &= \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s} f(\xi_t)\right] + \int_0^t \mathbf{E}_x\left[\mathrm{e}^{-\int_0^s \beta(\xi_u)\mathrm{d}u}\mathtt{H}[\mathtt{v}_{t-s}](\xi_s)\right]\mathrm{d}s + \mathbb{E}_{\delta_x}[\mathbb{1}(N_t = 0)] \\
&= \mathbf{E}_x[f(\xi_t)] + \mathbb{E}_{\delta_x}[\mathbb{1}(N_t = 0)] + \int_0^t \mathbf{E}_x[\mathtt{H}[\mathtt{v}_{t-s}] - \beta\mathtt{v}_{t-s}]\mathrm{d}s \\
&= \hat{\mathtt{P}}_t[f] + \int_0^t \mathtt{P}_{t-s}[\mathtt{G}[\mathtt{v}_{t-s}]](x)\mathrm{d}s.
\end{aligned}$$

Most literature focuses on the setting of local branching. This corresponds to the setting that all offspring are positioned at their parent's point of death (i.e. $x_i = x$ in the definition of $\mathtt{G}$). In that case, the branching mechanism reduces to

$$\mathtt{G}[s](x) = \beta(x)\left[\sum_{k=1}^{\infty} p_k(x)s^k - s\right], \qquad x \in E,$$

where $s \in [0,1]$ and $(p_k(x), k \geq 0)$ is the offspring distribution when a parent branches at site $x \in E$. The branching mechanism $\mathtt{G}$ may otherwise be seen in general as a mixture of local and nonlocal branching.

1.2.2 Perron Frobenius type assumption and Criticality condition

As we mentioned before, throughout this thesis we will make two main assumptions, the first one of them being that the linear semigroup of the process exhibits a Perron-Frobenius type asymptotic of the following form.

(H1): There exists an eigenvalue $\lambda \in \mathbb{R}$ and a corresponding right eigenfunction $\varphi \in B^+(E)$ and finite left eigenmeasure $\tilde{\varphi}$ such that, for $f \in B^+(E)$,

$$\langle \mathtt{T}_t[\varphi], \mu \rangle = \mathrm{e}^{\lambda t} \langle \varphi, \mu \rangle \text{ and } \langle \mathtt{T}_t[f], \tilde{\varphi} \rangle = \mathrm{e}^{\lambda t} \langle f, \tilde{\varphi} \rangle ,$$

for all $\mu \in N(E)$. Further let us define

$$\Delta_t = \sup_{x \in E, f \in B_1^+(E)} |\varphi(x)^{-1} \mathrm{e}^{-\lambda t} \mathtt{T}_t[f](x) - \langle f, \tilde{\varphi} \rangle |, \qquad t \geq 0.$$

We suppose that

$$\sup_{t \geq 0} \Delta_t < \infty \text{ and } \lim_{t \to \infty} \Delta_t = 0. \tag{1.10}$$

For a lot of literature surrounding spatial branching processes, there has been emphasis on results for which an underlying assumption of exponential ergodic growth in the first moment is present as in the first assumption; see e.g. [40, 17, 1, 38, 26, 22]. Due to this, we may characterise the process as supercritical if $\lambda > 0$, critical if $\lambda = 0$ and subcritical if $\lambda < 0$.

One way to understand (1.10), is through the martingale that comes hand-in-hand with the eigenpair (λ, φ), i.e.

$$M_t^{\varphi} := \mathrm{e}^{-\lambda t} \langle \varphi, X_t \rangle, \qquad t \geq 0. \tag{1.11}$$

Normalising this martingale and using it as a change of measure results in the ubiquitous spine decomposition; cf. [26, 23, 40]. Roughly speaking, under the change of measure, the process is equal in law to a copy of the original process with a superimposed process of immigration, which occurs both in space and time along the path of a single particle trajectory in E, the spine. If the process is issued from e.g. $\mu \in M(E)$, then the semigroup of the latter, is given by $\mathrm{e}^{-\lambda t} \langle \mathtt{T}_t[f\varphi], \mu \rangle / \langle \varphi, \mu \rangle$, $t \geq 0$. We see then that an assumption of the type (1.10) implies that the spine has a stationary limit with stationary measure $\varphi \tilde{\varphi}$.

1.2.3 Moment condition

(H2): The second assumption is a moment condition on the offspring distribution. Suppose $k \geq 1$. If $(X, \mathbb{P})$ is a branching Markov process, then

$$\sup_{x \in E} \mathcal{E}_x(\langle 1, \mathcal{Z} \rangle^k) < \infty. \tag{1.12}$$

This assumption is natural to ensure that k-moments are well defined for all $t \geq 0$. If not explicitly stated in the literature, their need to ensure that the functional moments $\mathtt{T}_t^{(k)}[f](x)$ are finite for all $t \geq 0$, $f \in B^+(E)$ and $x \in E$ is certainly folklore, where

$$\mathtt{T}_t^{(k)}\left[f\right](x) = \mathbb{E}_{\delta_x}\left[\langle f, X_t \rangle^k\right]. \tag{1.13}$$

1.2.4 Examples of branching Markov processes

We now give some examples to illustrate our results and the generality of this setting.

Example 1: Branching Brownian motion

A Branching Brownian motion (BBM), is a spatial branching process in which the particles move as Brownian motions, and, for each particle, after an independent and exponentially distributed random time they split into an independent random number of new particles that behaves stochastically in the same way as the previous ones.

More generally, we could take a branching diffusion in which particles move according to the diffusion with generator L in a bounded domain $D \subset \mathbb{R}^d$ satisfying the uniform exterior cone condition. For example, if ∂D is Lipschitz, the condition is satisfied.

$$L = \frac{1}{2} \sum_{i,j} \partial_{x_j}(a^{ij} \partial_{x_i}),$$

which is uniformly elliptic with coefficients $a^{ij} = a^{ji} \in C^1(D)$ for $1 \leq i, j \leq d$. This means there exist a constant $\theta > 0$ such that for all $\xi \in \mathbb{R}^d$ and almost every $x \in D$

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta |\xi|^2.$$

An example of this is the model given in [39], in which the author considers a branching

Brownian motion that lives on a compact domain $D \subset \mathbb{R}^d$, that is, those which particles are killed upon exiting D.

Example 2: Branching Brownian motion on the sphere

The natural way to described BBM on a sphere is to describe the evolution of particles in a regular BBM on $\mathbb{R}^d$ and write their spatial positions in terms of a skew product. That is to say, a particle at $x \in \mathbb{R}^d$ is identified as $x = |x| \arg(x)$. Roughly speaking, up to a time change, considering only the component $\arg(x)$ of each particle x in the system thus produces a BBM on $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$, the surface of a d dimensional sphere, for example in dimension 3 as shown in Figure 1-1. Ito and McKean in [32] define what they call the spherical Brownian motion, and then use it to give a skew product representation of a d-dimensional Brownian motion.

The spherical Brownian motion $\mathrm{BM}(\mathbb{S}^{d-1})$ is defined as the diffusion on the spherical surface $\mathbb{S}^{d-1}$ with generator $\frac{1}{2}\Delta^{d-1}$, the spherical Laplace operator defined recursively as

$$\Delta^d \phi = (\sin\phi)^{1-d}\frac{\partial}{\partial\phi}(\sin\phi)^{d-1}\frac{\partial}{\partial\phi} + (\sin\phi)^{-2}\Delta^{d-1},$$

for $\phi \in (0, \pi)$, where ϕ is the colatitude, and $\Delta^1 = \partial^2/\partial\vartheta^2$.

For example, $\mathrm{BM}(\mathbb{S}^1)$ is the projection modulo 2π of the standard 1-dimensional Brownian motion onto the unit circle $\mathbb{S}^1$. $\mathrm{BM}(\mathbb{S}^{d-1})$ is constructed as follows: given a $\mathrm{BM}(\mathbb{S}^{d-2})$ with sample paths $t \to \vartheta(t)$ and an independent Legendre process $\mathrm{LEG}(d-1)$ on $[0, \pi]$ with generator

$$\frac{1}{2}(\sin\phi)^{-d}\frac{\partial}{\partial\phi}(\sin\phi)^{d-2}\frac{\partial}{\partial\phi},$$

and sample paths $t \to \phi(t)$, the additive functional

$$\mathsf{l}(t) = \int_0^t (\sin\phi(s))^2 ds$$

converges if $0 < \phi(0) < \pi$ and the skew product $(\phi, \vartheta(\mathsf{l}))$ is a diffusion.

Example 3: Neutron Branching process (NBP)

As introduced in [4, 26], neutrons evolve in the configuration space $E = D \times V$, where $D \subset \mathbb{R}^3$ is a bounded, open set denoting the set of particle locations and $V := \{\upsilon \in \mathbb{R}^3 : \mathtt{v}_{\min} \leq |\upsilon| \leq \mathtt{v}_{\max}\}$ with $0 < \mathtt{v}_{\min} \leq \mathtt{v}_{\max} < \infty$, denotes the set of velocities. From an

Figure 1-1: A simulation of mass or a gas spreading over the surface of $\mathbb{S}^2$, based on a branching Brownian motion.

initial space-velocity configuration (r, υ), particles move according to piecewise deterministic Markov processes characterised by $\sigma_{\mathtt{s}}\pi_{\mathtt{s}}$, where $\sigma_{\mathtt{s}}(r, \upsilon)$, $r \in D, \upsilon \in V$ denotes the rate at which particles change velocity (also called *scattering* events) at (r, υ), and $\pi_{\mathtt{s}}(r, \upsilon, \upsilon')\mathrm{d}\upsilon'$ denotes the probability that such a scattering event results in a new outgoing velocity υ'. When at $(r, \upsilon) \in D \times V$, at rate $\sigma_{\mathtt{f}}(r, \upsilon)$, a branching (or *fission*) event occurs, resulting in the release of several new neutrons with configurations $(r, \upsilon_1), \ldots, (r, \upsilon_N)$, say and particles are absorbed at the boundary. The quantity $\pi_{\mathtt{f}}(r, \upsilon, \upsilon')$ gives the average number of neutrons produced with outgoing velocity υ' from a fission event at (r, υ). Thus, the NBP is an example of a branching Markov process with nonlocal branching, where the motion is a piecewise deterministic Markov process (see for example Figure 1-2).

In [26], under the assumptions

(A1) $\sigma_{\mathtt{s}}$, $\pi_{\mathtt{s}}$, $\sigma_{\mathtt{f}}$, $\pi_{\mathtt{f}}$ **are uniformly bounded from above.**

(A2) $\inf_{r \in D\upsilon, \upsilon' \in V}(\sigma_{\mathtt{s}}(r, \upsilon)\pi_{\mathtt{s}}(r, \upsilon, \upsilon') + \sigma_{\mathtt{f}}(r, \upsilon)\pi_{\mathtt{f}}(r, \upsilon, \upsilon')) > 0$**,**

it was shown that (H1) holds. Moreover, since only a finite number of neutrons can be produced at a fission event, the number of offspring is uniformly bounded from above and thus (H2) holds. Hence, the main results obtained in this work hold for the NBP.

Of particular interest in this setting is the notion of particle clustering that appears in Monte Carlo criticality calculations [8, 44, 9]. This phenomena occurs in critical reactors where particles exhibit strong spatial correlations.

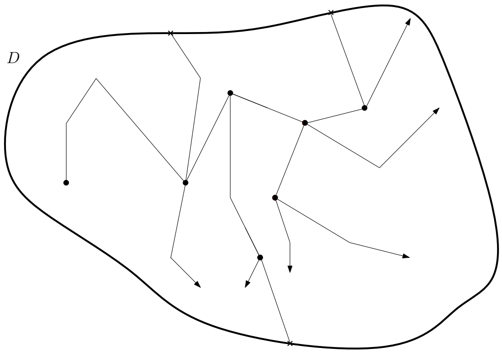

Figure 1-2: Branching process in a bounded domain D

Example 4: CMJ processes

Consider a branching process in which particles live for a random amount of time L and during their lifetime, give birth to a (possibly random) number of offspring at random times. The offspring reproduce and die as independent copies of the parent particle. Although this process is not covered in the present work, it is worth making some comparisons with the results and methods used in the literature.

First let us consider the case where the number of offspring, N, born to the initial individual during its lifetime satisfies $\mathbb{E}[N] = 1$, which [11] identifies as the critical case. Further, let N_t denote the number of offspring produced by the initial individual by time t and Z_t denote the number of individuals in the population at time t. Under the moment assumption that $\mathbb{E}[N(N-1)\cdots(N-k+1)] < \infty$ for some $k \geq 1$, then [11] showed that the factorial moments $m_k(t) := \mathbb{E}[Z_t(Z_t-1)\cdots(Z_t-k+1)]$ satisfy

$$\lim_{t\to\infty} \frac{m_k(t)}{t^{k-1}} = k! m_1 a^{k-1},$$

where $m_1(t)$ and a are known constants and assumed to be finite and positive. The proof follows similar ideas as in this work, albeit in time rather than space with a lower order of complexity[2]. Indeed, the author first presents the analogue of Lemma 3.10, i.e. a non-linear integral equation that describes the evolution of $m_k(t)$ in terms of the lower order moments, cf. [11, Theorem 1]. An inductive argument along with this evolution equation is then used to prove the above asymptotics. We must remark that the results presented in this thesis cover a wider and more general class of branching processes, and they are consistent with those found in the literature.

[2]We became aware of this paper during the writing up of the thesis.

It is worth noting that $\mathbb{E}[N] = 1$ is not the usual definition of critical for CMJ. If we write $\mu(\mathrm{d}u)$ for the intensity of offspring at time u into the lifetime of the parent, then the assumption $\mathbb{E}[N] = 1$ implies that

$$1 = \mathbb{E}[N] = \mathbb{E}[\mu(0,\infty)] = \int_{(0,\infty)} e^{-\alpha u}\mu(\mathrm{d}u), \tag{1.14}$$

with $\alpha = 0$, that is, the Malthusian parameter corresponds to that of the critical setting. Conversely, if the Malthusian parameter equals zero, then, from (1.14) we equivalently see that $\mathbb{E}[N] = 1$.

1.2.5 Main results for branching Markov processes

With these assumptions, in this work we show that, for $k \geq 2$ and any positive bounded measurable function f on E,

$$\lim_{t\to\infty} g_k(t)\mathbb{E}_{\delta_x}[\langle f, X_t\rangle^k] = C_k(x,f),$$

where the constant $C_k(x,f)$ can be identified in terms of the principal right eigen-function and left eigen-measure and $g(t)$ is an appropriate deterministic normalisation, which can be identified explicitly as either polynomial in t or exponential in t, depending on whether X is a critical, supercritical or subcritical process. More precisely, in this thesis we will show the following asymptotic behaviour in each of the three cases:

- Critical case ($\lambda = 0$): $\mathbb{E}_{\delta_x}\left[\langle f, X_t\rangle^k\right] \sim t^{k-1}C_k(f)$.
- Supercritical case ($\lambda > 0$): $\mathbb{E}_{\delta_x}\left[\langle f, X_t\rangle^k\right] \sim \mathrm{e}^{\lambda tk}C_k(f)$.
- Subcritical case ($\lambda < 0$): $\mathbb{E}_{\delta_x}\left[\langle f, X_t\rangle^k\right] \sim \mathrm{e}^{\lambda t}C_k(f)$.

Where f is a positive bounded measurable function on E and C_k will vary depending on the case.

1.2.6 Occupation moments

The method we employ is extremely robust and we are able to extract similarly precise results that additionally give us the moment growth with time of $\int_0^t\langle g, X_s\rangle\mathrm{d}s$, for bounded measurable g on E.

- Critical case ($\lambda = 0$): $\mathbb{E}_{\delta_x}\left[\left(\int_0^t \langle g, X_s\rangle\,\mathrm{d}s\right)^k\right] \sim t^{2k-1}C_k(g)$.

- Supercritical case ($\lambda > 0$): $\mathbb{E}_{\delta_x}\left[\left(\int_0^t \langle g, X_s\rangle\, \mathrm{d}s\right)^k\right] \sim \mathrm{e}^{\lambda tk} C_k(g)$.
- Subcritical case ($\lambda < 0$): $\mathbb{E}_{\delta_x}\left[\left(\int_0^t \langle g, X_s\rangle\, \mathrm{d}s\right)^k\right] \sim C_k(g)$.

Here is worth mentioning some heuristics for the critical case. Theorems 1.2 and 1.3 in [21] give us some insight to the reason for the polynomial behaviour. The first of them presents a Kolmogorov limit in the following form

$$t\mathbb{P}_{\delta_x}(\zeta > t) \to c,$$

where $c > 0$, and the second is a Yaglom type limit that reads

$$\mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\theta\frac{\langle f, X_t\rangle}{t}} | \zeta > t\right] \to \frac{\Lambda}{\Lambda + \theta},$$

where $\zeta := \inf_{t\geq 0}\{s \geq 0 : X_s = 0\}$ is the extinction time of the process $(X_t, t \geq 0)$ and Λ is a special constant. Hence,

$$\left\{\frac{\langle f, X_t\rangle}{t}\middle| \zeta > t\right\} \overset{d}{\sim} \mathsf{e}_\Lambda, \qquad \text{as } t \to \infty \tag{1.15}$$

with e_Λ being an exponential random variable with parameter Λ. Next note that the asymptotic (1.15)

$$\left(\int_0^t \langle f, X_s\rangle\, \mathrm{d}s\right)^k \middle| \zeta > t \approx \left(\int_0^t s\mathrm{d}s\right)^k = \left(\frac{t^2}{2}\right)^k = O(t^{2k}),$$

in which case,

$$\left(\int_0^t \langle f, X_s\rangle\, \mathrm{d}s\right)^k \approx O(t^{2k})\mathbb{P}_{\delta_x}(\zeta > t) = O(t^{2k-1}).$$

1.3 Superprocess setting

Superprocesses can be thought of as the high-density limit of a sequence of branching Markov processes, resulting in a new family of measure-valued Markov processes; see e.g. [37, 6, 46, 13, 7]. Just as branching Markov processes are Markovian in $N(E)$, the former are Markovian in the space of finite Borel measures on E denoted by $M(E)$ topologised by weak convergence. There is a large literature base for superprocesses, e.g. [37, 6, 46, 18, 16], with so-called local branching mechanisms. Moreover this has been broadened to include the more general setting of nonlocal branching mechanisms in [7, 37]. Let us now introduce these concepts more formally.

A Markov process $X := (X_t : t \geq 0)$ with state space $M(E)$ and probabilities $\mathbb{P} := (\mathbb{P}_\mu, \mu \in M(E))$ is called a $(\mathtt{P}, \psi, \phi)$-superprocess if it has transition semigroup $(\hat{\mathbf{E}}_t, t \geq 0)$ on $M(E)$ satisfying

$$(1.16) \qquad \mathbb{E}_\mu\left[\mathrm{e}^{-\langle f, X_t\rangle}\right] = \int_{M(E)} \mathrm{e}^{-\langle f,\nu\rangle}\hat{\mathbf{E}}_t(\mu, \mathrm{d}\nu) = \mathrm{e}^{-\langle \mathtt{V}_t[f],\mu\rangle}, \qquad \mu \in M(E), f \in B^+(E).$$

Here, we work with the inner product on $B^+(E) \times M(E)$ defined by $\langle f, \mu\rangle = \int_E f(x)\mu(\mathrm{d}x)$ and $(\mathtt{V}_t, t \geq 0)$ is a semigroup evolution that is characterised via the unique bounded positive solution to the evolution equation

$$(1.17) \qquad \mathtt{V}_t[f](x) = \mathtt{P}_t[f](x) - \int_0^t \mathtt{P}_s\left[\psi(\cdot, \mathtt{V}_{t-s}[f](\cdot)) + \phi(\cdot, \mathtt{V}_{t-s}[f])\right](x)\mathrm{d}s,$$

see for example Lemma 3.3 in [7]. In (1.17), ψ denotes the local branching mechanism

$$(1.18) \qquad \psi(x, \lambda) = -b(x)\lambda + c(x)\lambda^2 + \int_{(0,\infty)} \left(\mathrm{e}^{-\lambda y} - 1 + \lambda y\right)\nu(x, \mathrm{d}y), \quad \lambda \geq 0,\ x \in E,$$

where $b \in B(E)$, $c \in B^+(E)$ and $(x \wedge x^2)\nu(x, \mathrm{d}y)$ is a bounded kernel from E to $(0, \infty)$, and ϕ is the nonlocal branching mechanism

$$(1.19) \qquad \phi(x, f) = \beta(x)\left(f(x) - \zeta(x, f)\right), \quad x \in E,\ f \in B^+(E),$$

where $\beta \in B^+(E)$ and ζ has representation

$$(1.20) \qquad \zeta(x, f) = \gamma(x, f) + \int_{M(E)^\circ}(1 - \mathrm{e}^{-\langle f,\nu\rangle})\Gamma(x, \mathrm{d}\nu),$$

such that $\gamma(x, f)$ is a bounded function on $E \times B^+(E)$ and $\nu(1)\Gamma(x, \mathrm{d}\nu)$ is a bounded kernel from E to $M(E)^\circ := M(E) \setminus \{0\}$ with

$$(1.21) \qquad \gamma(x, f) + \int_{M(E)^\circ} \langle 1, \nu\rangle\, \Gamma(x, \mathrm{d}\nu) \leq 1.$$

Lemma 3.1 in [7] tells us that the functional $\zeta(x, f)$ has the following equivalent representation

$$(1.22) \qquad \zeta(x, f) = \int_{M_0(E)} \left[d(x, \pi)\langle f, \pi\rangle + \int_0^\infty \left(1 - \mathrm{e}^{-u\langle f,\pi\rangle}\right)n(x, \pi, \mathrm{d}u)\right]G(x, \mathrm{d}\pi),$$

where $M_0(E)$ denotes the set of probability measures on E, $d \geq 0$ is a bounded function on $E \times M_0(E)$, $un(x, \pi, \mathrm{d}u)$ is a bounded kernel from $E \times M_0(E)$ to $(0, \infty)$ and $G(x, \mathrm{d}\pi)$ is a probability kernel from E to $M_0(E)$ with

$$d(x, \pi) + \int_0^\infty un(x, \pi, \mathrm{d}u) \leq 1. \tag{1.23}$$

In the superprocess setting, we will have the analogue assumptions to (H1) and (H2), which we will call the same way for simplicity as they will not cause confusion.

(H1): Assume that the mean semigroup exhibits a Perron-Frobenius type asymptotic of the form

$$\mathtt{T}_t[f](x) \sim \mathrm{e}^{\lambda t}\varphi(x)\langle f, \tilde{\varphi}\rangle, \qquad t \to \infty, \tag{1.24}$$

uniformly on E in the same way as in the branching particle system setting, where λ is the lead eigenvalue of the mean the semigroup $(\mathtt{T}_t, t \geq 0)$ and φ, $\tilde{\varphi}$ are the associated right eigenfunction, left eigenmeasure respectively. The latter meaning $\langle \mathtt{T}_t[\varphi], \mu\rangle = \mathrm{e}^{\lambda t}\langle \varphi, \mu\rangle$, for $t \geq 0$, $\mu \in M(E)$, and $\langle \mathtt{T}_t[f], \tilde{\varphi}\rangle = \mathrm{e}^{\lambda t}\langle f, \tilde{\varphi}\rangle$, for $f \in B^+(E)$, $t \geq 0$, respectively.

(H2): The analogue of 1.10 in our superprocess setting is the assumption that

$$\sup_{x\in E}\left(\int_0^\infty |y|^k \nu(x, \mathrm{d}y) + \int_{M(E)^\circ} \langle 1, \nu\rangle^k \Gamma(x, \mathrm{d}\nu)\right) < \infty. \tag{1.25}$$

Whereas for superprocesses, it is usual to separate out the non-diffusive local branching behaviour from nonlocal behaviour via the measures $\nu(x, \mathrm{d}y)$ and $\Gamma(x, \mathrm{d}\nu)$, the analogous behaviour is captured in the single point process $\mathcal{Z}$ for branching Markov processes, introduced in (1.2).

1.3.1 Main results for superprocesses

Superprocesses exhibit the same asymptotic behaviour as branching Markov processes, albeit with different constants C_k. This is an expected result, as superprocesses are limits of branching particle systems.

The reader will notice that we have deliberately used some of the same notation for both branching Markov processes and superprocesses. In the sequel there should be no confusion over meaning. The motivation for this choice of repeated notation is that our main result is indifferent to which of the two processes we are talking about.

The robustness of our methods in the following sections means that the principal ideas used to prove the main theorems are essentially the same for both branching particle systems and superprocesses, regardless of the criticality. The main idea is to study the non-linear semigroup, $\mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta\langle f,X_t\rangle}]$, $\theta, t \geq 0, f \in B^+(E)$, associated with $(X,\mathbb{P})$. The relation

$$\mathtt{T}^{(k)}[f](x) = (-1)^k \frac{\partial}{\partial\theta}\mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta\langle f,X_t\rangle}]\bigg|_{\theta=0}, \quad x \in E, \tag{1.26}$$

means that we can use knowledge of the non-linear semigroup (1.5) or (1.16) to study the moments of $(X,\mathbb{P})$. Indeed, the first step is to write an evolution equation for $\mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta\langle f,X_t\rangle}]$ in terms of the linear semigroup $\mathtt{T}$. We will show that differentiating this equation and using (1.26) means that we can write the k-th moment of the process in terms of the lower order moments. An inductive argument will then yield the final results.

Despite this generic approach, the proof for each of the two processes requires slightly different technicalities due to the fact that, on the one hand, superprocesses have Lévy-Khintchine branching mechanisms but no particles, whereas, on the other hand, branching particle systems do have individual particles but less regular branching mechanisms. Due to these discrepancies, we require a different toolbox to deal with the evolution equation of the k-th moment.

To compute the derivatives in (1.26) for branching particle systems, we use the product Leibniz rule, however, in the case of superprocesses, we use Fàa di Bruno's rule. This yields different equations for the k-th moment evolutions, resulting in slightly different combinatorial arguments when completing the proofs of the theorems. Moreover, the different normalisation required in each of the three theorems, as well as the different limits requires some care.

1.4 Outline of the thesis

This thesis is divided in six chapters, the first one being this introductory chapter. In the next chapter, we present the first moments of a particular branching Markov process, with a so-called Many-to-one formula and a Many-to-two formula, similar to the ones that can be found in [24]. Its contents were part of an earlier stage of the research in this thesis and they serve as warm-up calculations for the more general setting. In Chapter 3 we present higher moments of a general class of branching Markov process as well as the main results concerning the asymptotic behaviour of these moments. These Theorems are contained in

[34], a joint work with Andreas Kyprianou and Emma Horton which has been accepted for publication in Probability Theory and Related Fields. Chapter 4 presents the integrated moments of branching Markov processes and the corresponding results for their limiting behaviour. These results are also contained in [34]. The fifth Chapter serves as the analogue of Chapter 2, as it presents the first moments of a particular class of superprocesses as well as some related problems that helped to build up the main ideas used to work on the more general case. The sixth and last chapter contains the main results for a general class of superprocesses, both for the moments and the integrated moments. We decided to include all the results concerning superprocesses in the same chapter to avoid too much repetition as they use very similar ideas to the branching particle system setting. We finish the thesis with an Appendix containing some key Lemmas used to prove the main results.

Chapter 2

First moments of branching Markov processes

The setting of first and second moments has received more attention than higher moments in the literature, see for example [18, 14, 23, 33, 20], but before exploring the properties and asymptotic behaviour of higher moments, let us devote some time looking at the first moments of a particular case of branching Markov process. The contents of this chapter were part of the first stage of the research project, but we decided to include them as they will serve us as warm-up calculations for a more general case and for higher moments in the following chapters.

Consider a $(\mathtt{P}, \psi)$ branching Markov process with local branching, which corresponds to the setting that all offspring are positioned at their parent's point of death (i.e. $x_i = x$ in the definition of $\mathtt{G}$), and constant rate of branching β. In this case the branching mechanism (1.3) reduces to

$$\mathtt{G}\left[f\right](x) = \beta\left(\mathbb{E}\left[f(x)^N\right] - f(x)\right), \tag{2.1}$$

and as we are not considering spatial dependence, setting $f \equiv s$, (2.1) is equal to

$$\mathtt{G}(s) = \beta\left(\mathbb{E}\left[s^N\right] - s\right) = \beta\left(\sum_{k=1}^{\infty} p_k s^k - s\right), \tag{2.2}$$

where $s \in [0, 1]$ and $(p_k, k \geq 0)$ is the offspring distribution. In this scenario, the number of particles that are created in E in a branching event is denoted by N, and we will define

$m_1 := \mathcal{E}(N)$.

2.1 A many-to-one formula

We begin with a many-to-one formula for this particular case.

Proposition 1. *(A many-to-one formula) Let* $(X_t, t \geq 0)$ *be a* $(\mathtt{P}, \psi)$ *branching Markov process as the one described above and* $f \in B^+(E)$*, then*

$$\mathtt{T}_t\left[f\right](x) = \mathbb{E}_{\delta_x}[\langle f, X_t\rangle] = \mathbb{E}_{\delta_x}\left[\sum_{i=1}^{N_t} f(x_i(t))\right] = \mathrm{e}^{(m_1-1)\beta t}\mathtt{P}_t\left[f\right](x). \tag{2.3}$$

Proof. The proof consist on conditioning on the first time a branching occurs, denoted by $\mathtt{e}$, which is an exponential random variable with parameter β. We have then, that

$$\begin{aligned}
\mathtt{T}_t\left[f\right](x) &= \int_0^t \beta \mathrm{e}^{-\beta s}\mathbb{E}_x[\ \langle f, X_t\rangle \ |\, \mathtt{e} = s]\,\mathrm{d}s + \int_t^\infty \beta \mathrm{e}^{-\beta s}\mathbb{E}_{\delta_x}[\ \langle f, X_t\rangle \ |\, \mathtt{e} = s]\,\mathrm{d}s \\
&= \int_0^t \beta \mathrm{e}^{-\beta s}\mathcal{E}(N)\mathtt{P}_s\left[\mathbb{E}_{\delta_\cdot}[\langle f, X_{t-s}\rangle]\right]\mathrm{d}s + \int_t^\infty \beta \mathrm{e}^{-\beta s}\mathtt{P}_t\left[f\right](x)\mathrm{d}s \\
&= \int_0^t \beta \mathrm{e}^{-\beta s} m_1 \mathtt{P}_s\left[\mathtt{T}_{t-s}\left[f\right]\right](x)\mathrm{d}s + \mathrm{e}^{-\beta t}\mathtt{P}_t\left[f\right](x)\mathrm{d}s.
\end{aligned}$$

Let us define now $\tilde{\mathtt{T}}_t\left[f\right](x) = \mathrm{e}^{(m_1-1)\beta t}\mathtt{P}_t\left[f\right](x)$, that is, the right hand side of (2.3). We will show that $\tilde{\mathtt{T}}_t$ is solves the equation

$$\tilde{\mathtt{T}}_t\left[f\right](x) = \mathrm{e}^{-\beta t}\mathtt{P}\left[f\right](x) + m_1\beta \int_0^t \mathrm{e}^{-\beta s}\mathtt{P}_s\left[\tilde{\mathtt{T}}_{t-s}\left[f\right]\right](x)\mathrm{d}s, \tag{2.4}$$

then Grönwall's inequality will give us the uniqueness of the solution, and this will finish the proof. Using the definition of $\tilde{\mathtt{T}}_t$ on the right hand side of (2.4), this is equal to

$$\begin{aligned}
& \mathrm{e}^{-\beta t}\mathtt{P}_t\left[f\right](x) + m_1\beta \int_0^t \mathrm{e}^{-\beta s}\mathtt{P}_s\left[\tilde{\mathtt{T}}_{t-s}\left[f\right](\cdot)\right](x)\mathrm{d}s \\
=\ & \mathrm{e}^{-\beta t}\mathtt{P}_t\left[f\right](x) + m_1\beta \int_0^t \mathrm{e}^{-\beta s}\mathtt{P}_s\left[\mathrm{e}^{(m_1-1)\beta t}\mathtt{P}_{t-s}\left[f\right](\cdot)\right](x)\mathrm{d}s \\
=\ & \mathrm{e}^{-\beta t}\mathtt{P}_t\left[f\right](x) + m_1\beta \mathrm{e}^{(m_1-1)\beta t}\int_0^t \mathrm{e}^{-m_1\beta s}\mathtt{P}_s\left[\mathtt{P}_{t-s}\left[f\right](\cdot)\right](x)\mathrm{d}s \\
=\ & \mathrm{e}^{-\beta t}\mathtt{P}_t\left[f\right](x) + m_1\beta \mathrm{e}^{(m_1-1)\beta t}\mathtt{P}_t\left[f\right](x)\left(1 - \mathrm{e}^{-m_1\beta t}\right) \\
=\ & \mathrm{e}^{(m_1-1)\beta t}\mathtt{P}_t\left[f\right](x) \\
=\ & \tilde{\mathtt{T}}_t\left[f\right](x),
\end{aligned}$$

so $\tilde{\mathtt{T}}$ satisfies (2.4). In order to see if this solution is unique, suppose that $\mathtt{T}_t$ and $\tilde{\mathtt{T}}_t$ solve (2.4), then define $w_t[f](x) := |\mathtt{T}_t[f](x) - \tilde{\mathtt{T}}_t[f](x)|$ and note that

$$\begin{aligned} w_t[f](x) &\leq m_1\beta \int_0^t \mathrm{e}^{-\beta s} |\mathtt{P}_s[\mathtt{T}_{t-s}[f](\cdot)](x) - \mathtt{P}_s\left[\tilde{\mathtt{T}}_{t-s}[f](\cdot)\right](x)| \mathrm{d}s \\ &\leq m_1\beta \int_0^t \mathrm{e}^{-\beta s} \mathtt{P}_s[w_{t-s}[f]](x)\mathrm{d}s. \end{aligned}$$

Using Grönwall's lemma, we get that $w_t[f](x) = 0$ for each $t \geq 0$ and then the solution is unique. □

2.2 Many-to-two formula

The following result is called many-to-two formula, and it comes from [24], and we prove it here for our particular case.

Proposition 2. *Suppose that $(X_t, t \geq 0)$ is a $(\mathtt{P}, \psi)$ branching Markov process with local branching and constant branching rate β, then*

$$\begin{aligned} \mathbb{E}_{\delta_x}[\langle f, X_t\rangle \langle g, X_t\rangle] &= \mathrm{e}^{(m_1-1)\beta t}\mathtt{P}_t[fg](x) \\ &\quad + \beta(m_2 - m_1)\int_0^t \mathrm{e}^{(2t-s)(m_1-1)\beta}\mathtt{P}_s[\mathtt{P}_{t-s}[f]\mathtt{P}_{t-s}[g]](x)\mathrm{d}s, \end{aligned} \tag{2.5}$$

where m_1 is defined as above and $m_2 = \mathcal{E}(N^2)$

Proof. We begin as we did in the previous result by conditioning on the first offspring time as follows.

$$\begin{aligned} \mathbb{E}_{\delta_x}[\langle f, X_t\rangle \langle g, X_t\rangle] &= \int_0^t \beta \mathrm{e}^{-\beta s}\mathbb{E}_x\left[\sum_{i=1}^{N_t} f(x_i(t)) \sum_{j=1}^{N_t} g(x_j(t)) \,\middle|\, \mathtt{e} = s\right] \mathrm{d}s \\ &\quad + \int_t^\infty \beta \mathrm{e}^{-\beta s}\mathtt{P}_t[fg](x)\mathrm{d}s, \end{aligned}$$

where the second term of the sum simplifies because before the first splitting time there is only one particle in the system, and then the second integral is equal to $\mathrm{e}^{-\beta t}\mathtt{P}_t[fg](x)$. For the first integral on the right hand side notice that when the particle splits at time s, it generates a number N of particles from which there will be formed N new independent sub-trees, so the terms in the sum inside the expectation can be grouped into two groups, one in which the particles at positions $x_i(t)$ and $x_j(t)$ come from the same sub-tree (there are N of these sums), and other in which the pair of particles come from two different, independent sub-trees (there are $N(N-1)$ of those sums), so we get that the first integral is equal to

$$\int_0^t \beta e^{-\beta s} \mathtt{P}_s \left[\mathbb{E}_{\delta_\cdot} \left[N \sum_{i=1}^{N_{t-s}} f(x_i(t-s)) \sum_{j=1}^{N_{t-s}} g(x_j(t-s)) \right] \right] (x) \mathrm{d}s$$
$$+ \int_0^t \beta e^{-\beta s} \mathtt{P}_s \left[\mathbb{E}_{\delta_\cdot} \left[N(N-1) \sum_{i=1}^{N^{(1)}_{t-s}} f(x_i(t-s)) \sum_{j=1}^{N^{(2)}_{t-s}} g(x_j(t-s)) \right] \right] (x) \mathrm{d}s,$$

where the superindex in $N^{(1)}$ and $N^{(2)}$ are to denote that this two quantities correspond to different (and independent) copies of the process, starting at the position of the particle that branched at time s. Then this is equal to

$$\int_0^t m\beta e^{-\beta s} \mathtt{P}_s \left[\mathbb{E}_{\delta_\cdot} \left[\langle f, X_{t-s} \rangle \langle g, X_{t-s} \rangle \right] \right] (x) \mathrm{d}s + \int_0^t (m_2 - m)\beta e^{-\beta s} \mathtt{P}_s \left[\mathtt{T}_{t-s}[f] \, \mathtt{T}_{t-s}[g] \right] (x) \mathrm{d}s,$$

where we split the second term as the product of expectations due to independence of the two sums. Now, using the many-to-one formula for $\mathtt{T}_t$ and putting the other term together we get the following recursive equation for $\mathtt{T}_t[f,g](x) := \mathbb{E}_{\delta_x}[\langle f, X_t \rangle \langle g, X_t \rangle]$:

$$\begin{aligned} \mathtt{T}_t[f,g](x) &= e^{-\beta t} \mathtt{P}_t[fg](x) + m\beta \int_0^t e^{-\beta s} \mathtt{P}_s \left[\mathtt{T}_{t-s}[f,g] \right] (x) \mathrm{d}s \\ &\quad + (m_2 - m)\beta \int_0^t e^{-\beta s + 2\beta(m-1)(t-s)} \mathtt{P}_s \left[\mathtt{P}_{t-s}[f] \, \mathtt{P}_{t-s}[g] \right] (x) \mathrm{d}s. \end{aligned} \tag{2.6}$$

Now, if we define $\tilde{\mathtt{T}}_t[f,g]$ as

$$\tilde{\mathtt{T}}_t[f,g](x) = e^{(m-1)\beta t} \mathtt{P}_t[fg](x) + \beta(m_2 - m) \int_0^t e^{\beta(m-1)(2t-s)} \mathtt{P}_s \left[\mathtt{P}_{t-s}[f] \, \mathtt{P}_{t-s}[g] \right] (x) \mathrm{d}s,$$

then using the identity in Lemma A.1 we obtain that $\tilde{\mathtt{T}}_t[f,g](x)$ satisfies the recursion in (2.6), so it is a solution. Uniqueness comes from a similar argument as the one given in the proof of the many-to-one formula using Grönwall inequality. □

2.3 Limit behaviour of the first moments

In this section we will describe the asymptotic behaviour of the first moments $\mathtt{T}_t[f]$ and $\mathtt{T}^{(}_t 2)[f]$. Recall that we have the Perron-Frobenius type assumption (H1) that gives us the asymptotic behaviour of the first moment. Besides that, (1.10) also gives us that there exists t_0 and a constant γ such that

$$\sup_{x\in D}|\varphi(x)^{-1}\mathrm{e}^{-\lambda t}\mathtt{T}_t\left[f\right](x)-\langle f,\tilde{\varphi}\rangle|\leq \mathrm{e}^{-\gamma t}\left\|f\right\|_\infty, \tag{2.7}$$

for all $t>t_0$. We are now ready to explore the asymptotic behaviour of the second moment of the sum over the living particles at time t.

Proposition 3. *Let $(X_t, t\geq 0)$ be a $(\mathtt{P},\psi)$ branching Markov process described as above, then for any $f\in B_1^+(E)$ we have the following asymptotic behaviour for the second moment in the critical, supercritical and sub-critical case*

i) If $\lambda=0$,

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_x\left[\langle f,X_t\rangle^2\right]=\beta(m_2-m)\langle\varphi,\tilde{\varphi}\rangle\langle f,\tilde{\varphi}\rangle^2\varphi(x). \tag{2.8}$$

ii) If $\lambda>0$,

$$\lim_{t\to\infty}\mathrm{e}^{-2\lambda t}\mathbb{E}_x\left[\langle f,X_t\rangle^2\right]=\beta(m_2-m)\langle f,\tilde{\varphi}\rangle^2\int_0^\infty \mathrm{e}^{-2\lambda s}\mathtt{T}_s\left[\varphi^2\right](x)\mathrm{d}s. \tag{2.9}$$

iii) If $\lambda<0$,

$$\lim_{t\to\infty}\mathrm{e}^{-\lambda t}\mathbb{E}_x\left[\langle f,X_t\rangle^2\right]=\varphi(x)\left(\langle f^2,\tilde{\varphi}\rangle+\beta(m_2-m)\int_0^\infty \mathrm{e}^{-\lambda s}\left\langle\mathtt{T}_s\left[f\right]^2,\tilde{\varphi}\right\rangle\mathrm{d}s\right). \tag{2.10}$$

Proof. For this proof, we will follow similar calculations as in [3]. First notice that from (1.10) we get that there exist a constant K such that

$$\sup_{t\geq 0}\left\|\mathrm{e}^{-\lambda_* t}\mathtt{T}_t\left[f\right]\right\|_\infty\leq K\left\|f\right\|_\infty. \tag{2.11}$$

Then, we recall that we can write the second moment of $\langle f,X_t\rangle$ in terms of the linear semi group $\mathtt{T}_t\left[f\right]$ as

$$\mathbb{E}_x\left[\langle f,X_t\rangle^2\right]=\mathtt{T}_t\left[f^2\right](x)+\beta(m_2-m)\int_0^t\mathtt{T}_s\left[\mathtt{T}_{t-s}\left[f\right]^2\right](x)\mathrm{d}s. \tag{2.12}$$

In the critical case, note that the first term in the right hand side of (2.12) is $o(t)$ because of (2.11) with $\lambda = 0$, so it is enough to show that

$$\left| \int_0^t \mathtt{T}_s \left[\mathtt{T}_{t-s} [f]^2 \right] (x) \mathrm{d}s - t \left\langle \langle f, \tilde{\varphi} \rangle^2 \varphi^2, \tilde{\varphi} \right\rangle \varphi(x) \right| = o(t).$$

To do so, we split the integral in the right hand side of (2.12) in two parts and look at the dominant growth rate as $t \to \infty$

$$\int_0^t \mathtt{T}_s \left[\mathtt{T}_{t-s} [f]^2 \right] (x) \mathrm{d}s = \int_0^{t-t_0} \mathtt{T}_s \left[\mathtt{T}_{t-s} [f]^2 \right] (x) \mathrm{d}s + \int_{t-t_0}^t \mathtt{T}_s \left[\mathtt{T}_{t-s} [f]^2 \right] (x) \mathrm{d}s. \tag{2.13}$$

Looking at the second term in the right hand side, we notice that

$$\begin{aligned} \int_{t-t_0}^t \mathtt{T}_s \left[\mathtt{T}_{t-s} [f]^2 \right] (x) \mathrm{d}s &\leq \int_{t-t_0}^t \mathtt{T}_s \left[K^2 \|f\|_\infty^2 \right] (x) \mathrm{d}s \qquad \text{by (2.11)} \\ &\leq \int_{t-t_0}^t K^3 \|f\|_\infty^2 \, \mathrm{d}s \qquad \text{by (2.11)} \\ &= t_0 K^3 \|f\|_\infty^2 \\ &= o(t). \end{aligned}$$

On the other hand, for the first term we have that

$$\begin{aligned} &\left| \int_0^{t-t_0} \mathtt{T}_s \left[\mathtt{T}_{t-s} [f]^2 \right] (x) \mathrm{d}s - \int_0^{t-t_0} \left\langle \varphi^2, \tilde{\varphi} \right\rangle \langle f, \tilde{\varphi} \rangle^2 \varphi(x) \mathrm{d}s \right| \\ &\qquad \leq \int_0^{t-t_0} \left| \mathtt{T}_s \left[\mathtt{T}_{t-s} [f]^2 \right] (x) - \varphi(x) \left\langle \mathtt{T}_{t-s} [f]^2, \tilde{\varphi} \right\rangle \right| \mathrm{d}s \\ &\qquad\quad + \int_0^{t-t_0} \varphi(x) \left| \left\langle \mathtt{T}_{t-s} [f]^2, \tilde{\varphi} \right\rangle - \left\langle (f, \tilde{\varphi})^2 \varphi^2, \tilde{\varphi} \right\rangle \right| \mathrm{d}s \\ &\qquad \leq \int_0^{t-t_0} \mathrm{e}^{-\gamma(t-s)} \left\| \mathtt{T}_{t-s} [f]^2 \right\|_\infty ds + \int_0^{t-t_0} \varphi(x) \left\langle \left| \mathtt{T}_{t-s} [f]^2 - \langle f, \tilde{\varphi} \rangle^2 \varphi^2 \right|, \tilde{\varphi} \right\rangle \mathrm{d}s, \end{aligned} \tag{2.14}$$

where the last inequality is given by (2.7) for the first integral, and this is bounded by $K^2 \|f\|_\infty^2 \int_0^{t-t_0} \mathrm{e}^{-\gamma(t-s)} ds$ by (2.11). To bound the difference inside the second integral, we use that

$$|f^2 - g^2| \leq \|f - g\|_\infty \left(\|f\|_\infty + \|g\|_\infty \right) \tag{2.15}$$

to get, for $0 \leq s \leq t - t_0$ that

$$
\begin{aligned}
\left|\mathtt{T}_{t-s}[f]^2 - \langle f, \tilde{\varphi}\rangle^2 \varphi^2\right| &\leq \left\|\mathtt{T}_{t-s}[f] - \langle f, \tilde{\varphi}\rangle \varphi\right\|_\infty \left(\left\|\mathtt{T}_{t-s}[f]\right\|_\infty + |\langle f, \tilde{\varphi}\rangle| \, \|\varphi\|_\infty\right) \\
&\leq \mathrm{e}^{-\gamma(t-s)} \|f\|_\infty \left(K \|f\|_\infty + |\langle f, \tilde{\varphi}\rangle| \, \|\varphi\|_\infty\right),
\end{aligned}
$$

where the last inequality comes from by (2.7) and (2.11), and then (2.14) is less than or equal to

$$
\left(K^2 \|f\|_\infty^2 + \|f\|_\infty \left(K \|f\|_\infty + |\langle f, \tilde{\varphi}\rangle| \, \|\varphi\|_\infty\right) \varphi(x) \, |\langle 1, \tilde{\varphi}\rangle|\right) \int_0^{t-t_0} \mathrm{e}^{-\gamma(t-s)} \mathrm{d}s,
$$

which is of order $o(t)$, so then the result follows in the critical case.

Now lets assume that $\lambda > 0$ and note that the first term in the right hand side of (2.12) is of order $o(\mathrm{e}^{2\lambda t})$ because of (2.11). So we only need to focus on the integral in the second term which we divide in two parts as in (2.13), in which we have for the second integral that

$$
\begin{aligned}
\int_{t-t_0}^{t} \mathtt{T}_s\left[(\mathtt{T}_{t-s}[f])^2\right](x) \mathrm{d}s &\leq \int_{t-t_0}^{t} \mathtt{T}_s\left[K^2 \mathrm{e}^{2\lambda(t-s)} \|f\|_\infty^2\right] \mathrm{d}s \qquad \text{by (2.11)} \\
&\leq K^3 \mathrm{e}^{2\lambda t} \|f\|_\infty^2 \int_{t-t_0}^{t} \mathrm{e}^{-2\lambda s} \mathrm{d}s \qquad \text{by (2.11)} \\
&= o(\mathrm{e}^{2\lambda t}),
\end{aligned}
$$

as $t \to \infty$. For the other integral we have that

$$
\begin{aligned}
&\left|\int_0^{t-t_0} \mathtt{T}_s\left[\mathtt{T}_{t-s}[f]^2\right](x) ds - \langle f, \tilde{\varphi}\rangle^2 \int_0^{t-t_0} \mathrm{e}^{2\lambda(t-s)} \mathtt{T}_s\left[\varphi^2\right](x) \mathrm{d}s\right| \\
&\leq \int_0^{t-t_0} \mathtt{T}_s\left[\left|\mathtt{T}_{t-s}[f]^2 - \langle f, \tilde{\varphi}\rangle^2 \mathrm{e}^{2\lambda(t-s)} \varphi^2\right|\right](x) \mathrm{d}s \\
&\leq \int_0^{t-t_0} \mathtt{T}_s\left[\left\|\mathtt{T}_{t-s}[f] - \mathrm{e}^{\lambda(t-s)} \langle f, \tilde{\varphi}\rangle \varphi\right\|_\infty \left(\left\|\mathtt{T}_{t-s}[f]\right\|_\infty + \left\|\mathrm{e}^{\lambda(t-s)} \langle f, \tilde{\varphi}\rangle \varphi\right\|_\infty\right)\right](x) \mathrm{d}s \\
&\hspace{30em} \text{by (2.15)} \\
&\leq \int_0^{t-t_0} \mathtt{T}_s\left[\mathrm{e}^{2\lambda(t-s)} \mathrm{e}^{-\gamma(t-s)} K \|f\|_\infty \left(\|f\|_\infty + |\langle f, \tilde{\varphi}\rangle| \, \|f\|_\infty\right)\right](x) \mathrm{d}s \\
&\hspace{26em} \text{by (2.7) and (2.11)} \\
&\leq K^2 \|f\|_\infty \left(\|f\|_\infty + |\langle f, \tilde{\varphi}\rangle| \, \|\varphi\|_\infty\right) \mathrm{e}^{2\lambda t} \mathrm{e}^{-\gamma t} \int_0^{t-t_0} \mathrm{e}^{-\lambda s} \mathrm{d}s \qquad \text{by (2.11)} \\
&= o(\mathrm{e}^{2\lambda t}),
\end{aligned}
$$

as $t \to \infty$. This implies that

$$\mathrm{e}^{-2\lambda t} \int_0^{t-t_0} \mathtt{T}_s \left[\mathtt{T}_{t-s}\left[f\right]^2\right](x)\mathrm{d}s \sim \langle f, \tilde{\varphi}\rangle^2 \int_0^{\infty} \mathrm{e}^{-2\lambda s}\mathtt{T}_s\left[\varphi^2\right](x)\mathrm{d}s,$$

from which the asymptotic behaviour follows for the super critical case.

Finally, in the sub-critical case ($\lambda < 0$) we begin similarly as above by noticing for the first term in the right hand side of (2.12) that by (2.7),

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda t}\mathtt{T}_t\left[f^2\right](x) = \varphi(x)\left\langle f^2, \tilde{\varphi}\right\rangle,$$

which corresponds to the first term in the right hand side of (2.10). For the second one, we begin with the following change of variable in the integral in (2.12)

$$\int_0^t \mathtt{T}_s\left[\mathtt{T}_{t-s}\left[f\right]^2\right](x)\mathrm{d}s = \int_0^t \mathtt{T}_{t-s}\left[\mathtt{T}_s\left[f\right]^2\right](x)\mathrm{d}s,$$

then we proceed as before by splitting this integral two parts and noticing that

$$\begin{aligned}
\int_{t-t_0}^t \mathtt{T}_{t-s}\left[\mathtt{T}_s\left[f\right]^2\right](x)\mathrm{d}s &\le \int_{t-t_0}^t \mathtt{T}_{t-s}\left[K^2\left\|f\right\|_\infty^2 \mathrm{e}^{2\lambda s}\right](x)\mathrm{d}s \qquad \text{by (2.11)}\\
&\le \int_{t-t_0}^t K^3\left\|f\right\|_\infty^2 \mathrm{e}^{2\lambda s}\mathrm{e}^{\lambda(t-s)}\mathrm{d}s \qquad \text{by (2.11)}\\
&\le K^3\left\|f\right\|_\infty^2 \mathrm{e}^{\lambda t}\int_{t-t_0}^t \mathrm{e}^{\lambda s}\mathrm{d}s\\
&= o(\mathrm{e}^{\lambda t}),
\end{aligned}$$

as $t \to \infty$. On the other hand, for the second term we have that

$$\begin{aligned}
&\left|\int_0^{t-t_0} \mathtt{T}_{t-s}\left[\mathtt{T}_s\left[f\right]^2\right](x)\mathrm{d}s - \varphi(x)\int_0^{t-t_0} \mathrm{e}^{\lambda(t-s)}\left\langle \mathtt{T}_s\left[f\right]^2, \tilde{\varphi}\right\rangle \mathrm{d}s\right|\\
\le &\int_0^{t-t_0} \left|\mathtt{T}_{t-s}\left[\mathtt{T}_s\left[f\right]^2\right](x) - \varphi(x)\mathrm{e}^{\lambda(t-s)}\left\langle \mathtt{T}_s\left[f\right]^2, \tilde{\varphi}\right\rangle\right|\mathrm{d}s\\
\le &\int_0^{t-t_0} \mathrm{e}^{\lambda(t-s)}\mathrm{e}^{-\gamma(t-s)}\left\|\mathtt{T}_s\left[f\right]^2\right\|_\infty \mathrm{d}s \qquad \text{by (2.7)}\\
\le &\ \mathrm{e}^{\lambda t}\mathrm{e}^{-\gamma t}K^2\left\|f\right\|_\infty^2 \int_0^{t-t_0} \mathrm{e}^{\lambda s}\mathrm{d}s \qquad \text{by (2.11)}\\
= &o(\mathrm{e}^{\lambda t}),
\end{aligned}$$

as $t \to \infty$, which implies that

$$\lim_{t\to\infty} \mathrm{e}^{\lambda t} \int_0^{t-t_0} \mathtt{T}_{t-s} \left[\mathtt{T}_s \left[f \right]^2 \right] (x) \mathrm{d}s = \varphi(x) \int_0^{\infty} \mathrm{e}^{-\lambda s} \left\langle \mathtt{T}_s \left[f \right]^2, \tilde{\varphi} \right\rangle \mathrm{d}s,$$

from which the asymptotic behaviour in the sub critical case follows, and then completes the proof. □

This proposition gives us a first approach to the more general problem of moments of branching Markov processes. Here we can see the linear behaviour of the first moment in the critical case, and the exponential scaling of the non-critical cases.

After looking at the first moments, we are ready to study the more general case, for higher moments of branching Markov processes.

Chapter 3

Moments of branching Markov processes

This chapter contains the results presented in [34] regarding branching Markov processes. Our main results concern understanding the growth of the k-th moment functional in time

$$\mathtt{T}_t^{(k)}[f](x) := \mathbb{E}_{\delta_x}\left[\langle f, X_t\rangle^k\right], \qquad x \in E, f \in B^+(E), k \geq 1, t \geq 0.$$

For convenience, we will write $\mathtt{T}$ in preference of $\mathtt{T}^{(1)}$ throughout.

Before stating our main theorem, recall the assumptions (HI) and $(H2)$ from Chapter 1, which will be crucial in analysing the moments defined above.

Theorem 1 (Critical, $\lambda = 0$)**.** *Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda = 0$. Define*

$$\Delta_t^{(\ell)} = \sup_{x\in E, f\in B_1^+(E)} \left| t^{-(\ell-1)}\varphi(x)^{-1}\mathtt{T}_t^{(\ell)}[f](x) - 2^{-(\ell-1)}\ell!\, \langle f, \tilde{\varphi}\rangle^{\ell} \langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle^{\ell-1}\right|,$$

where

$$\mathbb{V}[\varphi](x) = \beta(x)\mathcal{E}_x\left(\langle \varphi, \mathcal{Z}\rangle^2 - \langle \varphi^2, \mathcal{Z}\rangle\right).$$

Then, for all $\ell \leq k$

$$\sup_{t\geq 1} \Delta_t^{(\ell)} < \infty \textit{ and } \lim_{t\to\infty} \Delta_t^{(\ell)} = 0. \tag{3.1}$$

Remark: The choice $t \geq 1$ is arbitrary, and it could be any strictly positive number.

The novel contribution of Theorem 1 is the general and precise polynomial growth of the

k-th moment. The only other comparable results are that of [21], which inspired this work and deals with the special case of a general *critical* branching particle processes and that the test function f is specifically taken to be the eigenfunction φ. During the writing up of this work, we discovered some similar results in [12] for the critical case with local and non space dependant branching.

There are two facts that stand out in this result. The first is the polynomial scaling, which is quite a delicate conclusion given that there is no exponential growth to rely on. The second is that, for $k \geq 3$, the scaled moment limit is expressed not in terms of the k-th moments in (1.12), but rather the second order moments.

In some sense, however, both the polynomial growth and the nature of the limiting constant are not entirely surprising given the folklore for the critical setting. More precisely, in at least some settings (see e.g. [21]), one would expect to see a Yaglom-type result at criticality. The latter would classically see convergence of $t^{-1}\langle f, X_t\rangle$ in law to an exponentially distributed random variable as $t \to \infty$, whose parameter is entirely determined by the second moments of X.

The next results present a significantly different picture for the supercritical and subcritical cases. For those settings, the exponential behaviour of the first moment semigroup becomes a dominant feature of the higher expected moments.

Theorem 2 (Supercritical, $\lambda > 0$)**.** *Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda > 0$. Redefine*

$$\Delta_t^{(\ell)} = \sup_{x\in E, f\in B_1^+(E)} \left| \mathrm{e}^{-\ell\lambda t}\mathtt{T}_t^{(\ell)}[f](x) - \ell!\langle f, \tilde{\varphi}\rangle^\ell L_\ell(x) \right|,$$

where $L_1(x) = \varphi(x)$ and we define iteratively for $k \geq 2$

$$L_k(x) = \int_0^\infty \mathrm{e}^{-\lambda k s}\mathtt{T}_s\left[\beta\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]_k^2}\prod_{\substack{j=1\\ j:k_j>0}}^N L_{k_j}(x_j)\right]\right](x)\mathrm{d}s,$$

where $[k_1,\dots,k_N]_k^2$ is the set of all non-negative N-tuples $(k_1,\dots,k_N)$ such that $\sum_{i=1}^N k_i = k$ and at least two of the k_i are strictly positive[1]*. Then, for all $\ell \leq k$ (3.1) holds.*

As with the critical setting, we could not find any existing result of this kind in the literature.

[1]We interpret $\sum_\emptyset = 0$ and $\prod_\emptyset = 1$.

The growth of the expected k-th moments appears to be entirely controlled by the growth of the k-th moment of the linear semigroup. Jensen's inequality easily shows that this is the minimal rate of growth, it turns out that it is the exact rate of growth. If we again appeal to folklore then this is again not necessarily surprising. In a number of settings, we would expect X to obey a strong law of large numbers (cf. [22, 17, 1, 38]) in the sense that

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda t}\langle f, X_t\rangle = \langle\tilde{\varphi}, f\rangle M^{\varphi}_{\infty},$$

where $(M^{\varphi}_t, t \geq 0)$ was defined in (1.11) and the limit holds either almost surely or in the sense of L_p moments, for $p > 1$.

Finally we turn to the decay of moments in the subcritical setting, which offers the heuristically appealing result that the k-th moment decays slower than the k-th moment of the linear semigroup.

Theorem 3 (Subcritical, $\lambda < 0$)**.** *Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine*

$$\Delta^{(k)}_t = \sup_{x\in E, f\in B^+_1(E)} \left|\varphi^{-1}\mathrm{e}^{-\lambda t}\mathrm{T}^{(k)}_t[f](x) - L_k\right|,$$

where we define iteratively $L_1 = \langle f, \varphi\rangle$ and for $k \geq 2$,

$$L_k = \langle f^k, \tilde{\varphi}\rangle + \int_0^\infty \mathrm{e}^{-\lambda s}\left\langle \beta\mathcal{E}.\left[\sum_{[k_1,\dots,k_N]^2_k} \binom{k}{k_1,\dots,k_m} \prod_{\substack{j=1\\ j:k_j>0}}^{N} \mathrm{T}^{(k_j)}_s[f](x_j)\right], \tilde{\varphi}\right\rangle \mathrm{d}s.$$

Then, for all $\ell \leq k$ (3.1) holds.

As alluded to above, it is heuristically appealing that the the k-th moment does not grow at the rate $\exp(-k\lambda t)$. On the other hand the actual growth rate $\exp(-\lambda t)$ is slightly less obvious but nonetheless the obvious candidate. The decay in mass to zero in the branching system would suggest that the k-th moment similarly does so, but no slower that the first moment.

First of all, we study the linear and non-linear evolution equations associated with $(X, \mathbb{P})$. The relation (1.26) then yields an evolution equation for the k-th moment in terms of the lower order moments. Using this and an inductive argument, along with several crucial results that we house in the appendix, yields the result of the three theorems.

3.1 Linear and non-linear semigroup equations

Associated with every linear semigroup of a branching process is a so-called many-to-one formula. Many-to-one formulae are not necessarily unique and the one we will develop here is slightly different from the usual construction because of nonlocality.

Suppose that $\xi = (\xi_t, t \geq 0)$, with probabilities $\mathbf{P} = (\mathbf{P}_x, x \in E)$, is the Markov process corresponding to the semigroup $\mathtt{P}$. Let us introduce a new Markov process $\hat{\xi} = (\hat{\xi}_t, t \geq 0)$ which evolves as the process ξ but at rate $\beta(x)\mathtt{m}[1](x)$ the process is sent to a new position in E, such that for all Borel $A \subset E$, the new position is in A with probability $\mathtt{m}[\mathbf{1}_A](x)/\mathtt{m}[1](x)$. We will refer to the latter as *extra jumps*. Note the law of the extra jumps is well defined thanks to the assumption (1.12), which ensures that $\sup_{x\in E}\mathtt{m}[1](x) = \sup_{x\in E}\mathcal{E}_x(\langle 1, \mathcal{Z}\rangle) < \infty$. Accordingly we denote the probabilities of $\hat{\xi}$ by $(\hat{\mathbf{P}}_x, x \in E)$. We can now state our many-to-one formula.

Lemma 1. *Write* $\mathtt{B}(x) = \beta(x)(\mathtt{m}[1](x) - 1)$, $x \in E$. *For* $f \in B^+(E)$ *and* $t \geq 0$, *we have*

$$\mathtt{T}_t[f](x) = \hat{\mathbf{E}}_x\left[\exp\left(\int_0^t \mathtt{B}(\hat{\xi}_s)\mathrm{d}s\right) f(\hat{\xi}_t)\right]. \tag{3.2}$$

Proof. From (1.6), we have, for $f \in B^+(E)$,

$$\mathtt{T}_t[f](x) = \mathtt{P}_t[f](x) + \int_0^t \mathtt{P}_s\left[\beta(\mathtt{m}[\mathtt{T}_{t-s}[f]] - \mathtt{T}_{t-s}[f])\right](x)\mathrm{d}s, \qquad t \geq 0, x \in E, \tag{3.3}$$

At the same time, suppose we denote the right-hand side of (3.2) by $\hat{\mathtt{T}}_t[f](x)$, $t \geq 0$. By conditioning this expectation on the first extra jump, we get, for $f \in B^+(E)$, $x \in E$ and $t \geq 0$,

$$\begin{aligned}\hat{\mathtt{T}}_t[f](x) &= \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathtt{m}[1](\xi_s)\mathrm{d}s}\mathrm{e}^{\int_0^t \mathtt{B}(\xi_s)\mathrm{d}s} f(\xi_t)\right] \\ &\quad + \mathbf{E}_x\left[\int_0^t \beta(\xi_s)\mathtt{m}[1](\xi_s)\mathrm{e}^{-\int_0^s \beta(\xi_u)\mathtt{m}[1](\xi_u)\mathrm{d}u}\mathrm{e}^{\int_0^s \mathtt{B}(\xi_u)\mathrm{d}u}\frac{\mathtt{m}[\hat{\mathtt{T}}_{t-s}[f]](\xi_s)}{\mathtt{m}[1](\xi_s)}\mathrm{d}s\right] \\ &= \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s} f(\xi_t)\right] + \mathbf{E}_x\left[\int_0^t \beta(\xi_s)\mathrm{e}^{-\int_0^s \beta(\xi_u)\mathrm{d}u}\mathtt{m}[\hat{\mathtt{T}}_{t-s}[f]](\xi_s)\mathrm{d}s\right]\end{aligned} \tag{3.4}$$

Now using Lemma A.1 we deduce that (3.4) solves (3.3). Grönwall's Lemma, the fact that $\beta \in B^+(E)$ and (1.12) for $k = 1$ ensure that the relevant integral equations have unique solutions. □

We now define a variant of the non-linear evolution equation (1.5) associated with X via

$$\mathtt{u}_t[f](x) = \mathbb{E}_{\delta_x}\left[1 - \prod_{i=1}^{N_t} f(x_i(t))\right], \qquad t \geq 0,\ x \in E,\ f \in B_1^+(E). \tag{3.5}$$

For $f \in B_1^+(E)$, define

$$\mathtt{A}[f](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N}(1 - f(x_i)) - 1 + \sum_{i=1}^{N} f(x_i)\right], \qquad x \in E.$$

Our first preparatory result relates the two semigroups $(\mathtt{u}_t, t \geq 0)$ and $(\mathtt{T}_t, t \geq 0)$.

Lemma 2. *For all $g \in B_1^+(E)$, $x \in E$ and $t \geq 0$, the non-linear semigroup $\mathtt{u}_t[g](x)$ satisfies*

$$\mathtt{u}_t[u](x) = \mathtt{T}_t[1 - f](x) - \int_0^t \mathtt{T}_s\left[\mathtt{A}[\mathtt{u}_{t-s}[g]]\right](x)\mathrm{d}s. \tag{3.6}$$

Proof. By splitting the expectation in (3.5) on the first branching event, we get, for $g \in B^+(E)$, $t \geq 0$ and $x \in E$,

$$\begin{aligned}\mathtt{u}_t[g](x) = \mathbf{E}_x&\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s}(1 - g(\xi_t))\right] \\ &+ \mathbf{E}_x\left[\int_0^t \beta(\xi_s)\mathrm{e}^{-\int_0^s \beta(\xi_u)\mathrm{d}u}\left[1 - \mathtt{H}[1 - \mathtt{u}_{t-s}[g]](\xi_s)\right]\mathrm{d}s\right],\end{aligned}$$

where

$$\mathtt{H}[g](x) = \mathcal{E}_x\left(\prod_{i=1}^{N} g(x_i)\right), \qquad g \in B^+(E), x \in E.$$

Using Lemma A.1 we can move the multiplicative potential with rate β to an additive potential in the above evolution equation to obtain

$$\begin{aligned}\mathtt{u}_t[g](x) &= \mathtt{P}_t[1 - g](x) + \int_0^t \mathtt{P}_s\left[\beta(1 - \mathtt{H}[1 - \mathtt{u}_{t-s}[g]] - \mathtt{u}_{t-s}[g])\right](x)\mathrm{d}s \\ &= \mathtt{P}_t[1 - g](x) + \int_0^t \mathtt{P}_s[(\mathtt{F} - \mathtt{A})[\mathtt{u}_{t-s}[g]]](x)\mathrm{d}s. \qquad (3.7)\end{aligned}$$

with

$$\mathtt{F}[f](x) = \beta(x)\mathcal{E}_x\left[\sum_{i=1}^{N} f(x_i) - f(x)\right] =: \beta(x)(\mathtt{m}[f](x) - f(x)), \qquad x \in E.$$

Now define $(\tilde{\mathtt{u}}_t, t \geq 0)$ via

$$\tilde{\mathtt{u}}_t[g](x) = \hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t \mathtt{B}(\hat{\xi}_s)\mathrm{d}s}\left(1 - g(\hat{\xi}_t)\right)\right] - \hat{\mathbf{E}}_x\left[\int_0^t \mathrm{e}^{\int_0^s \mathtt{B}(\hat{\xi}_u)\mathrm{d}u}\mathtt{A}\left[\tilde{u}_{t-s}[g]\right](\hat{\xi}_s)\mathrm{d}s\right], \tag{3.8}$$

for $x \in E, t \geq 0$ and $g \in B^+(E)$ such that $\|g\|_\infty \leq 1$. Note that for the moment we don't claim a solution to (3.8) exists. Our existing notation allows us to write

$$\mathtt{T}_t[1-g](x) = \hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t \mathtt{B}(\hat{\xi}_s)\mathrm{d}s}\left(1 - g(\hat{\xi}_t)\right)\right]$$

and, for convenience, we will define

$$\mathtt{K}_t[g](x) = \hat{\mathbf{E}}_x\left[\int_0^t \mathrm{e}^{\int_0^s \mathtt{B}(\hat{\xi}_u)\mathrm{d}u}\mathtt{A}\left[\tilde{u}_{t-s}[g]\right](\hat{\xi}_s)\mathrm{d}s\right],$$

so that $\tilde{\mathtt{u}}_t[g](x) = \mathtt{T}_t[1-g](x) - \mathtt{K}_t[g](x)$. By conditioning the right-hand side of (3.8) on the first extra jump of $\hat{\xi}$ (bearing in mind the dynamics of $\hat{\xi}$ given just before Lemma 1) we can check with the help of the Markov property that

$$\begin{aligned}
\mathtt{T}_t[1-g](x) =& \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathtt{m}[1](\xi_s)\mathrm{d}s}\mathrm{e}^{\int_0^t \mathtt{B}(\xi_s)\mathrm{d}s}(1 - g(\xi_t))\right] \\
&+ \mathbf{E}_x\left[\int_0^t \beta(\xi_\ell)\mathtt{m}[1](\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_s)\mathtt{m}[1]\mathrm{d}s}\mathrm{e}^{\int_0^t \mathtt{B}(\xi_s)\mathrm{d}s}\frac{\mathtt{m}[\mathtt{T}_{t-\ell}[1-g]](\xi_\ell)}{\mathtt{m}[1](\hat{\xi}_\ell)}\mathrm{d}\ell\right] \\
=& \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s}(1 - g(\xi_t))\right] + \mathbf{E}_x\left[\int_0^t \beta(\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_s)\mathrm{d}s}\mathtt{m}[\mathtt{T}_{t-\ell}[1-g]](\xi_\ell)\mathrm{d}\ell\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathtt{K}_t\left[g\right](x) = &\mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathtt{m}[1](\xi_s)\mathrm{d}s}\int_0^t \mathrm{e}^{\int_0^s \mathtt{B}(\xi_u)\mathrm{d}u}\mathtt{A}\left[\tilde{u}_{t-s}[g]\right](\xi_s)\mathrm{d}s\right] \\
&+ \mathbf{E}_x\left[\int_0^t \beta(\xi_\ell)\mathtt{m}[1](\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_u)\mathtt{m}[1](\xi_u)\mathrm{d}u}\right. \\
&\left.\left(\int_0^\ell \mathrm{e}^{\int_0^s \mathtt{B}(\xi_u)\mathrm{d}u}\mathtt{A}\left[\tilde{u}_{t-s}[g]\right](\xi_s)\mathrm{d}s + \mathrm{e}^{\int_0^\ell \mathtt{B}(\xi_u)\mathrm{d}u}\frac{\mathtt{m}[\mathtt{K}_{t-\ell}[g]](\xi_\ell)}{\mathtt{m}[1](\xi_\ell)}\right)\mathrm{d}\ell\right].
\end{aligned}$$

Exchanging the order of integration in the double integral and simplifying terms we have

$$\begin{aligned}
\mathtt{K}\,[g]\,(x) &= \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathtt{m}[1](\xi_s)\mathrm{d}s}\int_0^t \mathrm{e}^{\int_0^s \mathtt{B}(\xi_u)\mathrm{d}u}\mathtt{A}\left[\tilde{u}_{t-s}[g]\right](\xi_s)\mathrm{d}s\right] \\
&+ \mathbf{E}_x\left[\int_0^t \mathrm{e}^{\int_0^s \mathtt{B}(\xi_u)\mathrm{d}u}\mathtt{A}\left[\tilde{u}_{t-s}[g]\right](\xi_s)\left(\int_s^t \beta(\xi_\ell)\mathtt{m}[1](\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_u)\mathtt{m}[1](\xi_u)\mathrm{d}u}\mathrm{d}\ell\right)\mathrm{d}s\right] \\
&+ \mathbf{E}_x\left[\int_0^t \beta(\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_u)\mathrm{d}u}\mathtt{m}[\mathtt{K}_{t-\ell}[g]](\xi_\ell)\mathrm{d}\ell\right] \\
&= \mathbf{E}_x\left[\int_0^t \mathrm{e}^{-\int_0^s \beta(\xi_u)\mathrm{d}u}\mathtt{A}\left[\tilde{u}_{t-s}\left[g\right]\right](\xi_s)\mathrm{d}s\right] + \mathbf{E}_x\left[\int_0^t \beta(\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_u)\mathrm{d}u}\mathtt{m}[\mathtt{K}_{t-\ell}[g]](\xi_\ell)\mathrm{d}\ell\right]
\end{aligned}$$

Gathering terms this simplifies to

$$\begin{aligned}
\tilde{\mathtt{u}}_t[g](x) =& \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s}(1-g(\xi_t))\right] + \mathbf{E}_x\left[\int_0^t \beta(\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_s)\mathrm{d}s}\mathtt{m}[\tilde{\mathtt{u}}_{t-\ell}[g](x)](\xi_\ell)\mathrm{d}\ell\right] \\
&- \mathbf{E}_x\left[\int_0^t \mathrm{e}^{-\int_0^s \beta(\xi_u)\mathrm{d}u}\mathtt{A}\left[\tilde{u}_{t-s}[g]\right](\xi_s)\mathrm{d}s\right].
\end{aligned}$$

Finally, appealing to Lemma A.1 we get

$$\begin{aligned}
\tilde{\mathtt{u}}_t[g](x) =& \mathtt{P}_t[1-g](x) + \int_0^t \mathtt{P}_t\left[\beta\mathtt{m}\left[\tilde{u}_{t-s}[g]\right] - \mathtt{A}\left[\tilde{u}_{t-s}[g]\right] - \beta\tilde{u}_{t-s}[g]\right](x)\mathrm{d}s \\
=& \mathtt{P}_t[1-g](x) + \int_0^t \mathtt{P}_t\left[(\mathtt{F}-\mathtt{A})\left[\tilde{u}_{t-s}[g]\right]\right](x)\mathrm{d}s
\end{aligned}$$

and hence $(\tilde{\mathtt{u}}_t, t \geq 0)$ is a solution to (3.7). Note that it is possible to reverse all these arguments and calculations to show that solutions to (3.7) solve (3.8). Now that we know a solution exist, using $\beta \in B^+(E)$, the assumption (1.12) for $k = 1$ and Grönwall's Lemma on the difference of two solutions of (3.7) we get that (3.8) has a unique solution. In conclusion, $(u_t[g], t \geq 0)$ and $(\tilde{u}_t[g], t \geq 0)$ agree, which gives us the statement of the Lemma. □

3.2 Evolution equation for the kth moment

Next we turn our attention to the evolution equation generated by the k-th moment functional $\mathtt{T}_t^{(k)}$, $t \geq 0$. To this end, we start by observing that

$$\mathtt{T}_t^{(k)}[f](x) = (-1)^{k+1}\frac{\partial^k}{\partial\theta^k}\mathtt{u}_t[\mathrm{e}^{-\theta f}](x)\bigg|_{\theta=0}. \tag{3.9}$$

The following result gives us an iterative approach to writing the k-th moment functional in terms of lower order moment functionals.

Proposition 1. *Fix $k \geq 2$. Under the assumptions of Theorem 1, with the additional assumption that*

$$\sup_{x\in E, s\leq t} \mathtt{T}_s^{(\ell)}[f](x) < \infty, \qquad \ell \leq k-1, f \in B^+(E), t \geq 0, \tag{3.10}$$

it holds that

$$\mathtt{T}_t^{(k)}[f](x) = \mathtt{T}_t[f^k](x) + \int_0^t \mathtt{T}_s\left[\beta\eta_{t-s}^{(k-1)}[f]\right](x)\,\mathrm{d}s, \qquad t \geq 0, \tag{3.11}$$

where

$$\eta_{t-s}^{(k-1)}[f](x) = \mathcal{E}_x\left[\sum_{[k_1,\ldots,k_N]_k^2} \binom{k}{k_1,\ldots,k_N} \prod_{j=1}^N \mathtt{T}_{t-s}^{(k_j)}[f](x_j)\right],$$

and $[k_1,\ldots,k_N]_k^2$ is the set of all non-negative N-tuples $(k_1,\ldots,k_N)$ such that $\sum_{i=1}^N k_i = k$ and at least two of the k_i are strictly positive.

Proof. Recall from (3.6) that

$$\mathtt{u}_t[\mathrm{e}^{-\theta f}](x) = \mathtt{T}_t[1-\mathrm{e}^{-\theta f}](x) - \int_0^t \mathtt{T}_s\left[\mathtt{A}[\mathtt{u}_{t-s}[\mathrm{e}^{-\theta f}]]\right](x)\mathrm{d}s, \qquad t \geq 0. \tag{3.12}$$

It is clear that differentiating the first term k times and setting $\theta = 0$ on the right-hand side of (3.12) yields

$$\frac{\partial^k}{\partial\theta^k}\mathtt{T}_t[1-\mathrm{e}^{-\theta f}](x)\bigg|_{\theta=0} = (-1)^{k+1}\mathtt{T}_t[f^k](x). \tag{3.13}$$

Thus it remains to differentiate the second term on the right-hand side of (3.12) k times. To this end, without concern for passing derivatives through expectations, using the product

Leibniz rule in Lemma A.3 of the Appendix, we have

$$
\begin{aligned}
&-\frac{\partial^k}{\partial\theta^k}\mathtt{A}[\mathtt{u}_t[\mathrm{e}^{-\theta f}]](x)\bigg|_{\theta=0}\\
&\quad=\frac{\partial^k}{\partial\theta^k}\beta(x)\mathcal{E}_x\left[1-\prod_{i=1}^N\mathbb{E}_{\delta_{x_i}}[\mathrm{e}^{-\theta\langle f,X_t\rangle}]-\sum_{i=1}^N\mathbb{E}_{\delta_{x_i}}[1-\mathrm{e}^{-\theta\langle f,X_t\rangle}]\right]\\
&\quad=-\beta(x)\mathcal{E}_x\left[\sum_{k_1+\cdots+k_N=k}\binom{k}{k_1,\ldots,k_m}\prod_{j=1}^N(-1)^{k_j}\mathtt{T}_t^{(k_j)}[f](x_j)+(-1)^{k+1}\sum_{i=1}^N\mathtt{T}_t^{(k)}[f](x_i)\right]
\end{aligned}
$$

$$
(3.14)\qquad =\beta(x)\mathcal{E}_x\left[(-1)^{k+1}\sum_{k_1+\cdots+k_N=k}\binom{k}{k_1,\ldots,k_m}\prod_{j=1}^N\mathtt{T}_t^{(k_j)}[f](x_j)+(-1)^{k}\sum_{i=1}^N\mathtt{T}_t^{(k)}[f](x_i)\right].
$$

where the sum is taken over all non-negative integers $k_1,\cdots,k_N$ such that $\sum_{i=1}^N k_i=k$.

Next let us look in more detail at the sum/product term on the right hand (3.14). Consider the terms where only one of the k_i in the sum is positive, in which case $k_i=k$ and

$$
\binom{k}{k_1,\ldots,k_m}=1.
$$

There are N ways this can happen in the sum of the sum-product term and hence

$$
\begin{aligned}
&\sum_{k_1+\cdots+k_N=k}\binom{k}{k_1,\ldots,k_m}\prod_{j=1}^N\mathtt{T}_t^{(k_j)}[f](x_j)\\
&\qquad=\sum_{i=1}^N\mathtt{T}^{(k)}[f](x_i)+\sum_{[k_1,\ldots,k_N]_k^2}\binom{k}{k_1,\ldots,k_N}\prod_{j=1}^N\mathtt{T}_t^{(k_j)}[f](x_j),
\end{aligned}
$$

where $[k_1,\ldots,k_N]_k^2$ is the set of all non-negative N-tuples $(k_1,\ldots,k_N)$ such that $\sum_{i=1}^N k_i=k$ and at least two of the k_i are strictly positive. Substituting this back into (3.14) yields

$$
-\frac{\partial^k}{\partial\theta^k}\mathtt{A}[\mathtt{u}_t[\mathrm{e}^{-\theta f}]]\bigg|_{\theta=0}=(-1)^{k+1}\beta(x)\mathcal{E}_x\left[\sum_{[k_1,\ldots,k_N]_k^2}\binom{k}{k_1,\ldots,k_N}\prod_{j=1}^N\mathtt{T}_t^{(k_j)}[f](x_j)\right].
$$

Now let us return to the justification that we can pass the derivatives through the expectation in the above calculation, we first note that derivatives are limits and so an 'epsilon-delta' argument will ultimately require dominated convergence. This is where the assumption (1.12) and (3.10) come in. On the right-hand side of (3.14), each of the $\mathtt{T}_t^{(k_j)}[f](x_j)$ in the sum term are uniformly bounded by the assumption of the lemma as the collection $[k_1, \ldots, k_N]_k^2$ means that $0 \leq k_j \leq k-1$ for each $j = 1, \cdots, N$. Moreover, there can be at most k items in the sum/product. Noting that

$$\sum_{k_1+\cdots+k_N=k} \binom{k}{k_1, \ldots, k_m} = N^k, \tag{3.15}$$

the assumption (1.12) allows us to use a domination argument with the k-th order moment. Note that

Combining this with (3.13) and (3.12), using an easy dominated convergence argument to pull the k derivatives through the integral in t, then dividing by $(-1)^{k+1}$, we get (3.11), as required. □

3.3 Proof of Theorem 1

We will prove Theorem 1 by induction, starting with the case $k = 1$. In this case, (3.1) reads

$$\sup_{t\geq 1} \Delta_t < \infty \text{ and } \lim_{t\to\infty} \Delta_t = 0,$$

which holds due to (1.10).

We now assume that the theorem holds true in the branching Markov process setting for some $k \geq 1$ and proceed to show that (3.1) holds for all $\ell \leq k+1$.

To this end, first note that the induction hypothesis implies that (3.10) holds. Hence Proposition 1 tells us that

$$
\begin{aligned}
&\varphi(x)^{-1}t^{-k}\mathtt{T}_t^{(k+1)}[f](x)\\
&= \varphi(x)^{-1}t^{-k}\mathtt{T}_t[f^{(k+1)}](x)\\
&\qquad + \varphi(x)^{-1}t^{-k}\int_0^t \mathtt{T}_s\left[\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]^2_{k+1}}\binom{k+1}{k_1,\dots,k_N}\prod_{j=1}^N \mathtt{T}^{(k_j)}_{t-s}[f](x_j)\right]\right](x)\mathrm{d}s\\
&= \varphi(x)^{-1}t^{-k}\mathtt{T}_t[f^{(k+1)}](x)\\
&\qquad + \varphi(x)^{-1}t^{-(k-1)}\int_0^1 \mathtt{T}_{ut}\left[\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]^2_{k+1}}\binom{k+1}{k_1,\dots,k_N}\prod_{j=1}^N \mathtt{T}^{(k_j)}_{t(1-u)}[f](x_j)\right]\right](x)\mathrm{d}u,
\end{aligned} \tag{3.16}
$$

where we have used the change of variables $s = ut$ in the final equality.

We now make some observations that will simplify the expression on the right-hand side of (3.16) as $t \to \infty$. First note that due to (1.10), the first term on the right hand side of (3.16) will vanish as $t \to \infty$. Next, note that, if more than two of the k_i in the sum are strictly positive, then the renormalising by t^{k-1} will cause the associated summand to go to zero as well. For example, suppose without loss of generality that k_1 and k_2 are both strictly positive, we can write $t^{k-1} = t^{(k+1)-2} = t^{k_1-1}t^{k_2-1}t^{k_3}\dots t^{k_N}$. Now the induction hypothesis tells us that the correct normalisation of each of the terms in the product is t^{k_j-1}, which means that the item $\mathtt{T}^{(k_j)}_{t(1-u)}$ for a third $k_j > 0$ will be 'over normalised to zero' in the limit.

To make this heuristic rigorous, we can employ Theorem A.5 from the Appendix. To this end, let us set

$$
F(x,u,t) := \frac{1}{\varphi(x)t^{k-1}}\mathcal{E}_x\left[\sum_{[k_1,\dots,k_N]^3_{k+1}}\binom{k+1}{k_1,\dots,k_N}\prod_{j=1}^N \mathtt{T}^{(k_j)}_{t(1-u)}[f](x_j)\right], \tag{3.17}
$$

for $t \geq 1$, where $[k_1,\dots,k_N]^3_{k+1}$ is the subset of $[k_1,\dots,k_N]^2_{k+1}$, for which at least three of the k_i are strictly positive (which can be an empty set). The condition $t \geq 1$ comes in because we cannot define $F(x,u,t)$ for $t = 0$, so it needs to be defined away from zero. We will show that conditions (A.5) and (A.6) are satisfied via

$$
\sup_{x\in E, u\in[0,1]} \varphi(x)F(x,u,t) < \infty \text{ and } \lim_{t\to\infty}\sup_{u\in[0,1],x\in E}\varphi(x)F(x,u,t) = 0. \tag{3.18}
$$

First note that there are no more than $k+1$ of the k_i that are strictly greater than 1 in the product in (3.17). This follows from the fact that it is not possible to partition the set $\{1,\dots,k+1\}$ into more than $k+1$ non-empty blocks. Next note that

$$\frac{1}{t^{k-1}}\prod_{\substack{j=1\\ j:k_j>0}}^{N}\mathtt{T}^{(k_j)}_{t(1-u)}[f](x_j)=\frac{(t(1-u))^{k+1-\#\{j:k_j>0\}}}{t^{k-1}}\prod_{\substack{j=1\\ j:k_j>0}}^{N}\varphi(x_j)\cdot\frac{1}{\varphi(x_j)}\frac{\mathtt{T}^{(k_j)}_{t(1-u)}[f](x_j)}{(t(1-u))^{k_j-1}}.$$

The product term on the right-hand side is uniformly bounded in x_j and $t(1-u)$ on compact intervals due to boundedness of φ and the fact that (3.1) is assumed to hold for all $\ell\le k$ by induction. Moreover, if $\#\{j:k_j>0\}\le 1$, the set $[k_1,\dots,k_N]^3_{k+1}$ is empty, otherwise, the term $(t(1-u))^{k+1-\#\{j:k_j>0\}}/t^{k-1}$ is finite for all $t\ge 1$, say. From (3.15) and (1.12), we also observe that

$$\sup_{x\in E}\mathcal{E}_x\left[\sum_{[k_1,\dots,k_N]^3_{k+1}}\binom{k+1}{k_1,\dots,k_N}\right]\le\sup_{x\in E}\mathcal{E}_x\left[\langle 1,\mathcal{Z}\rangle^{k+1}\right]<\infty.$$

Taking these facts into account, it is now straightforward to see that the earlier given heuristic an be made rigorous and (3.18) holds. In particular, we can use dominated convergence to pass the limit in t through the expectation in (3.17) to achieve the second statement in (3.18).

As F belongs to the class of functions $\mathcal{C}$, defined just before Theorem A.5 in the Appendix, the aforesaid theorem tells us that

$$\lim_{t\to\infty}\sup_{x\in E}\left|\frac{1}{\varphi(x)}\int_0^1\mathtt{T}_{ut}[\varphi F(\cdot,u,t)](x)\mathrm{d}u\right|=0. \tag{3.19}$$

Returning to (3.16), since the sum there requires that at least two of the k_i are positive, this means that the only surviving terms in the limit are those that are combinations of two strictly positive terms k_i and k_j such that $i\neq j$ and $k_i+k_j=k+1$. This can be thought of as choosing $i,j\in\{1,\dots N\}$ with $i\neq j$, choosing $k_i\in\{1,\dots,k\}$ and then setting $k_j=k+1-k_i$. One should take care however to avoid double counting each pair (k_i,k_j).

Thus, we have

$$\frac{1}{t^k\varphi(x)}\mathtt{T}_t^{(k+1)}[f](x) = \frac{1}{\varphi(x)}\int_0^1 \mathtt{T}_{ut}\left[\frac{\beta(\cdot)}{2t^{k-1}}\mathcal{E}_\cdot\left[\sum_{i=1}^N\sum_{\substack{j=1\\ j\neq i}}^N\sum_{k_i=1}^k\binom{k+1}{k_i,\ k+1-k_i}\right.\right.$$

$$\left.\left.\mathtt{T}_{t(1-u)}^{(k_i)}[f](x_i)\mathtt{T}_{t(1-u)}^{(k+1-k_i)}[f](x_j)\right]\right](x)\mathrm{d}u, \tag{3.20}$$

where the factor of 1/2 appears to compensate for the aforementioned double counting.

In order to show that the right-hand side above delivers the required finiteness and limit (3.1), we again turn to Theorem A.5. For $x \in E$, $t \geq 1$ and $0 \leq u \leq 1$, in anticipation of using this theorem, we now re-define

$$F(x,u,t) := \frac{\beta(x)}{2\varphi(x)t^{k-1}}\mathcal{E}_x\left[\sum_{i=1}^N\sum_{\substack{j=1\\ j\neq i}}^N\sum_{k_i=1}^k\binom{k+1}{k_i,\ k+1-k_i}\mathtt{T}_{t(1-u)}^{(k_i)}[f](x_i)\mathtt{T}_{t(1-u)}^{(k+1-k_i)}[f](x_j)\right].$$

After some rearrangement, we have

$$F(x,u,t)$$

$$= \frac{\beta(x)(1-u)^{k-1}}{2\varphi(x)}\mathcal{E}_x\left[\sum_{i=1}^N\sum_{\substack{j=1\\ j\neq i}}^N\sum_{k_i=1}^k\binom{k+1}{k_i,\ k+1-k_i}\right.$$

$$\left.\varphi(x_i)\varphi(x_j)\frac{\mathtt{T}_{t(1-u)}^{(k_i)}[f](x_i)}{\varphi(x_i)(t(1-u))^{k_i-1}}\frac{\mathtt{T}_{t(1-u)}^{(k+1-k_i)}[f](x_j)}{\varphi(x_j)(t(1-u))^{k-k_i}}\right], \tag{3.21}$$

Using similar arguments to those given previously in the proof of (3.19) may, again, combine the induction hypothesis, simple combinatorics and dominated convergence to pass the limit as $t \to \infty$ through the expectation and show that

$$F(x,u) := \lim_{t\to\infty} F(x,u,t)$$

$$=(k+1)!(\langle\tilde\varphi,\mathbb{V}[\varphi]\rangle/2)^{k-1}\langle\tilde\varphi,f\rangle^{k+1}k\frac{(1-u)^{k-1}}{2\varphi(x)}\mathcal{E}_x\left[\sum_{i=1}^N\sum_{\substack{j=1\\ j\neq i}}^N\varphi(x_i)\varphi(x_j)\right]$$

$$=(k+1)!(\langle\tilde\varphi,\mathbb{V}[\varphi]\rangle/2)^{k-1}\langle\tilde\varphi,f\rangle^{k+1}k\frac{(1-u)^{k-1}}{2\varphi(x)}\mathbb{V}[\varphi](x), \tag{3.22}$$

for which one uses that

$$\mathcal{E}_x\Bigg[\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{k_i=1}^{k}\binom{k+1}{k_i,\ k+1-k_i}\varphi(x_i)\varphi(x_j)$$

$$\frac{k_i!\ \langle f,\tilde{\varphi}\rangle^{k_i}\ \langle \mathbb{V}[\varphi],\tilde{\varphi}\rangle^{k_i-1}}{2^{(k_i-1)}}\frac{(k+1-k_i)!\ \langle f,\tilde{\varphi}\rangle^{k+1-k_i}\ \langle \mathbb{V}[\varphi],\tilde{\varphi}\rangle^{k-k_i}}{2^{(k-k_i)}}\Bigg]$$

$$= (k+1)!(\langle\tilde{\varphi},\beta\mathbb{V}[\varphi]\rangle/2)^{k-1}\langle\tilde{\varphi},f\rangle^{k+1}k\mathcal{E}_x\left[\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\varphi(x_i)\varphi(x_j)\right]$$

Note that, thanks to the assumption (H2), the expression for $F(s,x)$ clearly satisfies (A.5).

Subtracting the right-hand side of (3.22) from the right-hand side of (3.21), again appealing to the induction hypotheses, specifically the second statement in (3.1), it is not difficult to show that, for each $\varepsilon\in(0,1)$,

$$\lim_{t\to\infty}\sup_{x\in E,u\in[0,\varepsilon)}|\varphi(x)F(x,u,t)-\varphi(x)F(x,u)|=0.$$

On the other hand, the first statement in the induction hypothesis (3.1) also implies that three exists a constant $C_k>0$ (which depends on k but not ε) such that

$$\lim_{t\to\infty}\sup_{x\in E,u\in[\varepsilon,1]}|\varphi(x)F(x,u,t)-\varphi(x)F(x,u)|\le C_k(1-\varepsilon)^{k-1}.$$

Since we may take ε arbitrarily close to 1, we conclude that (A.6) holds.

In conclusion, since the conditions of Theorem A.5 are now met, we get the two statements of (3.1) as a consequence. □

3.4 Proofs for the non-critical cases

We now give an outline of the main steps in the proof of Theorem 1 for the sub and supercritical cases. As previously mentioned, the ideas used in this section will closely follow those presented in the previous section for the proof of the critical case and so we leave the details to the reader. We first note that the Perron Frobenius behaviour in (H1) ensures the base case for the induction argument, regardless of the value of λ. We thus turn to the inductive step, assuming the result holds for $k-1$.

Proof of Theorem 2. Suppose for induction that the result is true for all ℓ-th integer moments with $1\ell \leq k-1$. From the evolution equation in Proposition 1, noting that $\sum_{j=1}^{N} k_j = k$, when the limit exists, we have

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda k t} \int_0^t \mathtt{T}_s \left[\beta \mathcal{E}_\cdot \left[\sum_{[k_1,\ldots,k_N]_k^2} \binom{k}{k_1,\ldots,k_N} \prod_{j=1}^{N} \mathtt{T}_{t-s}^{(k_j)}[f](x_j) \right]\right](x)\mathrm{d}s$$

$$= \lim_{t\to\infty} t \int_0^1 \mathrm{e}^{-\lambda(k-1)ut} \mathrm{e}^{-\lambda u t} \mathtt{T}_{ut}\Big[H[f](x,u,t)\Big](x)\mathrm{d}u, \tag{3.23}$$

where

$$H[f](x,u,t) := \beta(x)\mathcal{E}_x \left[\sum_{[k_1,\ldots,k_N]_k^2} \binom{k}{k_1,\ldots,k_N} \prod_{j=1}^{N} \mathrm{e}^{-\lambda k_j t(1-u)} \mathtt{T}_{t(1-u)}^{(k_j)}[f](x_j) \right].$$

It is easy to see that, pointwise in $x \in E$ and $u \in [0,1]$, using the induction hypothesis and (H2),

$$H[f](x) := \lim_{t\to\infty} H[f](x,u,t) = k!\langle f, \tilde{\varphi}\rangle^k \beta(x)\mathcal{E}_x \left[\sum_{[k_1,\ldots,k_N]_k^2} \prod_{\substack{j=1\\ j:k_j>0}}^{N} L_{k_j}(x_j) \right],$$

where we have again used the fact that the k_js sum to k to extract the $\langle f, \tilde{\varphi}\rangle^k$ term.

Using the expressions for $H[f](x,u,t)$ and $H[f](x)$ together with the definition of $L_k(x)$, we have, for any $\epsilon > 0$, as $t \to \infty$,

$$\sup_{x\in E, f\in B_1^+(E)} |\mathrm{e}^{-k\lambda t}\mathtt{T}_t^{(k)}[f] - k!\langle f, \tilde{\varphi}\rangle^k L_k|$$

$$\leq t \int_0^1 \mathrm{e}^{-\lambda(k-1)ut} \sup_{x\in E, f\in B_1^+(E)} \Big|\mathrm{e}^{-\lambda u t}\mathtt{T}_{ut}\left[H[f](\cdot,u,t) - H[f]\right]\Big| \mathrm{d}u + \epsilon, \tag{3.24}$$

where ϵ is an upper estimate for

$$\sup_{x\in E, f\in B_1^+(E)} k!\langle f, \tilde{\varphi}\rangle^k \int_t^\infty \mathrm{e}^{-\lambda k s}\mathtt{T}_s \left[\beta\mathcal{E}_\cdot \left[\sum_{[k_1,\ldots,k_N]_k^2} \prod_{\substack{j=1\\ j:k_j>0}}^{N} L_{k_j}(x_j) \right]\right](x)\mathrm{d}s. \tag{3.25}$$

Note, convergence to zero as $t \to \infty$ in (3.25) follows thanks to the induction hypothesis (ensuring that $L_{k_j}(x)$ is uniformly bounded), (H2) and the uniform boundedness of β.

The induction hypothesis, (H2) and dominated convergence again ensures that

$$\lim_{t\to\infty} \sup_{x\in E, f\in B_1^+(E), u\in[0,\varepsilon]} |H[f](\cdot,u,t) - H[f]| = 0 \tag{3.26}$$

As such, in (3.24), we can split the integral on the right-hand side over $[0,\varepsilon]$ and $(\varepsilon,1]$, for $\varepsilon\in(0,1)$. Using (3.26), we can ensure that, for any arbitrarily small $\varepsilon'>0$, making use of the boundedness in (H1), there is a global constant $C>0$ such that, for all t sufficiently large,

$$\begin{aligned}
t\int_0^\varepsilon & \mathrm{e}^{-\lambda(k-1)ut} \sup_{x\in E, f\in B_1^+(E)} \left|\mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}\left[H[f](\cdot,u,t) - H[f]\right]\right| \mathrm{d}u \\
&\le \varepsilon' C t\int_0^\varepsilon \mathrm{e}^{-\lambda(k-1)ut}\mathrm{d}u \\
&= \frac{\varepsilon' C}{\lambda(k-1)}(1-\mathrm{e}^{-\lambda(k-1)\varepsilon t}).
\end{aligned} \tag{3.27}$$

On the other hand, we can also control the integral over $(\varepsilon,1]$, again appealing to (H1), (H2) and (3.15) to ensure that

$$\sup_{x\in E, f\in B_1^+(E), u\in(\varepsilon,1]} \left|\mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}\left[H[f](\cdot,u,t) - H[f]\right]\right| < \infty.$$

We can again work with a (different) global constant $C>0$ such that

$$\begin{aligned}
t\int_\varepsilon^1 & \mathrm{e}^{-\lambda(k-1)ut} \sup_{x\in E, f\in B_1^+(E)} \left|\mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}\left[H[f](\cdot,u,t) - H[f]\right]\right| \mathrm{d}u \\
&\le C t\int_\varepsilon^1 \mathrm{e}^{-\lambda(k-1)ut}\mathrm{d}u \\
&= \frac{C}{\lambda(k-1)}(\mathrm{e}^{-\lambda(k-1)\varepsilon t} - \mathrm{e}^{-\lambda(k-1)t}).
\end{aligned} \tag{3.28}$$

In conclusion, using (3.27) and (3.28), we can take limits as $t\to\infty$ in (3.24) and the statement of the theorem follows. □

Proof of Theorem 3. First note that since we only compensate by $\mathrm{e}^{-\lambda t}$, the term $\mathtt{T}_t[f^k](x)$ that appears in equation (3.11) does not vanish after the normalisation. Due to assumption

(H1), we have

$$\lim_{t\to\infty}\varphi^{-1}(x)\mathrm{e}^{-\lambda t}\mathtt{T}_t[f^k](x)=\langle f^k,\tilde\varphi\rangle.$$

Next we turn to the integral term in (3.11). Define $[k_1,\dots,k_N]_k^{(n)}$, for $2\le n\le k$ to be the set of tuples $(k_1,\dots,k_N)$ with exactly n positive terms and whose sum is equal to k. Similar calculations to those given above yield

$$\begin{aligned}
&\frac{\mathrm{e}^{-\lambda t}}{\varphi(x)}\int_0^t \mathtt{T}_s\left[\beta\mathcal{E}_x\left[\sum_{[k_1,\dots,k_N]_k^2}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathtt{T}_{t-s}^{(k_j)}[f](x_j)\right]\right](x)\mathrm{d}s\\
&=t\int_0^1\sum_{n=2}^k \mathrm{e}^{\lambda(n-1)ut}\frac{\mathrm{e}^{-\lambda(1-u)t}}{\varphi(x)}\\
&\qquad\times\mathtt{T}_{(1-u)t}\left[\beta\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]_k^{(n)}}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}^{(k_j)}[f](x_j)\right]\right](x)\mathrm{d}u
\end{aligned}\tag{3.29}$$

Now suppose for induction that the result holds for all ℓ-th integer moments with $1\le\ell\le k-1$. Roughly speaking the argument can be completed by noting that the integral in the definition of L_k can be written

$$\int_0^\infty\sum_{n=2}^k \mathrm{e}^{\lambda(n-1)s}\left\langle\beta\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]_k^{(n)}}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathrm{e}^{-\lambda s}\mathtt{T}_s^{(k_j)}[f](x_j),\tilde\varphi\right\rangle\mathrm{d}s\tag{3.30}$$

which is convergent by appealing to (H2), the fact that $\beta\in B^+(E)$ and the induction hypothesis. As a convergent integral, it can be truncated at $t>0$ and the residual of the integral over (t,∞) can be made arbitrarily small by taking t sufficiently large. By changing variables in (3.30) when the integral is truncated at arbitrarily large t, so it is of a similar form to that of (3.29), we can subtract it from (3.29) to get

$$t\int_0^1\sum_{n=2}^k \mathrm{e}^{\lambda(n-1)ut}\left(\frac{\mathrm{e}^{-\lambda(1-u)t}}{\varphi(x)}\mathtt{T}_{(1-u)t}[H_{ut}^{(n)}]-\langle H_{ut}^{(n)},\tilde\varphi\rangle\right)\mathrm{d}u,$$

where

$$H_{ut}^{(n)}(x)=\beta\mathcal{E}_x\left[\sum_{[k_1,\dots,k_N]_k^{(n)}}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}^{(k_j)}[f](x_j)\right]$$

One proceeds to splitting the integral of the difference over $[0,1]$ into two integrals, one over $[0,1-\varepsilon]$ and one over $(1-\varepsilon,1]$. For the aforesaid integral over $[0,1-\varepsilon]$, we can control the behaviour of $\varphi^{-1}\mathrm{e}^{-\lambda(1-u)t}\mathtt{T}_{(1-u)t}[H_{ut}^{(n)}] - \langle H_{ut}^{(n)}, \tilde{\varphi}\rangle$ as $t \to \infty$, making it arbitrarily small, by appealing to uniform dominated control of its argument in square brackets thanks to (H1). The integral over $[0, 1-\varepsilon]$ can thus be bounded, as $t \to \infty$, by $t(1 - \mathrm{e}^{\lambda(n-1)(1-\varepsilon)})/|\lambda|(n-1)$.

For the integral over $(1-\varepsilon, 1]$, we can appeal to the uniformity in (H1) and (H2) to control the entire term $\mathrm{e}^{-\lambda(1-u)t}\mathtt{T}_{(1-u)t}[H_{ut}^{(n)}]$ (over time and its argument in the square brackets) by a global constant. Up to a multiplicative constant, the magnitude of the integral is thus of order

$$t\int_{1-\varepsilon}^{1} \mathrm{e}^{\lambda(n-1)ut}\mathrm{d}u = \frac{1}{|\lambda|(n-1)}(\mathrm{e}^{\lambda(n-1)(1-\varepsilon)t} - \mathrm{e}^{\lambda(n-1)t}),$$

which tends to zero as $t \to \infty$. $\square$

Remark: In section 2.3 we give a result about the limit behaviour of the second moment of a local Markov branching process in the sub, super and critical cases. These are now easily obtained taking $k = 2$ in Theorems 1, 2 and 3.

Chapter 4

Integrated moments of branching Markov processes

In this chapter we present some results on some quantities that we call integrated moments. As mentioned before, the method we used to get the main results for the moments of branching Markov processes is remarkably robust. Indeed, as we will show, careful consideration of the proofs of Theorems 1, 2 and 3 demonstrate that we can also conclude results for the quantities

$$\mathtt{M}_t^{(k)}[g](x) := \mathbb{E}_{\delta_x}\left[\left(\int_0^t \langle g, X_s\rangle\,\mathrm{d}s\right)^k\right], \qquad x \in E, g \in B^+(E), k \geq 1, t \geq 0.$$

We can think of $\int_0^t \langle g, X_s\rangle\,\mathrm{d}s$ as characterising the running occupation measure $\int_0^t X_s(\cdot)\mathrm{d}s$ of the process X and hence we refer to $\mathtt{M}_t^{(k)}[g](x)$ as the k-th moment of the running occupation. The following results also emerge from our calculations, mirroring Theorems 1, 2 and 3 respectively.

4.1 Non-linear semigroup equation

We now define a variant of the non-linear evolution equation (1.5) associated with X via

$$\mathtt{u}_t[f,g](x) = \mathbb{E}_{\delta_x}\left[1 - \mathrm{e}^{-\langle f, X_t\rangle - \int_0^t \langle g, X_s\rangle \mathrm{d}s}\right], \qquad t \geq 0,\ x \in E,\ f, g \in B^+(E). \tag{4.1}$$

For $f \in B_1^+(E)$, define

$$\mathtt{A}[f](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^N (1-f(x_i)) - 1 + \sum_{i=1}^N f(x_i)\right], \qquad x \in E.$$

Our first preparatory result relates the two semigroups $(\mathtt{u}_t, t \geq 0)$ and $(\mathtt{T}_t, t \geq 0)$.

Lemma 3. *For all $f, g \in B^+(E)$, $x \in E$ and $t \geq 0$, the non-linear semigroup $\mathtt{u}_t[f,g](x)$ satisfies*

$$\mathtt{u}_t[f,g](x) = \mathtt{T}_t[1-\mathrm{e}^{-f}](x) - \int_0^t \mathtt{T}_s\left[\mathtt{A}[\mathtt{u}_{t-s}[f,g]] - g(1-\mathtt{u}_{t-s}[f,g])\right](x)\mathrm{d}s. \tag{4.2}$$

Proof. The proof uses standard techniques for integral evolution equations so we only sketch the proof. Instead of considering $\mathtt{u}_t[f,g]$, we will first work instead with

$$\mathtt{v}_t[f,g] = \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t\rangle - \int_0^t \langle g, X_s\rangle \mathrm{d}s}\right], \qquad t \geq 0,\ x \in E,\ f, g \in B^+(E), \tag{4.3}$$

which will turn out to be more convenient for technical reasons.

By splitting the expectation in (4.3) on the first branching event and appealing to the Markov property, we get, for $f, g \in B^+(E)$, $t \geq 0$ and $x \in E$,

$$\begin{aligned}\mathtt{v}_t[f,g](x) = \mathbf{E}_x&\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s}\mathrm{e}^{-f(\xi_t) - \int_0^t g(\xi_s)\mathrm{d}s}\right]\\ &+ \mathbf{E}_x\left[\int_0^t \beta(\xi_s)\mathrm{e}^{-\int_0^s \beta(\xi_u)+g(\xi_u)\mathrm{d}u}\mathtt{H}[\mathtt{v}_{t-s}[f,g]](\xi_s)\mathrm{d}s\right],\end{aligned}$$

where

$$\mathtt{H}[g](x) = \mathcal{E}_x\left[\prod_{i=1}^N g(x_i)\right], \qquad g \in B^+(E), x \in E.$$

Using Lemma A.1 we can move the multiplicative potential with rate $\beta + g$ to an additive potential in the above evolution equation to obtain

$$\mathtt{v}_t[f,g](x) = \hat{\mathtt{P}}_t[\mathrm{e}^{-f}](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathtt{v}_{t-s}[f,g]] - g\mathtt{v}_{t-s}[f,g]\right](x)\mathrm{d}s. \tag{4.4}$$

Now define

$$\mathtt{D}[f](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^N f(x_i) - \sum_{i=1}^N f(x_i)\right] = \beta(x)\left(\mathtt{H}[f](x) - \mathtt{m}[f](x)\right), \quad f \in B_1^+(E), x \in E$$

and $(\tilde{\mathtt{v}}_t, t \geq 0)$ via

$$
\begin{aligned}
\tilde{\mathtt{v}}_t[f,g](x) &= \mathtt{T}_t[\mathrm{e}^{-f}](x) + \int_0^t \mathtt{T}_s\left[\mathtt{D}\left[\tilde{\mathtt{v}}_{t-s}[f,g]\right] - g\tilde{\mathtt{v}}_{t-s}[f,g]\right](x)\mathrm{d}s \\
&= \hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t \mathtt{B}(\hat{\xi}_s)\mathrm{d}s}\mathrm{e}^{-f(\hat{\xi}_t)}\right] \\
&\quad + \hat{\mathbf{E}}_x\left[\int_0^t \mathrm{e}^{\int_0^s \mathtt{B}(\hat{\xi}_u)\mathrm{d}u}\left(\mathtt{D}\left[\tilde{\mathtt{v}}_{t-s}[f,g]\right](\hat{\xi}_s) - g(\hat{\xi}_s)\tilde{\mathtt{v}}_{t-s}[f,g](\hat{\xi}_s)\right)\mathrm{d}s\right],
\end{aligned}
\tag{4.5}
$$

for $x \in E, t \geq 0$ and $f, g \in B^+(E)$. Note that for the moment we don't claim a solution to (4.5) exists.

For convenience, we will define

$$
\mathtt{K}_t[f,g](x) = \hat{\mathbf{E}}_x\left[\int_0^t \mathrm{e}^{\int_0^s \mathtt{B}(\hat{\xi}_u)\mathrm{d}u}\left(\mathtt{D}\left[\tilde{\mathtt{v}}_{t-s}[f,g]\right](\hat{\xi}_s) - g(\hat{\xi}_s)\tilde{\mathtt{v}}_{t-s}[f,g](\hat{\xi}_s)\right)\mathrm{d}s\right],
$$

so that $\tilde{\mathtt{v}}_t[f,g](x) = \mathtt{T}_t[\mathrm{e}^{-f}](x) + \mathtt{K}_t[f,g](x)$. By conditioning the right-hand side of (4.5) on the first jump of $\hat{\xi}$ (bearing in mind the dynamics of $\hat{\xi}$ given just before Lemma 1) with the help of the Markov property (recalling that $\mathtt{B}(x) - \beta\mathtt{m}[1] = \beta$), we get

$$
\begin{aligned}
&\tilde{\mathtt{v}}_t[f,g](x) \\
&= \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s}\mathrm{e}^{-f(\xi_t)}\right] + \mathbf{E}_x\left[\int_0^t \beta(\xi_\ell)\mathtt{m}[1](\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_s)\mathrm{d}s}\frac{\mathtt{m}[\mathtt{T}_{t-\ell}[\mathrm{e}^{-f}]](\xi_\ell)}{\mathtt{m}[1](\xi_\ell)}\mathrm{d}\ell\right] \\
&+ \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_u)\mathtt{m}[1](\xi_u)\mathrm{d}u}\int_0^t \mathrm{e}^{\int_0^s \mathtt{B}(\xi_u)\mathrm{d}u}\left(\mathtt{D}\left[\tilde{\mathtt{v}}_{t-s}[f,g]\right](\xi_s) - g(\xi_s)[\tilde{\mathtt{v}}_{t-s}[f,g]\right)\mathrm{d}s\right] \\
&+ \mathbf{E}_x\Bigg[\int_0^t \beta(\xi_\ell)\mathtt{m}[1](\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_u)\mathtt{m}[1](\xi_u)\mathrm{d}u} \\
&\qquad\left(\int_0^\ell \mathrm{e}^{\int_0^s \mathtt{B}(\xi_u)\mathrm{d}u}\left(\mathtt{D}\left[\tilde{\mathtt{v}}_{t-s}[f,g]\right](\xi_s) - g(\xi_s)\mathtt{v}_{t-s}[f,g](\xi_s)\right)\mathrm{d}s\right. \\
&\qquad\qquad\left. + \mathrm{e}^{\int_0^\ell \mathtt{B}(\xi_u)\mathrm{d}u}\frac{\mathtt{m}[\mathtt{K}_{t-\ell}[g]](\xi_\ell)}{\mathtt{m}[1](\xi_\ell)}\right)\mathrm{d}\ell\Bigg].
\end{aligned}
$$

Gathering terms and exchanging the order of integration in the double integral, this simplifies to

$$
\begin{aligned}
&\tilde{\mathtt{v}}_t[f,g](x)\\
&= \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s}\mathrm{e}^{-f(\xi_t)}\right] + \mathbf{E}_x\left[\int_0^t \beta(\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_s)\mathrm{d}s}\mathtt{m}[\tilde{\mathtt{v}}_{t-\ell}[f,g](x)](\xi_\ell)\mathrm{d}\ell\right]\\
&+ \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_u)\mathtt{m}[1](\xi_u)\mathrm{d}u}\int_0^t \mathrm{e}^{\int_0^s \mathtt{B}(\xi_u)\mathrm{d}u}\left(\mathtt{D}\left[\tilde{\mathtt{v}}_{t-s}[f,g]\right](\xi_s) - g(\xi_s)[\tilde{\mathtt{v}}_{t-s}[f,g]\right)\mathrm{d}s\right]\\
&+ \mathbf{E}_x\left[\int_0^t\int_0^t \mathbf{1}_{(s\le\ell)}\beta(\xi_\ell)\mathtt{m}[1](\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_u)\mathtt{m}[1](\xi_u)\mathrm{d}u}\mathrm{e}^{\int_0^s \mathtt{B}(\xi_u)\mathrm{d}u}\right.\\
&\qquad\qquad\left.\left(\mathtt{D}\left[\tilde{\mathtt{v}}_{t-s}[f,g]\right](\xi_s) - g(\xi_s)\tilde{\mathtt{v}}_{t-s}[f,g](\xi_s))\right)\mathrm{d}\ell\,\mathrm{d}s\right]\\
=&\mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s}\mathrm{e}^{-f(\xi_t)}\right] + \mathbf{E}_x\left[\int_0^t \beta(\xi_\ell)\mathrm{e}^{-\int_0^\ell \beta(\xi_s)\mathrm{d}s}\mathtt{m}[\tilde{\mathtt{v}}_{t-\ell}[g](x)](\xi_\ell)\mathrm{d}\ell\right]\\
&+ \mathbf{E}_x\left[\int_0^t \mathrm{e}^{-\int_0^s \beta(\xi_u)\mathrm{d}u}\left(\mathtt{D}\left[\tilde{\mathtt{v}}_{t-s}[f,g]\right](\xi_s) - g(\xi_s)\tilde{\mathtt{v}}_{t-s}[f,g](\xi_s))\right)\mathrm{d}s\right].
\end{aligned}
$$

Finally, appealing Lemma A.1, we get

$$
\tilde{\mathtt{v}}_t[f,g](x) =\hat{\mathtt{P}}_t[\mathrm{e}^{-f}](x) + \int_0^t \mathtt{P}_t\left[\mathtt{G}\left[\tilde{\mathtt{v}}_{t-s}[f,g]\right] - g\tilde{\mathtt{v}}_{t-s}[f,g]\right](x)\mathrm{d}s,
$$

and hence $(\tilde{\mathtt{v}}_t, t\ge 0)$ is a solution to (4.4). A standard argument using using $\beta\in B^+(E)$, the assumption (1.12) for $k=1$ and Grönwall's Lemma tells us that all of the integral equations thus far have unique solutions. In conclusion, $(\mathtt{v}_t[g], t\ge 0)$ and $(\tilde{\mathtt{v}}_t[g], t\ge 0)$ agree.

To complete the lemma, note that

$$
1 - \mathtt{T}_t[\mathrm{e}^{-f}](x) = \mathtt{T}_t[1-\mathrm{e}^{-f}](x) + 1 - \mathtt{T}_t[1](x),
$$

moreover,

$$
1 - \mathtt{T}_t[1](x) = \hat{\mathbf{E}}_x\left[\int_0^t \mathtt{B}(\hat{\xi}_s)\mathrm{e}^{\int_0^s \mathtt{B}(\hat{\xi}_u)\mathrm{d}u}\mathrm{d}s\right] = \int_0^t \mathtt{T}_s[\mathtt{B}]\mathrm{d}s.
$$

Hence, working form (4.5) and the definitions of $\mathtt{D}$ and $\mathtt{A}$, which are related via

$$
\mathtt{D}[1-f](x) = \beta(x)\mathcal{E}_x\left[\prod_i(1-f(x_i)) - \sum_{i=1}^N(1-f(x_i))\right] = \mathtt{A}[f](x)+\mathtt{B}(x),\quad x\in E, f\in B_1^+(E),
$$

we get

$$\begin{aligned}\mathtt{u}_t[f,g](x) &= 1-\mathtt{v}_t[f,g](x)\\ &= 1-\mathtt{T}_t[\mathrm{e}^{-f}](x)-\int_0^t \mathtt{T}_s\left[\mathtt{D}\big[1-\mathtt{u}_{t-s}[f,g]\big]-g(1-\mathtt{u}_{t-s}[f,g])\right](x)\mathrm{d}s\\ &= \mathtt{T}_t[1-\mathrm{e}^{-f}](x)-\int_0^t \mathtt{T}_s\left[\mathtt{A}\big[\mathtt{u}_{t-s}[f,g]\big]-g(1-\mathtt{u}_{t-s}[f,g])\right](x)\mathrm{d}s,\end{aligned}$$

as required. □

Next we turn our attention to the evolution equation generated by the k-th moment functional $\mathtt{T}_t^{(k)}$, $t\geq 0$. To this end, we start by defining observing that

$$\mathtt{M}_t^{(k)}[g](x) = (-1)^{k+1}\frac{\partial^k}{\partial\theta^k}\mathtt{u}_t[0,\theta g](x)\bigg|_{\theta=0}. \tag{4.6}$$

Taking account of (4.2), we see that

$$\mathtt{u}_t[0,\theta g](x) = -\int_0^t \mathtt{T}_s\left[\mathtt{A}[\mathtt{u}_{t-s}[0,\theta g]]-\theta g(1-\mathtt{u}_{t-s}[0,\theta g])\right](x)\mathrm{d}s. \tag{4.7}$$

Given the proximity of (4.7) to (3.12), it is easy to see that we can apply the same reasoning that we used for $\mathtt{T}_t^{(k)}[f](x)$ to $\mathtt{M}_t^{(k)}[g](x)$ and conclude that, for $k\geq 2$,

$$\mathtt{M}_t^{(k)}[g](x) = \int_0^t \mathtt{T}_s\left[\beta\hat{\eta}_{t-s}^{(k-1)}[g]\right](x)-k\mathtt{T}_s[g\mathtt{M}_{t-s}^{(k-1)}[g]](x)\mathrm{d}s, \tag{4.8}$$

where $\hat{\eta}^k$ plays the role of η^k albeit replacing the moment operators $\mathtt{T}^{(j)}$ by the moment operators $\mathtt{M}^{(j)}$.

4.2 Limit behaviour of Integrated moments: critical case

Theorem 4 (Critical, $\lambda=0$). *Suppose that (H1) holds along with (H2) for $k\geq 2$ and $\lambda=0$. Define*

$$\Delta_t^{(\ell)} = \sup_{x\in E, f\in B_1^+(E)}\left|t^{-(2\ell-1)}\varphi(x)^{-1}\mathtt{M}_t^{(\ell)}[g](x)-2^{-(\ell-1)}\ell!\,\langle g,\tilde{\varphi}\rangle^{\ell}\,\langle\mathbb{V}[\varphi],\tilde{\varphi}\rangle^{\ell-1}L_\ell\right|,$$

where $L_1 = 1$ and L_k is defined through the recursion $L_k = (\sum_{i=1}^{k-1} L_i L_{k-i})/(2k-1)$. Then, for all $\ell \le k$

$$\sup_{t\ge 1} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t\to\infty} \Delta_t^{(\ell)} = 0. \tag{4.9}$$

Proof. We now proceed to prove Theorem 4, also by induction. First we consider the setting $k = 1$. In that case,

$$\frac{1}{t}\mathtt{M}^{(1)}[g](x) = \frac{1}{t}\mathbb{E}_{\delta_x}\left[\int_0^t \langle g, X_s\rangle \mathrm{d}s\right] = \frac{1}{t}\int_0^t \mathtt{T}_s[g](x)\mathrm{d}s = \int_0^1 \mathtt{T}_{ut}[g](x)\mathrm{d}u.$$

Referring now to Theorem A.5 in the Appendix, we can take $F(x, s, t) = f(x)/\varphi(x)$, since $f \in B^+(E)$, the conditions of the theorem are trivially met and hence

$$\lim_{t\to\infty} \sup_{x\in E, g\in B_1^+(E)} \left|\frac{1}{t}\mathtt{M}^{(1)}[g](x) - \langle g, \tilde{\varphi}\rangle\right| = 0.$$

Note that this limit sets the scene for the polynomial growth in $t^{n(k)}$ of the higher moments for some function $n(k)$. If we are to argue by induction, whatever the choice of $n(k)$, it must satisfy $n(1) = 1$.

Next suppose that Theorem 4 holds for all integer moments up to and including $k-1$. We have from (4.8) that

$$\frac{1}{t^{2k-1}}\mathtt{M}_t^{(k)}[g](x) = \frac{1}{t^{2k-1}}\int_0^t \mathtt{T}_s\left[\beta\hat{\eta}_{t-s}^{(k-1)}[g]\right](x)\mathrm{d}s - \frac{1}{t^{2k-1}}\int_0^t k\mathtt{T}_s[g\mathtt{M}_{t-s}^{(k-1)}[g]](x)\mathrm{d}s. \tag{4.10}$$

Let us first deal with the right most integral in (4.10). It can be written as

$$\frac{1}{t^{2k-2}}\int_0^1 k\mathtt{T}_{ut}\left[\varphi F(\cdot, u, t)\right](x)\mathrm{d}u := \int_0^1 (1-u)^{2k-2}k\mathtt{T}_{ut}\left[g\frac{1}{(t(1-u))^{2k-2}}\mathtt{M}_{t(1-u)}^{(k-1)}[g]\right](x)\mathrm{d}u.$$

Arguing as in the spirit of the proof of Theorem 1, our induction hypothesis ensures that

$$\lim_{t\to\infty} F[g](x, u, t) = \lim_{t\to\infty} g(1-u)^{2k-2}k\frac{1}{(t(1-u))^{2k-2}}\frac{\mathtt{M}_{t(1-u)}^{(k-1)}[g](x)}{\varphi(x)} = 0 =: F(x, u)$$

satisfies (A.5) and (A.6). Theorem A.5 thus tells us that, uniformly in $x \in E$ and $g \in B_1^+(E)$,

$$\lim_{t\to\infty} \frac{1}{t^{2k-1}}\int_0^t k\mathtt{T}_s[g\mathtt{M}_{t-s}^{(k-1)}[g]](x) = 0. \tag{4.11}$$

On the other hand, again following the style of the reasoning in the proof of Theorem 1, we can pull out the leading order terms, uniformly for $x \in E$ and $g \in B_1^+(E)$,

$$
\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t^{2k-1}} \int_0^t \mathtt{T}_s \left[\beta \hat{\eta}^{(k-1)}_{t-s}[g]\right](x)\mathrm{d}s \\
&= \lim_{t\to\infty} \int_0^1 \mathtt{T}_{ut} \left[\frac{\beta(\cdot)}{2} (1-u)^{2k-2} \mathcal{E}_\cdot \left[\sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \sum_{k_i=1}^{k-1} \binom{k}{k_i,\ k-k_i} \varphi(x_i)\varphi(x_j) \right.\right. \\
&\qquad\qquad \left.\left. \times \frac{\mathtt{M}^{(k_i)}_{t(1-u)}[g](x_i)}{\varphi(x_i)(t(1-u))^{2k_i-1}} \frac{\mathtt{M}^{(k-k_i)}_{t(1-u)}[g](x_j)}{\varphi(x_j)(t(1-u))^{2k-2k_i-1}} \right]\right](x)\mathrm{d}u. \qquad (4.12)
\end{aligned}
$$

It is again worth noting here that the choice of the polynomial growth in the form $t^{n(k)}$ also constrains the possible linear choices of $n(k)$ to $n(k) = 2k-1$ if we are to respect $n(1) = 1$ and the correct distribution of the index across (4.12).

Identifying

$$
\begin{aligned}
F[g](x,u,t) = \frac{\beta(x)}{2\varphi(x)} (1-u)^{2k-2} \mathcal{E}_x &\left[\sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \sum_{k_i=1}^{k-1} \binom{k}{k_i,\ k-k_i} \varphi(x_i)\varphi(x_j) \right. \\
&\left. \times \frac{\mathtt{M}^{(k_i)}_{t(1-u)}[g](x_i)}{\varphi(x_i)(t(1-u))^{2k_i-1}} \frac{\mathtt{M}^{(k-k_i)}_{t(1-u)}[g](x_j)}{\varphi(x_j)(t(1-u))^{2k-2k_i-1}} \right],
\end{aligned}
$$

our induction hypothesis allows us to conclude that $F[g](x,u) := \lim_{t\to\infty} F[g](x,u,t)$ exists and

$$
\varphi(x)F[g](x,u) = (1-u)^{2k-2} k! \frac{\beta(x)\mathbb{V}[\varphi](x)}{2^{k-1}} \langle g, \tilde{\varphi}\rangle^k \langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle^{k-1} \sum_{\ell=1}^{k-1} L_\ell L_{k-\ell}.
$$

Thanks to our induction hypothesis, we can also easily verify (A.5) and (A.6). Theorem A.5 now gives us the required uniform (in $x \in E$ and $g \in B_1^+(E)$) limit

$$
\lim_{t\to\infty} \frac{1}{t^{2k-1}} \int_0^t \mathtt{T}_s \left[\beta \hat{\eta}^{(k-1)}_{t-s}[g]\right](x)\mathrm{d}s = \frac{k! \langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle^{k-1} \langle g, \tilde{\varphi}\rangle^k}{2^{k-1}} L_k. \qquad (4.13)
$$

Putting (4.13) together with (4.11) we get the statement of Theorem 4. □

4.3 Non critical cases

Theorem 5 (Supercritical, $\lambda > 0$)**.** *Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda > 0$. Redefine*

$$\Delta_t^{(\ell)} = \sup_{x\in E, f\in B_1^+(E)} \left| \mathrm{e}^{-\ell\lambda t}\mathtt{M}_t^{(\ell)}[g](x) - \ \ell! \langle g, \tilde{\varphi}\rangle^\ell L_\ell(x)\right|,$$

where L_k was defined in Theorem 2, albeit that $L_1(x) = \varphi(x)/\lambda$.

Then, for all $\ell \leq k$

$$\sup_{t\geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t\to\infty} \Delta_t^{(\ell)} = 0. \tag{4.14}$$

Proof. For the case $k = 1$, we have

$$\begin{aligned}
&\left| \mathrm{e}^{-\lambda t} \int_0^t \mathtt{T}_s[g](x)\mathrm{d}s - \varphi(x)\frac{\langle g, \tilde{\varphi}\rangle}{\lambda}\right| \\
&\qquad = \left| \mathrm{e}^{-\lambda t} t \int_0^1 \mathrm{e}^{\lambda ut}\left(\mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}[g](x) - \varphi(x)\langle g, \tilde{\varphi}\rangle\right)\mathrm{d}u - \mathrm{e}^{-\lambda t}\frac{\langle g, \tilde{\varphi}\rangle}{\lambda}\right| \\
&\qquad \leq \mathrm{e}^{-\lambda t} t \int_0^1 \mathrm{e}^{\lambda ut}\left|\mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}[g](x) - \varphi(x)\langle g, \tilde{\varphi}\rangle\right|\mathrm{d}u + \mathrm{e}^{-\lambda t}\frac{\langle g, \tilde{\varphi}\rangle}{\lambda}.
\end{aligned} \tag{4.15}$$

Thanks to (H1) and similar arguments to those used in the proof of Theorem 2, we may choose t sufficiently large such that the modulus in the integral is bounded above by $\varepsilon > 0$, uniformly in $g \in B_1^+(E)$ and $x \in E$. Then, the right-hand side of (4.15) is bounded above by $\varepsilon\lambda^{-1}(1 - \mathrm{e}^{-\lambda t}) + \mathrm{e}^{-\lambda t}\langle g, \tilde{\varphi}\rangle/\lambda$. Since ε can be taken arbitrarily small, this gives the desired result and also pins down the initial value $L_1 = \varphi(x)/\lambda$.

Now assume the result holds for all $\ell \leq k - 1$. Reflecting on proof of Theorem 2, we note that in this setting the starting point is almost identical except that the analogue of (3.23), which is derived from (4.8), is now the need to evaluate

$$\begin{aligned}
\lim_{t\to\infty}\frac{\mathrm{e}^{-\lambda kt}}{\varphi(x)}\mathtt{M}_t^{(k)}[g](x) &= \lim_{t\to\infty}\frac{\mathrm{e}^{-\lambda kt}}{\varphi(x)}\int_0^t \mathtt{T}_s\left[\beta\mathcal{E}.\left[\sum_{[k_1,\dots,k_N]_k^2}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathtt{M}_{t-s}^{(k_j)}[g](x_j)\right]\right](x)\mathrm{d}s \\
&\quad - k\lim_{t\to\infty}\frac{\mathrm{e}^{-\lambda kt}}{\varphi(x)}\int_0^t \mathtt{T}_s[g\mathtt{M}_{t-s}^{(k-1)}[g]](x)\mathrm{d}s.
\end{aligned} \tag{4.16}$$

The first term on the right-hand side of (4.16) can be handled in essentially the same way as in the proof of Theorem 2. The second term on the right-hand side of (4.16) can easily be dealt with along the lines that we are now familiar with from earlier proofs, using the induction hypothesis. In particular, its limit is zero. Hence combined with the first term on the right-hand side of (4.16), we recover the same recursion equation for L_k. □

Theorem 6 (Subcritical, $\lambda < 0$)**.** *Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$.*

$$\Delta_t^{(\ell)} = \sup_{x\in E, f\in B_1^+(E)} \left|\mathtt{M}_t^{(\ell)}[g](x) - L_\ell(x)\right|,$$

where $L_1(x) = \int_0^\infty \mathtt{T}_s[g](x)\mathrm{d}s$ and for $k \geq 2$, the constants L_k are defined recursively via

$$L_k(x) = \int_0^\infty \mathtt{T}_s\left[\beta\mathcal{E}.\left[\sum_{[k_1,\dots,k_N]_k^2} \binom{k}{k_1,\dots,k_N} \prod_{\substack{j=1\\ j:k_j>0}}^N L_{k_j}(x_j)\right]\right](x)\,\mathrm{d}s$$
$$- k\int_0^\infty \mathtt{T}_s\left[gL_{k-1}(x)\right](x)\,\mathrm{d}s.$$

Then, for all $\ell \leq k$

$$\sup_{t\geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t\to\infty} \Delta_t^{(\ell)} = 0. \tag{4.17}$$

Proof. The case $k = 1$ is straightforward. Now suppose the result holds for all $\ell \leq k-1$. We again refer to (4.8), which means we are interested in handling a limit which is very similar to (4.16), now taking the form

$$\mathtt{M}_t^{(k)}[g](x) = t\int_0^1 \mathrm{e}^{\lambda ut}\mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}\left[\beta\mathcal{E}.\left[\sum_{[k_1,\dots,k_N]_k^2} \binom{k}{k_1,\dots,k_N} \prod_{j=1}^N \mathtt{M}_{t(1-u)}^{(k_j)}[g](x_j)\right]\right](x)\mathrm{d}u$$
$$- kt\int_0^1 \mathrm{e}^{\lambda ut}\mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}\left[g\mathtt{M}_{t(1-u)}^{(k-1)}[g]\right](x)\mathrm{d}u. \tag{4.18}$$

Again skipping the details, we can quickly see from (4.18) and the induction hypothesis that

$$\mathtt{M}_t^{(k)}[g](x) \sim \int_0^\infty \mathtt{T}_s\left[\beta\mathcal{E}.\left[\sum_{[k_1,\dots,k_N]_k^2} \binom{k}{k_1,\dots,k_N} \prod_{\substack{j=1\\ j:k_j>0}}^N L_{k_j}(x_j)\right]\right](x)\,\mathrm{d}s$$
$$- k\int_0^\infty \mathtt{T}_s\left[gL_{k-1}(x)\right](x)\,\mathrm{d}s.$$

which gives us the required recursion for L_k. □

The results in Theorems 4, 5 and 6 are slightly less predictable. For the supercritical case, the extra "linear" term arising from the time integral does not affect the exponential growth of the process, and hence the leading order behaviour is still dominated by $\mathrm{e}^{k\lambda t}$. In the critical case, we know from Theorem 1 that the first moment does not require normalisation, and hence integrating up to time t will induce a linear growth in time. As one can determine the k-th moment from a combination of the lower order moments, this linear growth propagates through the recursion, which along with the time integral, yields the t^{2k-1} scaling. Finally, in the subcritical case, we know that the total occupation $\int_0^\zeta \langle g, X_s\rangle \mathrm{d}s$, where $\zeta = \inf\{t > 0 : \langle 1, X_t\rangle = 0\}$, is finite, behaving like an average spatial distribution of mass, i.e. $\langle g, \tilde{\varphi}\rangle$, multiplied by ζ, meaning that no normalisation is required to control the "growth" of the running occupation moments in this case.

Chapter 5

First moments of superprocesses and a Yaglom limit for super-Brownian motion

In this chapter, we will consider a particular type of of superprocess. We are interested in superprocesses in a particular scenario, in this case we take a super-Brownian motion with local branching mechanism, that is, a $(\mathtt{P}, \psi)$ super-Brownian motion (SBM) for such $\phi = 0$, that is, no local branching. In this case, we have that the semigroup $(\mathtt{V}_t, t \geq 0)$ is characterised via the unique bounded positive solution to the evolution equation

$$\mathtt{V}_t[f](x) = \mathtt{P}_t[f](x) - \int_0^t \mathtt{P}_s \left[\psi(\cdot, \mathtt{V}_{t-s}[f](\cdot))\right](x)\mathrm{d}s. \tag{5.1}$$

We will use the corresponding version of the evolution equation (3.11) for our local branching setting to extrapolate the moments for SBM from the particle system moments. Finally we will see a Yaglom type limit based on results found in [39] for the particular case of the SBM.

Similarly to the contents of Chapter 2, these calculations formed an early stage of the research of this topic, and offer insight to the more general setting and corresponding results, that we deal with in Chapter 6.

5.1 Branching particle system with weights and Poisson initial particles

The purpose of this section is to have a first approach to the form of the moments of superprocesses. To do so, following the construction of superprocesses as a limit of branching particle systems as presented in [14] or [18], we take a particular branching particle system starting with a random number N of particles following a Poisson distribution with parameter $1/\rho$ at $t=0$, and in which each particle will carry a weight $\rho \geq 0$. Although the branching Markov process we are considering is very specific, it will help us to have an idea of the form of the moments that will be described in detail and for a more general case in later chapters. The following part will use some of the ideas presented in [14] and [36].

Let $\{X_t^\rho, t \geq 0\}$ be a branching Brownian motion on the space E with branching rate $\beta = 1/\rho$ and branching mechanism $\Phi_\rho(s) = \sum_k p_k s^k$ given by

$$\Phi_\rho(s) = s + \rho^2 \psi\left(\frac{1-s}{\rho}\right), \tag{5.2}$$

for $0 \leq s \leq 1$ and a target function ψ. Notice that $\Phi_\rho(s)$ is a generating function. In this branching process, each particle will carry a weight ρ, that is, if $X_t = \sum \delta_{x_i(t)}$, then $X_t^\rho = \sum \rho\delta_{x_i(t)}$. Let $\mathtt{G}_\rho(s) := (\Phi_\rho(s) - s)/\rho$ be the branching mechanism (2.2), then it is known that the non-linear semigroup

$$\mathtt{v}_t^\rho[f] := \mathbb{E}_{\delta_x}\left[\prod_{i=1}^{N_t} f(x_i(t)^\rho u(t))\right], \tag{5.3}$$

then

$$\frac{\partial}{\partial t}\mathtt{v}_t^\rho[f](x) = \frac{1}{2}\frac{\partial^2}{\partial x}\mathtt{v}_t^\rho[f](x) + G^\rho(\mathtt{v}_t^\rho[f](x)), \quad \mathtt{v}_0^\rho[f](x) = f(x). \tag{5.4}$$

We know that the super-Brownian motion can be seen as a limit of this branching particle system, if we take the limit when $\rho \to 0$. Remember we have a Poisson number of particles at time $t=0$ with parameter $1/\rho$.

5.1.1 First moments

Let us start looking at the first two moments of such particle system and what happens to the limit when we take $\rho \to 0$. Define $\mathsf{S}_t^\rho [f](x) := \mathbb{E}_{\delta_x}[\langle f, X_t^\rho \rangle]$ to be the first moment of the process, recalling that we start with a random number N of particles that follow a Poisson distribution with rate $1/\rho$ and law denoted by $\mathcal{P}^\rho$, then for $f \in B_1^+(E)$, $x \in E$,

$$
\begin{aligned}
\mathsf{S}_t^\rho [f](x) &= \mathcal{E}^\rho\left[\mathbb{E}_{\mathsf{N}\delta_x}[\langle f, X_t^\rho \rangle]\right] \\
&= \mathcal{E}^\rho\left[\mathsf{N}\mathbb{E}_{\delta_x}[\langle \rho f, X_t \rangle]\right] \\
&= \frac{1}{\rho}\mathbb{E}_{\delta_x}[\langle \rho f, X_t \rangle] \\
&= \mathbf{P}_x\left[\mathrm{e}^{(m^\rho - 1)t/\rho} f(\xi_t)\right],
\end{aligned}
$$

where $m^\rho = \mathcal{E}^\rho[\mathsf{N}]$. To take the limit when $\rho \to 0$ we need to know what is the order of m^ρ which can be obtained differentiating (5.2) with respect to s

$$
\begin{aligned}
m^\rho = \Phi_\rho'(s)\big|_{s\uparrow 1} &= \left(s + \rho^2 \psi\left(\frac{1-s}{\rho}\right)\right)'\Bigg|_{s\uparrow 1} \\
&= 1 - \rho\psi'\left(\frac{1-s}{\rho}\right)\Bigg|_{s\uparrow 1} \\
&= 1 - \rho\psi'(0+).
\end{aligned}
$$

Then we get that

$$
\mathsf{S}_t^\rho [f](x) = \mathbf{P}_x\left[\mathrm{e}^{-\psi'(0+)t} f(\xi_t)\right] =: \mathtt{T}_t[f](x). \tag{5.5}
$$

For the second moment, defined as $\mathsf{S}_t^{(2,\rho)} := \mathbb{E}_{\delta_x}\left[\langle f, X_t^\rho \rangle^2\right]$ it is a straightforward exercise to show that a simplification of equation (3.11) for $k = 2$ and our particular setting gives us

$$
\mathsf{S}_t^{(2,\rho)}[f](x) = \rho \binom{2}{2} \mathtt{T}_t^{(2),\rho}[f](x) + \frac{1}{2}\binom{2}{1,1}\left(\mathtt{T}_t^{(1),\rho}[f](x)\right)^2,
$$

where $\mathtt{T}^{(k),\rho}[f](x)$ correspond to the k-th moment of a particle branching process with all the characteristics but starting with a single particle at x and with a unit mass in each particle instead of ρ. Then using again (3.11) we have that this is equal to

$$
\begin{aligned}
&= \rho \mathtt{T}_t^{(1),\rho}\left[f^2\right](x) + \rho \frac{(m_2^\rho - m^\rho)}{\rho} \int_0^t \mathtt{T}_s^{(1),\rho}\left[\left(\mathtt{T}_{t-s}^{(1),\rho}[f]\right)^2\right](x)\mathrm{d}s + \left(\mathtt{T}_t^{(1),\rho}[f](x)\right)^2 \\
&= \rho \mathtt{T}_t\left[f^2\right] + (m_2^\rho - m^\rho) \int_0^t \mathtt{T}_s\left[(\mathtt{T}_{t-s}[f]^2)\right](x)\mathrm{d}s + (\mathtt{T}_t[f](x))^2,
\end{aligned}
$$

where we can obtain $m_2^\rho - m^\rho = \mathcal{E}^\rho\left[\mathsf{N}^2\right] - \mathcal{E}^\rho\left[\mathsf{N}\right] = \Phi_\rho''(1-)$, which is calculated via

$$
\Phi_\rho''(s)\Big|_{s=1-} = \psi''\left(\frac{1-s}{\rho}\right)\Big|_{s=1-} = \psi''(0+),
$$

then

$$
\mathtt{S}_t^{(2,\rho)}[f](x) = (\mathtt{T}_t[f](x))^2 + \rho \mathtt{T}_t\left[f^2\right](x) + \psi''(0+) \int_0^t \mathtt{T}_s\left[(\mathtt{T}_{t-s}[f])^2\right](x)\mathrm{d}s,
$$

and as $\rho \to 0$ we get that

$$
\mathtt{S}_t^{(2,\rho)}[f](x) \to (\mathtt{T}_t[f](x))^2 + \psi''(0+) \int_0^t \mathtt{T}_s\left[(\mathtt{T}_{t-s}[f])^2\right](x)\mathrm{d}s. \tag{5.6}
$$

Proposition 4. *Let $S_t^{(k,\rho)}$ denote the k-th moment of a branching Brownian motion with the characteristics described above, then*

$$
S_t^{(k,\rho)}[f](x) = \sum_{j=1}^{k} \frac{\rho^{k-j}}{j!} \sum_{[k_1,\ldots,k_j]_+} \binom{k}{k_1,\ldots,k_j} \prod_{i=1}^{j} \mathtt{T}_t^{(k_i),\rho}[f](x), \tag{5.7}
$$

where $\mathtt{T}^{(k_j),\rho}[f](x)$ correspond to the k-th moment of a particle branching process with all the characteristics but starting with a single particle at x and with a unit mass in each particle instead of ρ.

Proof. We have that

$$
\begin{aligned}
S_t^{(k,\rho)}[f](x) &= \mathcal{E}^\rho\left[\mathbb{E}_{\mathsf{N}\delta_x}\left[\langle f, X_t^\rho\rangle^k\right]\right] \\
&= \mathcal{E}^\rho\left[\mathbb{E}_{\delta_x}\left[\left(\sum_{i=1}^{\mathsf{N}}\langle f, X_t^\rho\rangle\right)^k\right]\right] \\
&= \mathcal{E}^\rho\left[\sum_{j=1}^{k}\frac{\mathsf{N}!}{(\mathsf{N}-j)!j!}\sum_{[k_1,\ldots,k_j]_+}\binom{k}{k_1,\ldots,k_j}\prod_{i=1}^{j}\mathbb{E}_{\delta x}\left[\left\langle f, X_t^{\rho,i}\right\rangle^{k_i}\right]\right],
\end{aligned}
\tag{5.8}
$$

where the last equality is due to (A.4) and the sum runs over the set $[k_1,\ldots,k_j]_+$ of all combinations of strictly positive $\{k_1,\ldots,k_j\}$ such that $\sum_{i=1}^{j} k_i = k$. Using the fact that the factorial moments of the Poisson distribution are equal to

$$
\mathcal{E}^\rho\left[\frac{(\mathsf{N}!}{(\mathsf{N}-j)!}\right] = \frac{1}{\rho^j},
$$

we have that (5.8) is equal to

$$
\begin{aligned}
&= \sum_{j=1}^{k}\frac{1}{\rho^j j!}\sum_{[k_1,\ldots,k_j]_+}\binom{k}{k_1,\ldots,k_j}\prod_{i=1}^{j}\rho^{k_j}\mathtt{T}_t^{(k_i),\rho}[f](x) \\
&= \sum_{j=1}^{k}\frac{\rho^{k-j}}{j!}\sum_{[k_1,\ldots,k_j]_+}\binom{k}{k_1,\ldots,k_j}\prod_{i=1}^{j}\mathtt{T}_t^{(k_i),\rho}[f](x),
\end{aligned}
$$

and we get the result. □

Appealing to (3.11), albeit simplified due to the local branching, we have that the moments $\mathtt{T}_t^{(k_j),\rho}[f]$ in (5.7) fulfull the following evolution equation

$$
\mathtt{T}_t^{(k),\rho}[f](x) = \mathtt{T}_t^{(1),\rho}\left[f^k\right](x) + \frac{1}{\rho}\int_0^t \mathtt{T}_s^{(1),\rho}\left[\tilde{\eta}\left[\mathtt{T}_{t-s}^{(1),\rho},\ldots,\mathtt{T}_{t-s}^{(k-1),\rho}; f\right]\right](x)ds,
$$

where

$$
\tilde{\eta}[g_1,\ldots,g_{k-1}; f] = \sum_{n=0}^{\infty} p_n \sum_{\substack{[k_1,\ldots,k_n] \\ k_j<k}}\binom{k}{k_1,\ldots,k_n}\prod_{i=1}^{n} g_{k_i}[f].
$$

We can observe an interesting behaviour in the critical case of higher moments of this process as t goes to ∞ (before taking the limit when $\rho \to 0$). It is straightforward to see that from

Theorem 1, we have that $\mathtt{T}_t^{(k),\rho}$ exhibits the following asymptotic behaviour

$$\lim_{t\to\infty}\frac{1}{t^{k-1}}\mathtt{T}_t^{(k),\rho}=\frac{k!}{2^{k-1}}\varphi(x)(\tilde{\varphi},f)^k\left[\frac{1}{\rho}\psi''(0+)(\tilde{\varphi},\varphi^2)\right]^{k-1}.$$

Then, using the formula for $S_t^{(k,\rho)}[f]$ given in (5.7), we have that

$$\begin{aligned}
\frac{1}{t^{k-1}}S_t^{(k,\rho)}[f](x) &= \sum_{j=1}^{k}\frac{\rho^{k-j}}{j!}\sum_{[k_1,\ldots,k_j]_+}\binom{k}{k_1,\ldots,k_j}\frac{1}{t^{j-1}}\prod_{i=1}^{j}\frac{1}{t^{k_i-1}}\mathtt{T}_t^{(k_i),\rho}[f](x)\\
&\overset{t\to\infty}{\sim}\sum_{j=1}^{k}\frac{\rho^{k-j}}{j!}\sum_{[k_1,\ldots,k_j]_+}\frac{k!}{k_1!\ldots k_j!}\frac{1}{t^{j-1}}\prod_{i=1}^{j}\frac{k_i!}{2^{k_i-1}}\varphi(x)(\tilde{\varphi},f)^{k_i}\left[\frac{1}{\rho}\psi''(0+)(\tilde{\varphi},\varphi^2)\right]^{k_i-1}\\
&=\sum_{j=1}^{k}\frac{\rho^{k-j}}{j!}\sum_{[k_1,\ldots,k_j]_+}\frac{k!}{t^{j-1}}\frac{1}{2^{k-j}}\varphi(x)^j(\tilde{\varphi},f)^k\left[\frac{1}{\rho}\psi''(0+)(\tilde{\varphi},\varphi^2)\right]^{k-j}\\
&\overset{t\to\infty}{\sim}\frac{k!}{2^{k-1}}\varphi(x)(\tilde{\varphi},f)^k\left[\psi''(0+)(\tilde{\varphi},\varphi^2)\right]^{k-1}.
\end{aligned}$$

In fact, we can take only the first term in the sum, as all the others will be "over scaled" and will tend to zero, so

$$\begin{aligned}
\frac{1}{t^{k-1}}S_t^{(k,\rho)}[f](x) &= \sum_{j=1}^{k}\frac{\rho^{k-j}}{j!}\sum_{[k_1,\ldots,k_j]_+}\binom{k}{k_1,\ldots,k_j}\frac{1}{t^{j-1}}\prod_{i=1}^{j}\frac{1}{t^{k_i-1}}\mathtt{T}_t^{(k_i),\rho}[f](x)\\
&=\rho^{k-1}\frac{1}{t^{k-1}}\mathtt{T}_t^{(k),\rho}[f](x)\\
&\quad+\sum_{j=2}^{k}\frac{\rho^{k-j}}{j!}\sum_{[k_1,\ldots,k_j]_+}\binom{k}{k_1,\ldots,k_j}\frac{1}{t^{j-1}}\prod_{i=1}^{j}\frac{1}{t^{k_i-1}}\mathtt{T}_t^{(k_i),\rho}[f](x)\\
&\overset{t\to\infty}{\to}\rho^{k-1}\frac{k!}{2^{k-1}}\varphi(x)(\tilde{\varphi},f)^k\left[\frac{1}{\rho}\psi''(0+)(\tilde{\varphi},\varphi^2)\right]^{k-1}\\
&=\frac{k!}{2^{k-1}}\varphi(x)(\tilde{\varphi},f)^k\left[\psi''(0+)(\tilde{\varphi},\varphi^2)\right]^{k-1},
\end{aligned}$$

which gives us a limit to a constant that does not depend on the weights ρ. We will see this limit in the following chapters. Notice that we have taken the limit when $t\to\infty$ before taking the limit when $\rho\to 0$ and still have obtained an explicit limit. We will explore in detail the asymptotic behaviour of the moments of superprocesses and comment on the relation they hold with their corresponding cases for branching particle systems in later chapters of this work.

5.2 Super-Brownian motion (SBM)

One can think of the SBM to be a rescaled limit of a BBM, where the mass of each particle tends to 0 and the expected lifetime of each goes to 0 and the number of particles at time $t = 0$ goes to infinity in a "commensurate" way. As mentioned above, this is achieved by considering a branching particle system starting with a random number of particles that follow a Poisson distribution which parameter will tend to infinity, see for example [14] or [18]. The process known as the Dawson Watanabe superprocess or super-Brownian motion (SBM), is a finite-measure-valued strong Markov process $\{X_t, t \geq 0\}$ whose evolution is characterised via its log-Laplace semi-group

$$-\log \mathbb{E}_\mu[\exp\{-\langle f, X_t\rangle\}] = \langle \mathtt{V}_t[f](x), \mu\rangle, \quad t \geq 0, \tag{5.9}$$

for all $f \in C_b^+(\mathbb{R}^d)$, the space of positive, uniformly bounded, continuous functions on $\mathbb{R}^d$, $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, the space of finite measures on $\mathbb{R}^d$, $\langle f, \mu\rangle$ is the inner product $\int f d\mu$, and $\mathtt{V}_t[f](x)$ the unique positive solution to the evolution equation

$$\frac{\partial}{\partial t}\mathtt{V}_t[f](x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\mathtt{V}_t[f](x) - \psi(\mathtt{V}_t[f](x)), \tag{5.10}$$

with initial condition $\mathtt{V}_0[f](x) = f(x)$. It is also known (see for example [13]) that the evolution equation of the mass of a SBM can be written as

$$\mathtt{V}_t[f](x) + \int_0^t \mathtt{P}_s[\psi(\mathtt{V}_{t-s}[f])]\,\mathrm{d}s(x) = \mathtt{P}_t[f](x), \tag{5.11}$$

with branching mechanism ψ defined in (1.18).

The following definition and result will give us a first look to the form of the first two moments of a SBM, which we will study in detail in the following section.

Definition 1. *Let $X \geq 0$ be a finite random measure on $\mathbb{R}^d$. We say X (or equivalently its distribution) is infinitely divisible if for each random number n, we have*

$$X = X_1 + X_2 + \cdots + X_n,$$

in distribution, where X_i are independent, identically distributed (i.i.d.) random variables.

Theorem 1. *Canonical representation theorem. Let $X \geq 0$ be an infinitely divisible random measure on $(E, \mathcal{E})$. Then there exist exit measures $\gamma \in \mathcal{M}(E)$ and $m \in \mathcal{M}(\mathcal{M}(E))$, with*

$m \neq 0$, such that for all $f \in C_b(E)$,

$$\int \left(1 - \mathrm{e}^{-\langle f,\nu\rangle}\right) m(\mathrm{d}\nu) < \infty,$$

and

$$-\log \mathbb{E}\left(\mathrm{e}^{-\langle f,X\rangle}\right) = \langle f, \gamma\rangle + \int \left(1 - \mathrm{e}^{-\langle f,\nu\rangle}\right) m(\mathrm{d}\nu). \tag{5.12}$$

If $m(\{0\}) = 0$, then γ and m are unique.

The idea behind the proof is to take, by infinite divisibility, $X = X_1 + \ldots + X_n$ in distribution for some $m \in \mathbb{N}$ where $X_1, \ldots, X_n$ are i.i.d. Then define the empirical measure

$$\tilde{X}_n := \sum_{j=1}^{n} \delta_{X_j} \mathbf{1}(X_j \neq 0) \in \mathcal{M}(\mathcal{M}(E)), \tag{5.13}$$

which gives us that $X = \int \mu \tilde{X}_m(\mathrm{d}\mu)$ in distribution. Now if we define $m_n := \mathbb{E}\left[\tilde{X}_n\right]$ the idea is to show that the m_n are tight in $\mathcal{M}(\mathcal{M}(E))$ and then the laws of $\tilde{X}_n$ are tight in the weak topology. We can then choose a convergent subsequence $\{m_{n_k}\}_{k\geq 1}$ such that $m_{n_k} \to m$ in $\mathcal{M}(\mathcal{M}(E))$ and also $\tilde{X}_{n_k} \to \tilde{X}_\infty$, where the law of $\tilde{X}_\infty$ is that of a Poisson random measure with intensity measure m. The remaining details can be found in the proof of Theorem 3.4.1 in [5]. In the superprocess setting, this idea is equivalent to "divide" the initial mass into n parts and note that each one of them behaves independently. Hence we can write

$$X_t = \sum_{i=1}^{n} X_t^{(i)},$$

where $X_t^{(i)}$ are the i.i.d and denote the mass at time t released from $X_0^{(i)}$.

As the SBM falls into this category, we can write $\mathtt{V}_t\,[f]$ in (5.9) as

$$\langle \mathtt{V}_t\,[f]\,, \mu\rangle = \langle f, \gamma_t\rangle + \int_{\mathcal{M}(\mathbb{R}^d)} \left(1 - \mathrm{e}^{-\langle f,\nu\rangle}\right) m_t(\mathrm{d}\nu).$$

Let $\vartheta(\theta) = \langle \mathtt{V}_t\,[\theta f]\,, \mu\rangle$, as $\mathbb{E}_\mu\left[\langle f, X_t\rangle\right]^2 < \infty$ due to (H2), then by (5.11),

$$\mathrm{e}^{-\vartheta(\theta)} = 1 - \theta \mathbb{E}_\mu\left[\langle f, X_t\rangle\right] + \frac{1}{2}\theta^2 \mathbb{E}_\mu\left[\langle f, X_t\rangle\right]^2 + o(\theta^2) \tag{5.14}$$

and, by (5.2),

$$\vartheta(\theta) = \theta \langle f, \gamma_t \rangle + \theta \int \langle f, \nu \rangle \, m_t(\mathrm{d}\nu) - \frac{1}{2}\theta^2 \int \langle f, \nu \rangle^2 \, m_t(\mathrm{d}\nu) + o(\lambda^2). \tag{5.15}$$

Comparing the two expressions in (5.14) and (5.15) we have that

$$\mathbb{E}_\mu \left[\langle f, X_t \rangle \right] = \langle f, \gamma_t \rangle + \int \langle f, \nu \rangle \, m_t(\mathrm{d}\nu),$$

and

$$\mathbb{E}_\mu \left[\langle f, X_t \rangle \right]^2 = \left(\mathbb{E}_\mu \left[\langle f, X_t \rangle \right] \right)^2 + \int \langle f, \nu \rangle^2 \, m_t(\mathrm{d}\nu).$$

This later expressions will be clearer and make more sense in the following section.

5.3 First and second moments of the SBM

To give the precise expressions of the first and second moments of this process, we recall a well known result, the so called Feynman-Kac formula, which states that if we consider the partial differential equation

$$\frac{\partial}{\partial t} u(x,t) = \mu(x,t) \frac{\partial}{\partial x} u(x,t) + \frac{1}{2}\sigma^2(x,t) \frac{\partial^2}{\partial x^2} u(x,t) - h(x,t)u(x,t) + k(x,t),$$

with initial condition $u(x,0) = g(x)$, and the functions μ, σ, h, k and g satisfy some growth and global Lipschitz conditions. Then the solution is unique and can be written as

$$u(x,t) = \mathbb{E}^*_x \left[\int_0^t \mathrm{e}^{-\int_0^s h(X_r, t-r)\mathrm{d}r} k(X_s, t-s)\mathrm{d}s + \mathrm{e}^{-\int_0^t h(X_s, t-s)\mathrm{d}s} g(X_t) \right],$$

where under $\mathbb{E}^*_x$, $(X_t, t \geq 0)$ is an Itô process released from x, driven by the equation

$$dX_t = \mu(X_t, t) + \sigma(X_t, t) dB_t, \quad t \geq 0,$$

with $(B_t, t \geq 0)$ a Brownian motion starting in x under $\mathbb{P}^*_x$.

Proposition 5. *The first and second moments of X are given by the following expressions*

$$\mathbb{E}_\mu \left[\langle f, X_t \rangle \right] = \langle \mathtt{T}_t f, \mu \rangle,$$

$$\mathbb{E}_\mu \left[\langle f, X_t \rangle^2 \right] = \langle \mathtt{T}_t [f], \mu \rangle^2 + \left\langle \int_0^t \psi''(0+) \mathtt{T}_s \left[(\mathtt{T}_{t-s} [f])^2 \right] \mathrm{d}s, \mu \right\rangle,$$

where $\mathtt{T}_t f(x) = \mathbf{P}_x\left[\mathrm{e}^{-\psi'(0+)t} f(\xi_t)\right]$ *is the heat semi-group, that is, under* $\mathbf{P}_x$*,* $(\xi_t, t \geq 0)$ *is a standard Brownian motion released from* x *.*

Proof. Following same calculations as in [36], take for $x \in \mathbb{R}$, $g \in C_b^+(\mathbb{R})$, $\theta, t \geq 0$

$$\mathtt{V}_t\left[\theta g\right] = -\log \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\theta\langle g, X_t\rangle}\right].$$

With limits understood to be as $\theta \downarrow 0$, we have that $\mathtt{V}_t\left[\theta g\right]|_{\theta=0} = 0$ and that $\mathtt{v}_t\left[g\right](x) := \mathbb{E}_{\delta_x}\left[\langle g, X_t\rangle\right] = \partial \mathtt{V}_t\left[\theta g\right](x)/\partial\theta|_{\theta=0}$. Now, if we differentiate with respect to θ in the corresponding evolution equation for $\mathtt{V}_t\left[\theta g\right]$, we get that $\mathtt{v}_t\left[g\right]$ satisfies the following equation

$$\frac{\partial}{\partial t}\mathtt{v}_t\left[g\right](x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\mathtt{v}_t\left[g\right](x) - \psi'(0+)\mathtt{v}_t\left[g\right](x), \tag{5.16}$$

with the initial condition $\mathtt{v}_0\left[g\right](x) = g(x)$. Feynman-Kac formula tells us that (5.16) has a unique solution and is equal to $\mathbf{P}_x\left[\mathrm{e}^{-\psi'(0+)t} f(\xi_t)\right]$, which gives us the first moment.

For the second moment, note that if the look at the second derivative of $\mathtt{V}_t\left[\theta g\right]$, we get that

$$\left.\frac{\partial^2 \mathtt{V}_t\left[\theta g\right]}{\partial\theta^2}\right|_{\theta=0} = \mathbb{E}_{\delta_x}\left[\langle g, X_t\rangle\right]^2 - \mathbb{E}_{\delta_x}\left[\langle g, X_t\rangle^2\right] =: \mathtt{w}_t\left[g\right](x).$$

On the other hand, deriving the evolution equation twice with respect to θ, we get that $\mathtt{w}_t\left[g\right]$ satisfies the equation

$$\frac{\partial}{\partial t}\mathtt{w}_t\left[g\right](x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\mathtt{w}_t\left[g\right](x) - \psi'(0+)\mathtt{w}_t\left[g\right](x) - \psi''(0+)(\mathtt{v}_t\left[g\right](x))^2, \tag{5.17}$$

and initial condition $\mathtt{w}_0\left[g\right](x) = 0$. Using Feynman-Kac formula once again, we have that (5.17) has a unique solution and is equal to

$$\mathtt{w}_t\left[g\right](x) = -\mathbf{P}_x\left[\int_0^t \mathrm{e}^{-\psi'(0+)s}\psi''(0+)(\mathtt{v}_{t-s}\left[g\right](\xi_s))^2 \mathrm{d}s\right].$$

Using this with the previous observation, we get that

$$\mathbb{E}_{\delta_x}\left[\langle g, X_t\rangle^2\right] = \mathbf{P}_x\left[\mathrm{e}^{-\psi'(0+)t} g(\xi_t)\right]^2 + \psi''(0+)\mathbf{P}_x\left[\int_0^t \mathrm{e}^{-\psi'(0+)s}\mathbf{P}_{\xi_s}\left[\mathrm{e}^{-\psi'(0+)(t-s)} g(\xi_{t-s})\right]^2 \mathrm{d}s\right],$$

which gives us the second moment. □

Note that this result implies that, when taking the limit when $\rho \to 0$,

$$\mathtt{S}_t^{i,\rho}[f](x) \to \mathtt{T}_t^{(i)}[f](x),$$

for each $t \geq 0$, $i = 1, 2$ that is, the first moments of the weighted branching particle system converge to the first moments of a superprocess.

5.4 Particle systems and supercritical SBM

Now that we have explored the relation between the first moments of a branching particle system and a super-Brownian motion, we will explore another way these two processes are related. Consider this time a super critical ψ super-Brownian motion, that is, with branching mechanism ψ such that $\psi'(0+) < 0$. Note that, as ψ is convex, it has at most two roots on $[0, \infty)$ (see Figure 5-1). Let $\kappa^* > 0$ be the largest root of the equation $\psi(\kappa) = 0$. According to [36], it is possible to transform (5.10) into the setting (5.4) by choosing the branching generator of the particle system with branching rate β as

$$\mathtt{G}(s) = \beta\left(\sum_{k=0}^{\infty} p_k s^k - s\right) = \frac{1}{\kappa^*}\psi(\kappa^*(1-s)), \qquad s \in [0,1], \tag{5.18}$$

with the particular branching generator given by (5.18), $\beta = \psi'(\kappa^*)$, $p_0 = p_1 = 0$ and for $k \geq 2$, $p_k := p_k[0,\infty)$, where for $y \geq 0$, we define the measure $p_k(\cdot)$ on $\{2,3,4,\ldots\} \times [0,\infty)$ by

$$p_k(dy) = \frac{1}{\kappa^*\psi'(\kappa^*)}\left(\beta(\kappa^*)^2\delta_0(dy)\mathbb{1}(k=2) + (\kappa^*)^k\frac{y^k}{k!}\mathrm{e}^{-\kappa^* y}\nu(dy)\right).$$

Then taking $\mathtt{V}_t[f](x) = \kappa^*(1 - \mathtt{v}_t[g](x))$. It follows that $\mathtt{v}_t[g](x)$ is a solution to (5.4) if and only if $\mathtt{V}_t[f]$ is a solution to (5.10).

We can check this by computing the derivative with respect to t of $\mathtt{V}_t[f](x)$ and using the above definition and that $\mathtt{v}_t[g](x) = 1 - \mathtt{V}_t[f](x)/\kappa^*$

$$\begin{aligned}
\frac{\partial}{\partial t}\mathtt{V}_t[f](x) &= -\kappa^*\frac{\partial}{\partial t}\mathtt{v}_t[g](x) \\
&= -\kappa^*\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\mathtt{v}_t[g](x) + \frac{1}{\kappa^*}\psi(\kappa^*(1-\mathtt{v}_t[g](x)))\right) \\
&= -\kappa^*\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(1 - \frac{1}{\kappa^*}\mathtt{V}_t[f](x)\right)\right) - \psi(\mathtt{V}_t[f](x)) \\
&= \frac{1}{2}\frac{\partial^2}{\partial x^2}\mathtt{V}_t[f](x) - \psi(\mathtt{V}_t[f](x)).
\end{aligned}$$

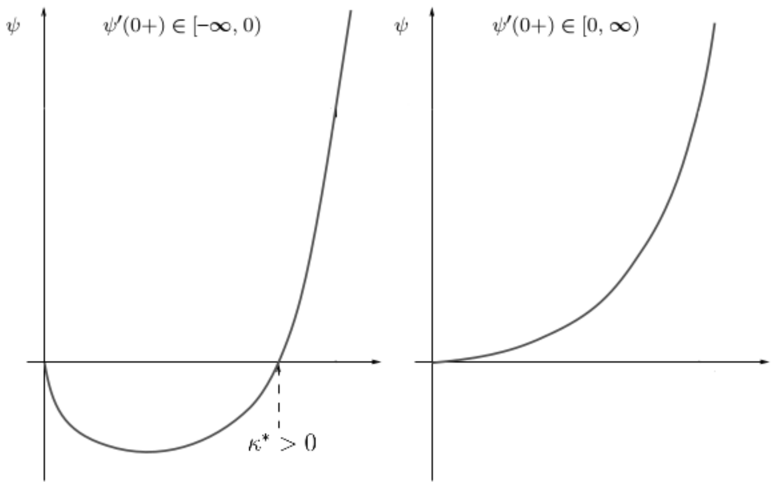

Figure 5-1: Branching mechanisms in the supercritical case (left) and critical or subcritical cases (right). This image can be originally found in [35].

5.5 Limit behaviour of SBM

In this section we use what it was discussed in previous sections to use some known results about branching Brownian motion in the super process framework.

5.5.1 An invariance principle

We want to study the limit behaviour of the probability of survival of the process up to time t when t goes to infinity in the critical case.

First, lets define $u(x,t) = -\log \mathbb{P}_{\delta_x}(|X_t| = 0)$, where $|X_t| = \langle 1, X_t\rangle$ for $t > 0$ and $x \in E$. Under the assumption that $\mathbb{P}_{\delta_x}(|X_t| = 0) > 0$ for each $x \in E$ and $t > 0$, then $u(x,t)$ is well defined. From the definition of $\mathtt{V}_t[f](x)$ and monotone convergence, we have for $f \in B_1^+(E)$

$$u(x,t) = \lim_{\theta\to\infty} \mathtt{V}_t[\theta f](x), \qquad t > 0,\ x \in E, \tag{5.19}$$

so if we allow extended values, we have that the semigroup $u(x,t)$ satisfies the evolution equation

$$\frac{\partial}{\partial t}u(x,t+s) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(x,t+s) - \psi(u(x,t+s)) \qquad s,t > 0, x \in D.$$

Using the relation $\mathtt{V}_t[f](x) = \kappa^*(1 - \mathtt{v}_t[g](x))$ with $g = \mathrm{e}^{-f}$, the above allows us to define the corresponding operator for the superprocess from the operator $v(x,t) := \mathtt{v}_t[0](x)$ to get that

$$u(x,t) = \kappa^*(1 - v(x,t)). \tag{5.20}$$

Now we are going to use the above equation and some results found in [39] along with the above observations to get the corresponding limit behaviour of a super process.

Proposition 6.

$$\lim_{t\to\infty} t\, \mathbb{P}_{\delta_x}\left(|X_t| \neq 0\right) = \frac{2\varphi(x)}{\langle \varphi^2, \tilde{\varphi}\rangle\, \psi''(0+)} \tag{5.21}$$

Proof. We know from [39] that if we define $a_g(t) := \int_E \varphi(x)(1 - \mathtt{v}_t\,[g]\,(x))\mathrm{d}x$, then

$$\frac{1 - \mathtt{v}_t\,[g]\,(x)}{\varphi(x)a_g(t)} \to 1,$$

and that

$$a_0(t) := a(t) \sim t^{-1}K,$$

with

$$K = \frac{2\varphi(x)}{\beta(m_2 - m)\,\langle \varphi^2, \tilde{\varphi}\rangle} = \frac{2\varphi(x)}{\langle \varphi^2, \tilde{\varphi}\rangle\, \kappa^*\psi''(0+)}. \tag{5.22}$$

If we take $\mathtt{V}_t\,[f]\,(x) = \kappa^*(1 - \mathtt{v}_t\,[g]\,(x))$ with $g(x) = \mathrm{e}^{-f(x)}$, then this implies that

$$\frac{\mathtt{V}_t\,[f]\,(x)}{\kappa^*\varphi(x)a_g(t)} \to 1. \tag{5.23}$$

Using now the above with u and v as in (5.20), we get that

$$\frac{u(x,t)}{\kappa^*\varphi(x)a(t)} \sim \frac{tu(x,t)}{\kappa^*\varphi(x)ta(t)} \sim 1, \tag{5.24}$$

which means that

$$u(x,t) \sim t^{-1}\kappa^*\varphi(x)K,$$

and then, we get that

$$u(x,t) \sim \frac{1}{t}\frac{2\varphi(x)}{\langle \varphi^2, \tilde{\varphi}\rangle\, \psi''(0+)}.$$

Now, note that if we define de death time of the process, at which $|X_t| = \langle 1, X_t\rangle = 0$

$$\zeta = \inf\{t \geq 0 : |X_t| = 0\},$$

then

$$u(x,t) = -\log \mathbb{P}_{\delta_x}(\zeta < t). \tag{5.25}$$

Now, using that as t goes to infinity $\mathbb{P}_{\delta_x}(\zeta > t)$ goes to zero, then

$$-\log \mathbb{P}_{\delta_x}(\zeta < t) = -\log(1 - \mathbb{P}_{\delta_x}(\zeta > t)) \sim \mathbb{P}_{\delta_x}(\zeta > t), \tag{5.26}$$

and then we get that

$$\mathbb{P}_{\delta_x}(\zeta > t) \sim \frac{1}{t}\frac{2\varphi(x)}{\langle \varphi^2, \tilde{\varphi}\rangle \psi''(0+)},$$

proving the result. □

5.5.2 Yaglom-type limit

Now we are going to use a Yaglom-type limit result form [39] to prove the following limit result for a superprocess in the critical case

Proposition 7.

$$\left\{\frac{\langle f, X_t\rangle}{t}\middle| \zeta > t\right\} \to \frac{1}{2}\langle \varphi^2, \tilde{\varphi}\rangle \psi''(0+)\langle f, \tilde{\varphi}\rangle\, \mathtt{e}_1 \tag{5.27}$$

in distribution, where $\mathtt{e}_1$ *is an exponential random variable with mean* 1.

Proof. We begin with observing the limit behaviour of the probability of survival of the process. According to [39] (Theorem 1.3), $v(x,t) = 1 - \mathbf{P}_x(|N_t| > 0) \to 1$ as $t \to \infty$ uniformly in D, which from (5.20) implies that $u(x,t) \to 0$. Using this with (5.25) gives that

$$\mathbb{P}_{\delta_x}(\zeta > t) \to 0, \tag{5.28}$$

as $t \to \infty$ uniformly in E. Theorem 1.6 in [39] tells us that

$$\mathbf{E}_x\left[\exp\left\{-\frac{\theta}{t}\langle f, Z_t\rangle\right\}\middle| |N_t| > 0\right] = \frac{\mathbf{E}_x\left[\exp\left\{-\frac{\theta}{t}\langle f, Z_t\rangle\right\}; |N_t| > 0\right]}{\mathbf{P}_x(|N_t| > 0)} \to \frac{\theta^*}{\theta^* + \theta}, \tag{5.29}$$

for $\theta < \theta^*$ with

$$\theta^* = \left(\frac{1}{2}\langle f, \tilde{\varphi}\rangle \langle \varphi^2, \tilde{\varphi}\rangle \kappa^* \psi''(0+)\right)^{-1}.$$

Writing this in terms of v means that

$$\frac{\mathtt{v}_t\left[\mathrm{e}^{-\theta f/t}\right](x) - v(x,t)}{1 - v(x,t)} \to \frac{\theta^*}{\theta^* + \theta}.$$

Now let $s(t) := 1 - \mathbb{E}_{\delta_x}\left[\exp\left\{-\frac{\theta}{t}\langle f, X_t\rangle\right\}\right]$, so Theorem 1.3 in [39] gives us that $s(t)$ goes to zero as t goes to infinity, so $-\log(1 - s(t)) \sim s(t)$ as t goes to infinity which implies that

$$\mathtt{V}_t\left[\theta f/t\right](x) = -\log \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\frac{\theta}{t}\langle f, X_t\rangle}\right] \sim 1 - \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\frac{\theta}{t}\langle f, X_t\rangle}\right]. \tag{5.30}$$

On the other hand we have that

$$\mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\frac{\theta}{t}\langle f, X_t\rangle}\right] = \left(\mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\frac{\theta}{t}\langle f, X_t\rangle}\middle| \zeta > t\right]\mathbb{P}_{\delta_x}(\zeta > t)\right) + \mathbb{P}_{\delta_x}(\zeta \le t),$$

which means that

$$\mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\frac{\theta}{t}\langle f, X_t\rangle}; \zeta > t\right] = \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\frac{\theta}{t}\langle f, X_t\rangle}\right] - 1 + \mathbb{P}_{\delta_x}(\zeta > t) \sim -\mathtt{V}_t\left[\theta f/t\right](x) + u(x,t), \tag{5.31}$$

using (5.30) and (5.26). To finish the proof we write $\mathtt{V}_t\left[\theta f/t\right](x)$ in terms of $\mathtt{v}_t\left[\mathrm{e}^{\theta f/t}\right](x)$ to get that

$$\begin{aligned}\mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\frac{\theta}{t}\langle f, X_t\rangle}\middle| \zeta > t\right] &\sim \frac{-\mathtt{V}_t\left[\theta f/t\right](x) + u(x,t)}{u(x,t)} \\ &= \frac{\mathtt{v}_t\left[\mathrm{e}^{-\theta f/t}\right](x) - v(x,t)}{1 - v(x,t)},\end{aligned}$$

which yields the result from (5.23) and (5.24). $\square$

As mentioned above, these calculations serve as a warm-up excercise and inspired some of the main ideas used to prove the main results for a general $(\mathtt{P}, \psi, \phi)$-superprocess, presented in the next chapter. More results of this type can be found in e.g. [21] and [36].

Chapter 6

Moments of Nonlocal branching superprocesses

In this chapter, we present the main results regarding the so-called nonlocal branching superproceses, which are contained in [34]. As stated above, the robustness of our methods in previous sections means that the principal ideas used to prove the above theorems are essentially the same for both branching particle systems and superprocesses, regardless of the criticality.

Let $(X_t, t \geq 0)$ be a $(\mathtt{P}, \psi, \phi)$- superprocess with probabilities $\mathbb{P}$ and transition semigroup $(\tilde{\mathbf{E}}_t, t \geq 0)$ on $M(E)$ defined in Section 1.3. We will being with some properties of its linear and non-linear semigroups, and then use a many-to-one formula to get an evolution equation for the kth moments which will be useful to prove out main results for their asymptotic behaviour.

6.1 Linear and non-linear semigroup equations

The evolution equations for the expectation semigroup $(\mathtt{T}_t, t \geq 0)$ is well known and satisfies

$$\mathtt{T}_t\left[f\right](x) = \mathtt{P}_t[f](x) + \int_0^t \mathtt{P}_s\left[\beta(\mathtt{m}[\mathtt{T}_{t-s}[f]] - 1) + b\right](x)\mathrm{d}s, \tag{6.1}$$

for $t \geq 0$, $x \in E$ and $f \in B^+(E)$, where, with a meaningful abuse of our branching Markov process notation, we now define

$$\begin{aligned} \mathtt{m}[f](x) &= \int_{M_0(E)} \left[\gamma(x,\pi) \langle f, \pi\rangle + \int_0^\infty u \langle f,\pi\rangle\, n(x,\pi,\mathrm{d}u) \right] G(x,\mathrm{d}\pi) \\ &= \gamma(x,f) + \int_{M(E)^\circ} \langle f,\nu\rangle \Gamma(x,\nu). \end{aligned} \tag{6.2}$$

See for example equation (3.24) of [7].

6.1.1 Many-to-one formula

In the spirit of Lemma 1 we can give a second representation of $\mathtt{T}_t[f]$ in terms of an auxiliary process, the so called many-to-one formula. To this end, if, as before, we work with the process $(\xi, \mathbf{P})$ to represent the Markov process associated to the semigroup $(\mathtt{P}_t, t \geq 0)$, then, although we have redefined the quantity $\mathtt{m}[f](x)$, we can still meaningfully work with the process $(\hat{\xi}, \hat{\mathbf{P}})$ as defined just before Lemma 1.

To this end, define the function $\vartheta := a + \beta$, then we have that

$$\mathtt{T}_t f(x) = \mathtt{P}_t f(x) + \int_0^t \mathtt{P}_s \left[\beta(\cdot) m(\cdot, \mathtt{T}_{t-s}[f])\right](x)\mathrm{d}s - \int_0^t \mathtt{P}_s \left[\vartheta(\cdot)\mathtt{T}_{t-s}[f]\right](x)\mathrm{d}s, \tag{6.3}$$

and also define the function

$$q(x) := \beta(x) \int_{M_0(E)} \left[d(x,\pi) + \int_0^\infty un(x,\pi,\mathrm{d}u)\right] G(x,\mathrm{d}\pi), \tag{6.4}$$

and note that is bounded by β as G is a probability Kernel. Now define the process $(\eta_t, t \geq 0)$ and denote its probabilities by $(\hat{\mathbf{E}}_x, x \in E)$. This process is described by the following SDE

$$d\eta_t = d\xi_t + J \circ \eta_{t-}, \tag{6.5}$$

where $J \circ \eta_{t-}$ is an operator that sends $\eta_{t-} \mapsto \eta_t$ when there is a jump at time t. Jumps occur at rate $q(\eta_{t-})$ and $J \circ \eta_{t-} : \eta_{t-} \mapsto y$, where y is selected with distribution $\pi^{(+)}_{\eta_{t-}}$, and $\pi^{(+)}_x$ occurs at rate

$$\frac{\beta(x)}{q(x)} m(x,\mathrm{d}y) = \frac{1}{q(x)}\beta(x) \int_{M_0(E)} \left[d(x,\pi) + \int_0^\infty un(x,\pi,\mathrm{d}u)\right] G(x,\mathrm{d}\pi)\pi(\mathrm{d}y). \tag{6.6}$$

Another description of this process is that η evolves from η_0 as ξ_t until time τ_1, which has survival probability $\mathrm{e}^{-\int_0^t q(\xi_s)}$. At time τ_1, η jumps form ξ_{t-} to η_T which is randomly distributed according to

(6.7)
$$\pi_{\xi_{t-}}(\mathrm{d}y) = \frac{\beta(\xi_{t-})}{q(\xi_{t-})}\int_{M_0(E)}\left[d(\xi_{t-},\pi) + \int_0^\infty un(\xi_{t-},\pi,\mathrm{d}u)\right] G(\xi_{t-},\mathrm{d}\pi)\pi(\mathrm{d}y) = \frac{\beta}{q}m(\xi_{t-},\mathrm{d}y)$$

and set $\eta_0 \leftarrow \eta_T$ and repeat.

Lemma 4. *Let $\vartheta(x) = B(x)+b(x) = \beta(x)(\mathtt{m}[1](x)-1)+b(x)$, then, for $t \geq 0$ and $f \in B^+(E)$,*

$$\mathtt{T}_t\left[f\right](x) = \hat{\mathbf{E}}_x\left[\exp\left(\int_0^t \vartheta(\hat{\xi}_s)\mathrm{d}s\right) f(\hat{\xi}_t)\right]. \tag{6.8}$$

Proof. As with Lemma 1, the proof requires only that we take the right-hand side of (6.8) and condition on the first extra jump of $(\hat{\xi},\hat{\mathbf{P}})$ to show that it also solves (6.2). It is a straightforward application of Grönwall's inequality to show that (6.2) has a unique solution and hence (6.1) holds. Note that because we have separated out the local and nonlocal branching mechanisms of the superprocess, the deliberate repeat definition of $\mathtt{m}[f]$ for superprocesses is only the analogue of its counter part for branching Markov processes in the sense of nonlocal activity. The mean local branching rate has otherwise been singled out as the term b.

Let $\widetilde{\mathtt{T}}_t\left[f\right](x)$ be the right hand side of (6.8). We will show that $\widetilde{\mathtt{T}}_t f(x)$ solves the equation (6.3) for $f \in B_1^+(E)$.

$$\begin{aligned}
\widetilde{\mathtt{T}}_t\left[f\right](x) =& \hat{\mathbf{E}}_x\left[\mathrm{e}^{-\int_0^t(\vartheta-q)(\hat{\xi}_s))\mathrm{d}s} f(\hat{\xi}_t)\mathbb{1}(t<\tau_1)\right] + \hat{\mathbf{E}}_x\left[\mathrm{e}^{-\int_0^t(\vartheta-q)(\hat{\xi}_s))\mathrm{d}s} f(\hat{\xi}_t)\mathbb{1}\left(t \geq \tau_1\right)\right]\\
=& \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \vartheta(\xi_s)\mathrm{d}s} f(\xi_t)\right]\\
&+ \mathbf{E}_x\left[\int_0^t q(\xi_s)\mathrm{e}^{-\int_0^s q(\xi_u)du}\mathrm{e}^{-\int_0^s(\vartheta-q)(\xi_u)du}\hat{\mathbf{E}}_{J\circ\xi_s}\left[\mathrm{e}^{-\int_0^{t-s}(\vartheta-q)(\xi_u)du} f(\xi_{t-s})\right]\mathrm{d}s\right]\\
=& \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \vartheta(\xi_s)\mathrm{d}s} f(\xi_t)\right] + \mathbf{E}_x\left[\int_0^t q(\xi_s)\mathrm{e}^{-\int_0^s \vartheta(\xi_u)du}\frac{\beta(\xi_s)}{q(\xi_s)}m(\xi_s,\widetilde{\mathtt{T}}_{t-s})\mathrm{d}s\right]\\
=& \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \vartheta(\xi_s)\mathrm{d}s} f(\xi_t)\right] + \mathbf{E}_x\left[\int_0^t \mathrm{e}^{-\int_0^s \vartheta(\xi_u)du}\beta(\xi_s)m(\xi_s,\widetilde{\mathtt{T}}_{t-s})\mathrm{d}s\right].
\end{aligned}$$

Finally, using Lemma A.1 we get that

$$\widetilde{\mathtt{T}}_t\left[f\right](x) = \mathtt{P}_t[f](x) + \int_0^t \mathtt{P}_s\left[\beta(\cdot)m(\cdot,\widetilde{\mathtt{T}}_{t-s}\left[f\right])\right](x)\mathrm{d}s - \int_0^t \mathtt{P}_s\left[\vartheta(\cdot)\widetilde{\mathtt{T}}_{t-s}\left[f\right](\cdot)\right](x)\mathrm{d}s, \tag{6.9}$$

which means $\widetilde{\mathtt{T}}_t f(x)$ solves equation (6.3). Uniqueness of the solution come from Grönwall's inequality, implying the desired result. $\square$

Similarly to the branching Markov process setting, let us re-write an extended version of the non-linear semigroup evolution $(\mathtt{V}_t, t \geq 0)$, defined in (1.17), i.e. the natural analogue of (4.2), in terms of the linear semigroup $(\mathtt{T}_t, t \geq 0)$. To this end, define

$$\mathtt{V}_t\left[f,g\right](x) = \mathbb{E}_x\left[\mathrm{e}^{-\langle f, X_t\rangle - \int_0^t \langle g, X_s\rangle \mathrm{d}s}\right],$$

Analogously to Theorem 3 we have the following result.

Lemma 5. *For all $f, g \in B^+(E)$, $x \in E$ and $t \geq 0$, the non-linear semigroup $\mathtt{V}_t[f,g](x)$ satisfies*

$$\mathtt{V}_t[f,g](x) = \mathtt{T}_t[f](x) - \int_0^t \mathtt{T}_s\left[\mathtt{J}[\mathtt{V}_{t-s}[f,g]] - g\mathtt{V}_{t-s}\left[f,g\right]\right](x)\mathrm{d}s, \tag{6.10}$$

where, for $h \in B^+(E)$ and $x \in E$,

$$\mathtt{J}[h](x) = \psi(x, h(x)) + \phi(x, h) + \beta(x)(\mathtt{m}[h](x) - h(x)) + b(x)h(x).$$

The proof is essentially the same as the proof of Lemma 3 and hence we leave the details to the reader.

6.1.2 Evolution equations for the k-th moment of a superprocesses

Recall that we defined $\mathtt{T}_t^{(k)}\left[f\right](x) := \mathbb{E}_{\delta_x}[\langle f, X_t\rangle^k]$, $t \geq 0$, $f \in B^+(E)$, $k \geq 1$. As with the setting of branching Markov processes, we want to establish an evolution equation for $(\mathtt{T}_t^{(k)}, t \geq 0)$, from which we can establish the desired asymptotics. To this end, let us introduce the following notation.

For $x \in E$, $k \geq 2$ and $t \geq 0$, define

$$R_k(x,t) = \sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{k!}{m_1!\dots m_{k-1}!}(-1)^{m_1+\dots+m_{k-1}-1}$$

$$(m_1 + \dots + m_{k-1} - 1)! \prod_{j=1}^{k-1}\left(\frac{(-1)^j \mathtt{T}_t^{(j)}\left[f\right](x)}{j!}\right)^{m_j}, \tag{6.11}$$

and

$$K_k(x,t) = \sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{k!}{m_1!\dots m_{k-1}!}\psi^{(m_1+\dots+m_{k-1})}(x,0+)\left(\mathtt{T}_t\left[f\right](x)\right)^{m_1}$$

$$\prod_{j=2}^{k-1}\left(\frac{(-1)^{j+1}\mathtt{T}_t^{(j)}\left[f\right](x) - R_j(x,t)}{j!}\right)^{m_j}, \tag{6.12}$$

and finally

$$S_k(x,t) = \int_{M(E)^\circ} \sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{k!}{m_1!\dots m_{k-1}!}(-1)^{m_1+\dots+m_{k-1}}\left\langle \mathtt{T}_t\left[f\right], \nu\right\rangle^{m_1}$$

$$\prod_{j=2}^{k-1}\left(\frac{\left\langle (-1)^{j+1}\mathtt{T}_t^{(j)}\left[f\right] - R_j(\cdot,t), \nu\right\rangle}{j!}\right)^{m_j} \Gamma(x,\mathrm{d}\nu), \tag{6.13}$$

and the sums run over the set of non-negative integers $\{m_1,\dots,m_{k-1}\}$ such that $m_1+2m_2+\dots+(k-1)m_{k-1} = k$.

Proposition 8. *Fix $k \geq 2$. Suppose that (H1) and (H2) hold, with the additional assumption that*

$$\sup_{x\in E, s\leq t} \mathtt{T}_s^{(\ell)}[f](x) < \infty, \qquad \ell \leq k-1, f\in B^+(E), t\geq 0. \tag{6.14}$$

Then,

$$\mathtt{T}_t^{(k)}\left[f\right](x) = (-1)^{k+1}R_k(x,t) + (-1)^k\int_0^t \mathtt{T}_s\left[U_k(\cdot,t-s)\right](x)\mathrm{d}s, \tag{6.15}$$

where

$$U_k(x,t) = K_k(x,t) + \beta(x)S_k(x,t). \tag{6.16}$$

Proof. For the proof of this result, recall the definition (1.16) and let

$$\mathtt{v}_t^{(k)}[f](x) := \frac{\partial^k}{\partial\theta^k}\mathtt{V}_t[\theta f,0](x)\bigg|_{\theta=0}, \qquad t\geq 0, f\in B^+(E), k\geq 1$$

as well as

$$\mathtt{e}_t[f](x) := \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\langle f,X_t\rangle}\right], \qquad t\geq 0, f\in B^+(E).$$

In that case, $\mathtt{V}_t[\theta f, 0](x) = -\log \mathtt{e}_t[\theta f](x)$ and $\mathtt{e}_t[0](x) = 1$, so that

$$\mathtt{e}_t^{(k)}[\theta f](x) := \frac{\partial^k}{\partial \theta^k}\mathtt{e}_t[\theta f](x) = (-1)^k \mathbb{E}_{\delta_x}\left[\langle f, X_t\rangle^k\, \mathrm{e}^{-\theta\langle f, X_t\rangle}\right]$$

and

$$\mathtt{e}_t^{(k)}[\theta f](x)|_{\theta=0} = (-1)^k \mathtt{T}_t^{(k)}\left[f\right](x). \tag{6.17}$$

Next we can us Faà di Bruno's Lemma A.4 to get

$$\begin{aligned}
&\mathtt{v}_t^{(k)}[f](x)\\
&= \frac{\partial^k}{\partial\theta^k} - \log \mathtt{e}_t[\theta f](x)\Big|_{\theta=0}\\
&= -\sum_{\{m_1,\dots,m_k\}_k} \frac{k!}{m_1!\dots m_k!}\frac{(-1)^{m_1+\dots+m_k-1}(m_1+\dots+m_k-1)!}{\mathtt{e}_t[\theta f](x)^{m_1+\dots+m_k}}\prod_{j=1}^k\left(\frac{\mathtt{e}^{(j)}[\theta f])}{j!}\right)^{m_j}\Bigg|_{\theta=0}\\
&= -\sum_{\{m_1,\dots,m_k\}_k} \frac{k!}{m_1!\dots m_k!}(-1)^{m_1+\dots+m_k-1}(m_1+\dots+m_k-1)!\prod_{j=1}^k\left(\frac{(-1)^j\mathtt{T}_t^{(j)}\left[f\right](x)}{j!}\right)^{m_j},
\end{aligned}$$

where the sum runs over the set of non-negative integers $\{m_1,\dots,m_k\}_k$ such that

$$m_1 + 2m_2 + \dots + km_k = k.$$

Note that $m_k > 0$ if and only if $m_k = 1$ and $m_1 = m_2 = \dots = m_{k-1} = 0$, so the k-th moment term $\mathtt{T}_t^{(k)}[f]$ appears only once and with a factor $(-1)^{k+1}$, that is,

$$\mathtt{v}_t^{(k)}[f](x) = (-1)^{k+1}\mathtt{T}_t^{(k)}\left[f\right](x) - R_k(x,t), \tag{6.18}$$

where all the terms in $R_k(x,t)$ are products of two or more lower order moments.

Now, we differentiate the evolution equation (6.10) k times at $\theta = 0$, momentarily not worrying about passing derivatives through integrals, to get that

$$\mathtt{v}_t^{(k)}[f](x) = -\int_0^t \mathtt{T}_s\left[\frac{\partial^k}{\partial\theta^k}\Big(\psi(\cdot,\mathtt{V}_{t-s}\left[\theta f,0\right](\cdot)) + \phi(\cdot,\mathtt{V}_{t-s}\left[\theta f,0\right]) + \mathtt{F}[\mathtt{V}_{t-s}[\theta f,0]]\Big)\Big|_{\theta=0}\right](x)\mathrm{d}s,$$

where

$$\mathtt{F}[g](x) = \beta(x)(m[g] - g) + b(x)g, \qquad x \in E, g \in B^+(E).$$

For the kth derivative of $\psi(x, \mathtt{V}_t[\theta f, 0](x))$ at $\theta = 0$ we again use Faà di Bruno, Lemma A.4, to get

$$\begin{aligned}
&\frac{\partial^k}{\partial\theta^k}\psi\left(x, \mathtt{V}_t[\theta f, 0](x)\right)\Big|_{\theta=0} \\
&= \sum_{\{m_1,\dots,m_k\}_k} \frac{k!}{m_1!\dots m_k!}\psi^{(m_1+\dots+m_k)}(x, \mathtt{V}_t[\theta f, 0]) \prod_{j=1}^k \left(\frac{\frac{\partial^j}{\partial\theta^j}\mathtt{V}_t[\theta f, 0](x)}{j!}\right)^{m_j}\Bigg|_{\theta=0} \\
&= \sum_{\{m_1,\dots,m_k\}_k} \frac{k!}{m_1!\dots m_k!}\psi^{(m_1+\dots+m_k)}(x, 0+) \prod_{j=1}^k \left(\frac{\mathtt{v}_t^{(j)}[f](x)}{j!}\right)^{m_j} \\
&= -b(x)\mathtt{v}_t^{(k)}[f](x) + K_k(x,t),
\end{aligned}$$

where the last equality holds because $m_k = 1$ if and only if $m_1 = \ldots = m_{k-1} = 0$ and $\psi'(x, 0+) = -b(x)$. Similarly, for the the kth derivative of the remaining terms recalling (1.19), (1.20) and (6.2),

$$\begin{aligned}
&\frac{\partial^k}{\partial\theta^k}\Big(\phi(x, \mathtt{V}_t[\theta f, 0]) + \mathtt{F}[\mathtt{V}_t[\theta f, 0]]\Big) \\
&= b(x)\frac{\partial^k}{\partial\theta^k}\mathtt{V}_t[\theta f, 0] \\
&\quad - \beta(x)\int_{M_0(E)}\int_0^\infty \frac{\partial^k}{\partial\theta^k}\left(1 - \mathrm{e}^{-u\langle \mathtt{V}_t[\theta f, 0], \pi\rangle} - u\left\langle \mathtt{V}_t[\theta f, 0], \pi\right\rangle\right) n(x, \pi, \mathrm{d}u)G(x, \mathrm{d}\pi),
\end{aligned}$$

and using Lemma A.4 we have

$$\begin{aligned}
&\frac{\partial^k}{\partial\theta^k}\left(1 - \mathrm{e}^{-u\langle \mathtt{V}_t[\theta f, 0], \pi\rangle} - u\left\langle \mathtt{V}_t[\theta f, 0], \pi\right\rangle\right) \\
&= \sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{k!}{m_1!\dots m_k!}(-1)^{m_1+\dots+m_k+1}\mathrm{e}^{-u\langle \mathtt{V}_t[\theta f, 0], \pi\rangle} \prod_{j=1}^k \left(\frac{u\left\langle \frac{\partial^j}{\partial\theta^j}\mathtt{V}_t[\theta f, 0], \pi\right\rangle}{j!}\right)^{m_j} \\
&\qquad + \left(\mathrm{e}^{-u\langle \mathtt{V}_t[\theta f, 0], \pi\rangle} - 1\right) u\left\langle \frac{\partial^k}{\partial\theta^k}\mathtt{V}_t[\theta f, 0], \pi\right\rangle,
\end{aligned}$$

where, in the final equality, we have singled out the case that $m_k = 1$ and $m_1 = \cdots = m_{k-1} = 0$ in the Faà di Bruno formula. and then, using the definition of $\mathtt{m}[f](x)$ in (6.2) and the same observation as above about the m_j's, we get

$$\frac{\partial^k}{\partial\theta^k}\Big(\phi(x,\mathtt{V}_t[\theta f,0])+\mathtt{F}[\mathtt{V}_t[\theta f,0]]\Big)\Big|_{\theta=0}=b(x)\mathtt{v}_t^{(k)}[f](x)+\beta(x)S_k(x,t). \tag{6.19}$$

Putting the pieces together, we get

$$\mathtt{v}_t^{(k)}[f](x)=-\int_0^t \mathtt{T}_s\left[U_k(\cdot,t-s)\right](x)\mathrm{d}s. \tag{6.20}$$

Combining this with equation (6.18) we get that

$$(-1)^{k+1}\mathtt{T}_t^{(k)}\left[f\right](x)=R_k(x,t)-\int_0^t \mathtt{T}_s\left[U_k(\cdot,t-s)\right](x)\mathrm{d}s,$$

which is the desired result.

There is one final matter we must attend to, which is the ability to move derivatives through integrals. In this setting, this is easier to deal with thanks to the the assumption (6.14), (H2) and the Lévy-Khintchine-type formulae for ψ and ϕ. $\square$

6.2 Limit behaviour of Moments of nonlocal superprocesses: critical case

Now we are ready to state the main results of this chapter. The following theorem gives us the analogue to the result presented above.

Theorem 7 (Critical, $\lambda=0$). *Suppose that (H1) holds along with (H2) for some $k\geq 2$ and $\lambda=0$. Define*

$$\Delta_t^{(\ell)}=\sup_{x\in E,f\in B_1^+(E)}\left|t^{-(\ell-1)}\varphi(x)^{-1}\mathtt{T}_t^{(\ell)}[f](x)-2^{-(\ell-1)}\ell!\,\langle f,\tilde\varphi\rangle^{\ell}\langle\mathbb{V}[\varphi],\tilde\varphi\rangle^{\ell-1}\right|,$$

where

$$\mathbb{V}[\varphi](x)=\psi''(x,0+)\varphi(x)^2+\beta(x)\int_{M(E)^\circ}\langle\varphi,\nu\rangle^2\Gamma(x,\mathrm{d}\nu).$$

Then, for all $\ell\leq k$

$$\sup_{t\geq 1}\Delta_t^{(\ell)}<\infty \text{ and } \lim_{t\to\infty}\Delta_t^{(\ell)}=0. \tag{6.21}$$

Proof. We will give a proof of this result using induction, similarly to the setting of branching Markov processes. The case $k = 1$ follows from assumption (H1).

Now assume that the statement of Theorem 1 holds in the superprocess setting for all $\ell \leq k$. Our aim is to prove that the result holds for $k + 1$. Using Theorem 8 and a change of variables, we have that

$$\frac{1}{\varphi(x)t^k}\mathtt{T}_t^{(k+1)}[f](x) = \frac{(-1)^k}{\varphi(x)t^k}R_{k+1}(x,t) + \frac{(-1)^{k+1}}{\varphi(x)t^{k-1}}\int_0^1 \mathtt{T}_{st}\left[U_{k+1}(\cdot, t(1-s))\right](x)\mathrm{d}s, \tag{6.22}$$

where R and U were defined in equations (6.11) and (6.16), respectively. We will prove first that, for each $x \in E$,

$$\begin{aligned}\lim_{t\to\infty}&\frac{1}{\varphi(x)t^k}\mathtt{T}^{(k+1)}[f](x)\\ &= \lim_{t\to\infty}\frac{1}{2\varphi(x)t^{k-1}}\int_0^1 \mathtt{T}_{st}\left[K_{k+1}^{(2)}(\cdot, t(1-s)) + \beta(\cdot)S_{k+1}^{(2)}(\cdot, t(1-s))\right](x)\mathrm{d}s,\end{aligned} \tag{6.23}$$

where

$$K_{k+1}^{(2)}(x,t) := \sum_{\{k_1,k_2\}^+}\frac{(k+1)!}{k_1!k_2!}\psi''(x,0+)\mathtt{T}_t^{(k_1)}[f](x)\mathtt{T}_t^{(k_2)}[f](x) \tag{6.24}$$

and

$$S_{k+1}^{(2)}(x,t) = \int_{M(E)^\circ}\sum_{\{k_1,k_2\}^+}\frac{(k+1)!}{k_1!k_2!}\langle \mathtt{T}_t^{(k_1)}[f],\nu\rangle\langle \mathtt{T}_t^{(k_2)}[f],\nu\rangle\Gamma(x,\mathrm{d}\nu), \tag{6.25}$$

such that $\{k_1, k_2\}^+$ is defined to be the set of positive integers k_1, k_2 such that $k_1+k_2 = k+1$.

To this end, writing $c(m_1, \ldots, m_k)$ for the constants preceding the product summands in (6.11), observe that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t^k}R_{k+1}(x,t) &= \lim_{t\to\infty}\frac{(k+1)!}{t^k}\sum_{\{m_1,\ldots,m_k\}_{k+1}} c(m_1,\ldots,m_k)\prod_{j=1}^k\left(\frac{(-1)^j\mathtt{T}_t^{(j)}[f](x)}{j!}\right)^{m_j}\\ &= (-1)^k(k+1)!\lim_{t\to\infty}\sum_{\{m_1,\ldots,m_k\}_{k+1}}\frac{c(m_1,\ldots,m_k)}{t^{m_1+\ldots+m_k-1}}\prod_{j=1}^k\left(\frac{1}{j!}\frac{\mathtt{T}_t^{(j)}[f](x)}{t^{j-1}}\right)^{m_j}\\ &= 0,\end{aligned}$$

where the final equality is due to the induction hypothesis and the fact that $m_1+\ldots+m_k > 1$, which follows from the fact that $m_1 + 2m_2 + \ldots + \ldots + km_k = k + 1$. Note, moreover that the induction hypothesis ensures that the limit is uniform in $x \in E$ and, in fact, that

$$\sup_{t\geq 0, x\in E} \frac{1}{t^{\ell-1}} R_\ell(x,t) < \infty \text{ and } \lim_{t\to\infty} \sup_{x\in E} \frac{1}{t^{\ell-1}} R_\ell(x,t) = 0 \qquad \ell = 1, \cdots, k+1. \tag{6.26}$$

We now return to (6.22), to deal with the term involving U_{k+1}, which we recall is a linear combination of R_{k+1} and S_{k+1}, which were defined in (6.11) and (6.13), respectively. Note that if any of the summands in either R_{k+1} or S_{k+1} have more than two of the m_j positive, the limit of that summand, when renormalised by $1/t^{k-1}$, will be zero. In essence, the argument here is analogous to those that led to (3.19) in the branching Markov process setting. This implies that the only terms in the sums of (6.11) and (6.13) that remain in the limit of (6.22) are those for which $m_{k_1} = m_{k_2} = 1$ and $m_j = 0$ for all $j \neq k_1, k_2$, with $k_1 < k_2$ such that $k_1 + k_2 = k + 1$, and if $k + 1$ is even, the terms in which $m_{(k+1)/2} = 2$ and $m_j = 0$ for all $j \neq (k+1)/2$.

Let us now convert all of the above heuristics into rigorous computation. We write

$$F(x, s, t) := \frac{1}{\varphi(x)t^{k-1}} \left(K^{(3+)}_{k+1}(x, t(1-s)) + \beta(x) S^{(3+)}_{k+1}(x, t(1-s)) \right), \tag{6.27}$$

where $K^{(3+)}_{k+1}$ and $S^{(3+)}_{k+1}$ contain the terms in K_{k+1} and S_{k+1}, respectively, for which the sum $m_1 + \ldots + m_k$ is greater than or equal to 3. We will prove that $\lim_{t\to\infty} F(x, s, t) = 0$ and that (A.5) and (A.6) hold.

Due to (1.25) and boundness of φ, dominated convergence implies that

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{\varphi(x)t^{k-1}} K^{(3+)}_{k+1}(x, t(1-s)) \\
&\quad = \frac{(k+1)!}{\varphi(x)} \sum_{\{m_1,\ldots,m_k\}^3_{k+1}} \frac{\psi^{(m_1+\ldots+m_k)}(x, 0+)}{m_1! \ldots m_k!} \\
&\qquad\qquad \lim_{t\to\infty} \frac{1}{t^{m_1+\ldots+m_k-2}} \prod_{j=1}^{k} \left(\frac{(-1)^{j+1} \mathrm{T}^{(j)}_{t(1-s)} [f](x) - R_j(x, t(1-s))}{j! t^{j-1}} \right)^{m_j},
\end{aligned}$$

where the set $\{m_1, \ldots, m_k\}^3_{k+1}$ is the subset of $\{m_1, \ldots, m_k\}_{k+1}$ for which $m_1 + \ldots + m_k \geq 3$. Using the induction hypothesis and (6.26), we get that the right-hand side above is zero. The same arguments also imply that the limit of $S^{(3+)}_{k+1}$ is zero. Thus $F(x, s) := \lim_{t\to\infty} F(x, s, t) = 0$. The condition (A.5) trivially holds. For (A.6), the required uniformity follows from the induction hypothesis and (6.26).

Using Theorem A.5 in the Appendix, we conclude that

$$\lim_{t\to\infty}\sup_{x\in E}\left|\frac{1}{\varphi(x)t^{k-1}}\int_0^1 \mathtt{T}_{ts}\left[\varphi F(\cdot,s,t)\right](x)\mathrm{d}s\right| = 0.$$

Let us now define $\{k_1 < k_2\}$ to be the elements in K_{k+1} for which $m_{k_1} = m_{k_2} = 1$ with $k_1 < k_2$ such that $k_1 + k_2 = k+1$ and $m_j = 0$ for all other indices and, in the case where $k+1$ is even, $m_{(k+1)/2} = 2$ and $m_j = 0$ for all $j \neq (k+1)/2$. Restricting the sum to this set in K_{k+1} we get the following expression

$$\begin{aligned}
K^{(2)}_{k+1}(x,t) = &\sum_{\{k_1<k_2\}} \frac{(k+1)!}{k_1!k_2!}\psi''(x,0+)(-1)^k\mathtt{T}^{(k_1)}_t[f](x)\mathtt{T}^{(k_2)}_t[f](x)\\
&\qquad + \mathbf{1}_{(k+1\text{ is even})}\frac{1}{2}\binom{k+1}{k/2}\psi''(x,0+)(-1)^{k+1}\left(\mathtt{T}^{(k/2)}_t[f](x)\right)^2\\
=&\frac{(-1)^{k+1}}{2}\sum_{\{k_1,k_2\}^+}\frac{(k+1)!}{k_1!k_2!}\psi''(x,0+)\mathtt{T}^{(k_1)}_t[f](x)\mathtt{T}^{(k_2)}_t[f](x),
\end{aligned}$$

where we recall $\{k_1,k_2\}^+$ of positive integers k_1, k_2 such that $k_1+k_2 = k+1$. Similarly, we obtain the following expression for S_{k+1}:

$$\begin{aligned}
K^{(2)}_{k+1}(x,t) = &\int_{M(E)^\circ}\sum_{k_1<k_2}\frac{(k+1)!}{k_1!k_2!}(-1)^{k+1}\langle\mathtt{T}^{(k_1)}_t[f],\nu\rangle\langle\mathtt{T}^{(k_2)}_t[f],\nu\rangle\Gamma(x,\mathrm{d}\nu)\\
&\qquad + \mathbf{1}_{(k+1\text{ is even})}\int_{M(E)^\circ}\binom{k+1}{k/2}\frac{(-1)^{k+1}}{2}\langle\mathtt{T}^{((k+1)/2)}_t[f],\nu\rangle^2\Gamma(x,\mathrm{d}\nu)\\
= &\frac{(-1)^{k+1}}{2}\int_{M(E)^\circ}\sum_{\{k_1,k_2\}^+}\frac{(k+1)!}{k_1!k_2!}\langle\mathtt{T}^{(k_1)}_t[f],\nu\rangle\rangle\mathtt{T}^{(k_2)}_t[f],\nu\rangle\Gamma(x,\mathrm{d}\nu).
\end{aligned}$$

Combining this with (6.22), we obtain (6.23). To conclude the proof define

$$F(x,s,t) := \frac{1}{2\varphi(x)t^{k-1}}\left(K^{(2)}_{k+1}(x,t(1-s)) + \beta(x)S^{(2)}_2(x,t(1-s))\right). \tag{6.28}$$

Due to (1.25) and the induction hypothesis,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{2\varphi(x)t^{k-1}}K^{(2)}_{k+1}(x,t(1-s))\\
&\qquad=\frac{(1-s)^{k-1}}{2\varphi(x)}\sum_{\{k_1,k_2\}^+}\frac{(k+1)!}{k_1!k_2!}\psi''(x,0+)\lim_{t\to\infty}\frac{\mathtt{T}^{(k_1)}_{t(1-s)}f(x)}{(t(1-s))^{k_1-1}}\frac{\mathtt{T}^{(k_2)}_{t(1-s)}f(x)}{(t(1-s))^{k_2-1}}\\
&\qquad=(1-s)^{k-1}\varphi(x)\sum_{\{k_1,k_2\}^+}(k+1)!2^{-k}\psi''(x,0+)\left\langle f,\tilde\varphi\right\rangle^{k+1}\left\langle\beta\mathbb{V}\left[\varphi\right],\tilde\varphi\right\rangle^{k-1}\\
&\qquad=k(1-s)^{k-1}\varphi(x)(k+1)!2^{-k}\psi''(x,0+)\left\langle f,\tilde\varphi\right\rangle^{k+1}\left\langle\beta\mathbb{V}\left[\varphi\right],\tilde\varphi\right\rangle^{k-1},
\end{aligned}$$

where the last equality holds because the total number of ways of splitting one set of size $k+1$ into two non empty sets is equal to k. To obtain the limit for $S^{(2)}_{k+1}$, we use (1.25), the induction hypothesis, dominated convergence and linearity to obtain

$$\begin{aligned}
&\lim_{t\to\infty}\frac{S^{(2)}_{k+1}(x,t(1-s))}{2\varphi(x)t^{k-1}}\\
&=\frac{(1-s)^{k-1}}{2\varphi(x)}\int_{M(E)^\circ}\sum_{\{k_1,k_2\}^+}\frac{(k+1)!}{k_1!k_2!}\lim_{t\to\infty}\frac{\langle\mathtt{T}^{(k_1)}_{t(1-s)}\left[f\right],\nu\rangle}{(t(1-s))^{k_1-1}}\frac{\langle\mathtt{T}^{(k_2)}_{t(1-s)}\left[f\right],\nu\rangle}{(t(1-s))^{k_2-1}}\Gamma(x,d\nu)\\
&=\frac{(1-s)^{k-1}}{2^k\varphi(x)}\int_{M(E)^\circ}\sum_{\{k_1,k_2\}^+}(k+1)!\left\langle f,\tilde\varphi\right\rangle^{k+1}\left\langle\beta\mathbb{V}\left[\varphi\right],\tilde\varphi\right\rangle^{k-1}\left\langle\varphi,\nu\right\rangle^2\Gamma(x,\mathrm{d}\nu)\\
&=\frac{k(1-s)^{k-1}}{2^k\varphi(x)}(k+1)!\left\langle f,\tilde\varphi\right\rangle^{k+1}\left\langle\beta\mathbb{V}\left[\varphi\right],\tilde\varphi\right\rangle^{k-1}\int_{M(E)^\circ}\left\langle\varphi,\nu\right\rangle^2\Gamma(x,\mathrm{d}\nu).
\end{aligned}$$

Combining these two limits, we get that

$$F(x,s):=\lim_{t\to\infty}F(x,s,t)=\frac{k(1-s)^{k-1}}{\varphi(x)}\frac{(k+1)!}{2^k}\left\langle f,\tilde\varphi\right\rangle^{k+1}\left\langle\beta\mathbb{V}\left[\varphi\right],\tilde\varphi\right\rangle^{k-1}\mathbb{V}\left[\varphi\right](x).$$

In order to complete the proof, we will use Theorem A.5 to deal with (6.23). By now the reader will be familiar with the arguments required to verify assumptions (A.5) and (A.6) and thus, we exclude the details. Hence, it follows that

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{\varphi(x)t^k}\mathtt{T}^{(k+1)}\left[f\right](x)&=\frac{(k+1)!}{2^k}\left\langle f,\tilde\varphi\right\rangle^{k+1}\left\langle\beta\mathbb{V}\left[\varphi\right],\tilde\varphi\right\rangle^{k-1}\int_0^1k(1-s)^{k-1}\left\langle\beta\mathbb{V}\left[\varphi\right],\tilde\varphi\right\rangle\mathrm{d}s\\
&=\frac{(k+1)!}{2^k}\left\langle f,\tilde\varphi\right\rangle^{k+1}\left\langle\beta\mathbb{V}\left[\varphi\right],\tilde\varphi\right\rangle^{k},
\end{aligned}$$

where the limit is uniform in $x\in E$. Moreover, $\sup_{t\geq0,x\in E}\mathtt{T}^{(k+1)}\left[f\right](x)/\varphi(x)t^k<\infty$. □

6.3 Non-critical cases

In this section we present the corresponding results for the non-critical case.

Theorem 8 (Supercritical, $\lambda > 0$)**.** *Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda > 0$. Redefine*

$$\Delta_t^{(\ell)} = \sup_{x \in E, f \in B_1^+(E)} \left| \mathrm{e}^{-\ell\lambda t} \mathrm{T}_t^{(\ell)}[f](x) - \ell! \langle f, \tilde{\varphi} \rangle^{\ell} L_\ell(x) \right|,$$

and define iteratively with $L_1(x) = \varphi(x)$ and $\mathfrak{I}_2(x) = \frac{1}{2}\int_0^\infty \mathrm{e}^{-2\lambda s} \mathrm{T}_s \left[\mathbb{V}\left[\varphi\right]\right](x) \mathrm{d}s$, for $k \geq 2$

$$L_k(x) = \mathfrak{R}_k(x) + \mathfrak{I}_k(x),$$

where

$$\mathfrak{R}_k(x) = \sum_{\{m_1,\ldots,m_{k-1}\}_k} \frac{1}{m_1! \ldots m_{k-1}!} (m_1 + \ldots + m_{k-1} - 1)! \prod_{j=1}^{k-1} (-L_j(x))^{m_j}$$

and

$$\mathfrak{I}_k(x) =$$
$$\int_0^\infty \mathrm{e}^{-\lambda k t} \mathrm{T}_s \left[\sum_{\{m_1,\ldots,m_{k-1}\}_k} \frac{1}{m_1! \ldots m_{k-1}!} \left(\psi^{(m_1+\ldots+m_{k-1})}(\cdot, 0+) (-\varphi(\cdot))^{m_1} \prod_{j=2}^{k-1} (-\mathfrak{I}_j(\cdot))^{m_j} \right.\right.$$
$$\left.\left. + \beta(\cdot) \int_{M(E)^\circ} \langle \varphi, \nu \rangle^{m_1} \prod_{j=2}^{k-1} \langle \mathfrak{I}_j, \nu \rangle^{m_j} \Gamma(\cdot, \mathrm{d}\nu) \right) \right] (x) \mathrm{d}s.$$

Here the sums run over the set $\{m_1, \ldots, m_k - 1\}_k$ of nonnegative integers such that $m_1 + 2m_2 + \ldots + (k-1)m_{k-1} = k$. Then, for all $\ell \leq k$ (6.21) holds.

It is worth remarking on the fact that there is dependency on x for the limit in the supercritical case but not in the critical. Thinking in terms of the skeletal decomposition (see e.g. [15, 2]), in principle the superprocess issued from a unit mass at x can be seen as the aggregation of a Poisson point process of 'superprocess excursions'. In the supercritical setting, a finite Poisson number of these will contribute to the overall growth of the process which are sampled at a rate proportional to $p(x)\delta_x$, where $p(x)$ is rate of survival of an excursion issued from $x \in E$. This may go part way to explaining the dependency of L_k on x in that setting. This excursion decomposition can also be developed in a subcritical setting, albeit that $p(x)$ is replaced by $p(x,t)$ the rate an excursion survives to time t (cf. [19]).

Finally we turn to the decay of moments in the subcritical setting, which offers the heuristically appealing result that the k-th moment decays slower than the k-th moment of the linear semigroup.

Theorem 9 (Subcritical, $\lambda < 0$). *Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine*

$$\Delta_t^{(k)} = \sup_{x\in E, f\in B_1^+(E)} \left|\varphi^{-1}(x)\mathrm{e}^{-\lambda t}\mathtt{T}_t^{(k)}[f](x) - L_k\right|,$$

where we define iteratively $L_1 = \langle f, \varphi\rangle$ and for $k \geq 2$,

$$L_k = \int_0^\infty \mathrm{e}^{-\lambda s}\left\langle \mathbb{V}_k\left[f\right], \tilde{\varphi}\right\rangle \mathrm{d}s,$$

with

$$\begin{aligned}\mathbb{V}_k[f](x) = &\sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{k!}{m_1!\dots m_{k-1}!} \\ &\times\left[\psi^{(m_1+\dots+m_{k-1})}(x,0+)(-\mathtt{T}_s\left[f\right](x))^{m_1}\prod_{j=2}^{k-1}\left(\frac{1}{j!}\left(-\mathtt{T}^{(j)}\left[f\right](x) + (-1)^{j+1}R_j(x,s)\right)\right)^{m_j}\right. \\ &\left. +\beta(x)\int_{M(E)^\circ}\left\langle \mathtt{T}_s\left[f\right], \nu\right\rangle^{m_1}\prod_{j=2}^{k-1}\left(\frac{1}{j!}\left\langle \mathtt{T}_s^{(j)}\left[f\right] + (-1)^j R_j(\cdot,s),\nu\right\rangle\right)^{m_j}\Gamma(x,\mathrm{d}\nu)\right].\end{aligned}$$

Here the sums run over the set $\{m_1,\dots,m_k-1\}_k$ of nonnegative integers such that $m_1 + 2m_2 + \ldots + (k-1)m_{k-1} = k$. Then, for all $\ell \leq k$ (6.21) holds.

The base case is given by the Perron Frobenius behaviour in (H1) for both sub and supercritical cases. Thus, we assume the result for $k-1$ and proceed to give the outline of the inductive step of the argument.

Proof of Theorem 8 (supercritical case). The main difference now compared to the critical case is that all the terms in $R_k(x,t)$ will survive after the normalization $\mathrm{e}^{-\lambda kt}$ as the exponential term shares across the product. From the evolution equation (6.15) and the definition of L_k we have that

$$
\begin{aligned}
&|\mathrm{e}^{-\lambda kt}\mathtt{T}_t^{(k)}[f](x) - k!\,\langle f,\tilde{\varphi}\rangle^k L_k(x)| \\
&\qquad \leq \Bigg|\varphi(x)^{-1}\mathrm{e}^{-\lambda kt}(-1)^{k+1}R_k(x,t) \\
&\qquad\quad - \langle f,\tilde{\varphi}\rangle^k \sum_{\{m_1,\dots,m_{k-1}\}} \frac{k!}{m_1!\dots m_{k-1}!}(m_1+\dots+m_{k-1}-1)!\prod_{j=1}^{k-1}(-L_j(x))^{m_j}\Bigg| \\
&\qquad\quad + \left|\mathrm{e}^{-\lambda kt}(-1)^k\int_0^t \mathtt{T}_s\left[U_k(\cdot,t-s)\right](x)\mathrm{d}s - k!\,\langle f,\tilde{\varphi}\rangle^k \mathfrak{I}_k\right|. \qquad (6.29)
\end{aligned}
$$

The first terms in the right hand side goes to zero uniformly since

$$
\begin{aligned}
\mathrm{e}^{-\lambda kt}(-1)^{k+1}R_k(x,t) = &\sum_{\{m_1,\dots,m_k\}_k}\frac{k!}{m_1!\dots m_{k-1}!}(-1)^{m_1+\dots+m_{k-1}} \\
&\times(m_1+\dots+m_{k-1}-1)!\prod_{j=1}^{k-1}\left(\frac{\mathrm{e}^{-\lambda jt}\mathtt{T}_t^{(j)}[f](x)}{j!}\right)^{m_j},
\end{aligned}
$$

and the induction hypothesis implies that

$$
\begin{aligned}
\lim_{t\to\infty}\mathrm{e}^{-\lambda kt}(-1)^{k+1}R_k(x,t) = \langle f,\tilde{\varphi}\rangle^k &\sum_{\{m_1,\dots,m_k\}_k}\frac{k!}{m_1!\dots m_{k-1}!} \\
&\times(m_1+\dots+m_{k-1}-1)!\prod_{j=1}^{k-1}(-L_j(x))^{m_j}. \qquad (6.30)
\end{aligned}
$$

For the term in (6.29), notice that we have

(6.31)

$$
\lim_{t\to\infty}\mathrm{e}^{-\lambda kt}(-1)^k\int_0^t \mathtt{T}_s\left[U_k(\cdot,t-s)\right](x)\mathrm{d}s = \lim_{t\to\infty} t\int_0^1 \mathrm{e}^{-\lambda(k-1)ut}\mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}\Big[H[f](x,u,t)\Big](x)\mathrm{d}u,
$$

where

$$
H[f](x,u,t) = (-1)^k\mathrm{e}^{-\lambda kt(1-u)}U_k(x,t(1-u)),
$$

that is,

$$H[f](x,u,t) := \sum_{\{m_1,\dots,m_{k-1}\}} \frac{k!}{m_1!\dots m_{k-1}!}\Bigg[\psi^{(m_1+\dots+m_{k-1})}(x,0+)(-\mathrm{e}^{-\lambda t(1-u)}\mathtt{T}_{t(1-u)}[f](x))^{m_1}$$
$$\prod_{j=2}^{k-1}\left(-\frac{\mathrm{e}^{-\lambda j t(1-u)}}{j!}\left(\mathtt{T}^{(j)}_{t(1-u)}[f](x)+(-1)^j R_j(x,t(1-u))\right)\right)^{m_j}$$
$$+\beta(x)\int_{M(E)^\circ}\left\langle \mathrm{e}^{-\lambda t(1-u)}\mathtt{T}_{t(1-u)}[f],\nu\right\rangle^{m_1}$$
$$\prod_{j=2}^{k-1}\left\langle\frac{\mathrm{e}^{-\lambda j t(1-u)}}{j!}\left(\mathtt{T}^{(j)}_{t(1-u)}[f]+(-1)^j R_j(\cdot,t(1-u))\right),\nu\right\rangle^{m_j}\Gamma(x,\mathrm{d}\nu)\Bigg].$$

Induction hypothesis and (H1) allows us to get

$$\begin{aligned}H[f](x) &:= \lim_{t\to\infty} H(x,u,t)\\ &= \sum_{\{m_1,\dots,m_{k-1}\}_k}\frac{1}{m_1!\dots m_{k-1}!}\Bigg(\psi^{(m_1+\dots+m_{k-1})}(x,0+)(-\varphi(x))^{m_1}\prod_{j=2}^{k-1}(-\mathfrak{I}_j(x))^{m_j}\\ &\qquad +\beta(x)\int_{M(E)^\circ}\langle\varphi,\nu\rangle^{m_1}\prod_{j=2}^{k-1}\langle\mathfrak{I}_j,\nu\rangle^{m_j}\Gamma(x,\mathrm{d}\nu)\Bigg).\end{aligned}$$

Using the expressions for $H[f](x,u,t)$ and $H[f](x)$ together with the definition of $L_k(x)$, we have, for any $\epsilon>0$, as $t\to\infty$,

$$\begin{aligned}&\sup_{x\in E, f\in B_1^+(E)}\left|\mathrm{e}^{-\lambda k t}(-1)^k\int_0^t \mathtt{T}_s[U_k(\cdot,t-s)](x)\mathrm{d}s - k!\langle f,\tilde\varphi\rangle^k\mathfrak{I}_k\right|\\ &\le t\int_0^1 \mathrm{e}^{-\lambda(k-1)ut}\sup_{x\in E, f\in B_1^+(E)}\left|\mathrm{e}^{-\lambda u t}\mathtt{T}_{ut}[H[f](\cdot,u,t)-H[f]]\right|\mathrm{d}u+\epsilon,\end{aligned}\tag{6.32}$$

where ϵ is an upper estimate for

$$\int_t^\infty \mathrm{e}^{-\lambda k t}\mathtt{T}_s\Bigg[\sum_{\{m_1,\dots,m_{k-1}\}_k}\frac{k!\langle f,\tilde\varphi\rangle^k}{m_1!\dots m_{k-1}!}\Bigg(\psi^{(m_1+\dots+m_{k-1})}(\cdot,0+)(-\varphi(\cdot))^{m_1}\prod_{j=2}^{k-1}(-\mathfrak{I}_j(\cdot))^{m_j}+$$
$$\beta(\cdot)\int_{M(E)^\circ}\langle\varphi,\nu\rangle^{m_1}\prod_{j=2}^{k-1}\langle\mathfrak{I}_j,\nu\rangle^{m_j}\Gamma(\cdot,\mathrm{d}\nu)\Bigg)\Bigg](x)\mathrm{d}s.$$

Convergence to zero as $t\to\infty$ in the term above follows thanks to the induction hypothesis, (H2) and the uniform boundedness of β. Following the same calculations as the proof of

Theorem 2 from equation (3.26) onwards, we get the desired result. $\square$

Proof of Theorem 9 (subcritical case). We now outline the proof for the subcritical case. Again we use an inductive argument. The case $k = 1$ follows from (H1) and the fact that $\langle \varphi, \tilde{\varphi} \rangle = 1$. Now assume the result to be true for $\ell = 1, \ldots, k-1$. We first note first that the term $R_k(x,t)$ in (6.15) vanishes in the limit after the normalisation $\mathrm{e}^{-\lambda t}$. To see this, note that

$$\left| \frac{\mathrm{e}^{-\lambda t}}{\varphi(x)} R_k(x,t) \right| \le \sum_{\{m_1,\ldots,m_{k-1}\}_k} c(m_1,\ldots,m_{k-1}) \prod_{j=1}^{k-1} \left| \frac{\mathrm{e}^{-\lambda t} \mathtt{T}_t^{(j)}[f](x)}{\varphi(x) j!} \right|^{m_j} \times \varphi(x)^{m_1+\ldots+m_{k-1}} \mathrm{e}^{\lambda(m_1+\ldots+m_{k-1}-1)},$$

where $c(m_1,\ldots,m_{k-1})$ is a constant depending only on $m_1,\ldots,m_{k-1}$. Since each of the terms in the product us bounded, $\lambda < 0$, and $m_1 + \ldots + m_{k-1} > 1$ for any partition, the limit of the right-hand side above is zero. Next we turn to the integral term in (6.15). Similar calculations to those given above yield

$$\frac{\mathrm{e}^{-\lambda t}}{\varphi(x)} \int_0^t \mathtt{T}_s \left[U_k(\cdot, t-s) \right](x) \mathrm{d}s$$

(6.33)

$$= t \int_0^1 \sum_{\{m_1,\ldots,m_{k-1}\}_k} \frac{k!}{m_1! \ldots m_{k-1}!} \mathrm{e}^{\lambda t u (m_1+\ldots+m_{k-1}-1)} \frac{\mathrm{e}^{-\lambda t(1-u)}}{\varphi(x)} \mathtt{T}_{t(1-u)} \left[H_{ut}^{(m_1,\ldots,m_{k-1})} \right](x) \mathrm{d}u$$

where $H_{ut}^{(m_1,\ldots,m_{k-1})}$ is defined as

$$H_{ut}^{(m_1,\ldots,m_{k-1})}(x) :=$$

$$\psi^{(m_1+\ldots+m_{k-1})}(x,0+)(-\mathrm{e}^{-\lambda ut} \mathtt{T}_{ut}[f](x))^{m_1} \prod_{j=2}^{k-1} \left(-\frac{\mathrm{e}^{-\lambda ut}}{j!} \left(\mathtt{T}_{ut}^{(j)}[f](x) + (-1)^j R_j(x, ut) \right) \right)^{m_j}$$

$$+ \beta(x) \int_{M(E)^\circ} \left\langle \mathrm{e}^{-\lambda ut} \mathtt{T}_{ut}[f], \nu \right\rangle^{m_1} \prod_{j=2}^{k-1} \left\langle \frac{\mathrm{e}^{-\lambda ut}}{j!} (\mathtt{T}_{ut}^{(j)}[f] + (-1)^j R_j(\cdot, ut)), \nu \right\rangle^{m_j} \Gamma(x, \mathrm{d}\nu).$$

Note that L_k can be written as

$$\int_0^\infty \sum_{\{m_1,\ldots,m_{k-1}\}^k} \frac{k!}{m_1! \ldots m_{k-1}!} \mathrm{e}^{\lambda(m_1+\ldots+m_{k-1}-1)s} \left\langle H_s^{(m_1,\ldots,m_{k-1})}[f], \tilde{\varphi} \right\rangle \mathrm{d}s \tag{6.34}$$

which is also convergent by appealing to (H2). As a convergent integral, it can be truncated at $t > 0$ and the residual of the integral over (t, ∞) can be made arbitrarily small by taking t sufficiently large. By changing variables in (6.34) when the integral is truncated at arbitrarily large t, so it is of a similar form to that of (6.33), we can subtract it from (6.33) to get

$$t \int_0^1 \sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{k! \mathrm{e}^{\lambda tu(m_1+\dots+m_{k-1}-1)}}{m_1! \dots m_{k-1}!} \left(\frac{\mathrm{e}^{-\lambda(1-u)t}}{\varphi(x)} \mathtt{T}_{(1-u)t}[H_{ut}^{(m_1,\dots,m_{k-1})}] - \langle H_{ut}^{(m_1,\dots,m_{k-1})}, \tilde{\varphi} \rangle \right) \mathrm{d}u,$$

One proceeds to splitting the integral of the difference over $[0, 1]$ into two integrals, one over $[0, 1-\varepsilon]$ and one over $(1-\varepsilon, 1]$. For the aforesaid integral over $[0, 1-\varepsilon]$, we can control the behaviour of $\varphi^{-1}\mathrm{e}^{-\lambda(1-u)t}\mathtt{T}_{(1-u)t}[H_{ut}^{(m_1,\dots,m_{k-1})}] - \langle H_{ut}^{(m_1,\dots,m_{k-1})}, \tilde{\varphi} \rangle$ as $t \to \infty$, making it arbitrarily small, by appealing to uniform dominated control of its argument in square brackets thanks to (H1). The integral over $[0, 1-\varepsilon]$ can thus be bounded, as $t \to \infty$, by $t(1 - \mathrm{e}^{\lambda(m_1+\dots+m_{k-1}-1)(1-\varepsilon)})/|\lambda|(m_1 + \dots + m_{k-1} - 1)$.

For the integral over $(1-\varepsilon, 1]$, we can appeal to the uniformity in (H1) and (H2) to control the entire term $\mathrm{e}^{-\lambda(1-u)t}\mathtt{T}_{(1-u)t}[H_{ut}^{(m_1,\dots,m_{k-1})}]$ (over time and its argument in the square brackets) by a global constant. Up to a multiplicative constant, the magnitude of the integral is thus of order

$$t \int_{1-\varepsilon}^1 \mathrm{e}^{\lambda(m_1+\dots+m_{k-1}-1)ut} \mathrm{d}u = \frac{(\mathrm{e}^{\lambda(m_1+\dots+m_{k-1}-1)(1-\varepsilon)t} - \mathrm{e}^{\lambda(m_1+\dots+m_{k-1}-1)t})}{|\lambda|(m_1 + \dots + m_{k-1} - 1)},$$

which tends to zero as $t \to \infty$.

□

6.4 Integrated Moments of superprocesses

We finish this chapter with the main results for integrated moments of superprocesses, which are the analogue to the branching Markov processes and exhibit very similar behaviour as in the moments' results.

For $x \in E$, $k \geq 2$ and $t \geq 0$, Define

$$\tilde{R}_k(x,t) = \sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{k!}{m_1!\dots m_{k-1}!}(-1)^{m_1+\dots+m_{k-1}-1}$$
$$(m_1+\dots+m_{k-1}-1)!\prod_{j=1}^{k-1}\left(\frac{(-1)^j \mathtt{M}_t^{(j)}[g](x)}{j!}\right)^{m_j}, \tag{6.35}$$

and

$$\tilde{K}_k(x,t) = \sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{k!}{m_1!\dots m_{k-1}!}\psi^{(m_1+\dots+m_{k-1})}(x,0+)$$
$$\prod_{j=1}^{k-1}\left(\frac{(-1)^{j+1}\mathtt{M}_t^{(j)}[g](x) - \tilde{R}_j(x,t)}{j!}\right)^{m_j}, \tag{6.36}$$

and finally

$$\tilde{S}_k(x,t) = \int_{M(E)^\circ}\sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{k!}{m_1!\dots m_{k-1}!}(-1)^{m_1+\dots+m_{k-1}}$$
$$\prod_{j=1}^{k-1}\left(\frac{\left\langle(-1)^{j+1}\mathtt{M}_t^{(j)}[g] - \tilde{R}_j(\cdot,t),\nu\right\rangle}{j!}\right)^{m_j}\Gamma(x,\mathrm{d}\nu), \tag{6.37}$$

and the sums run over the set of non-negative integers $\{m_1,\dots,m_{k-1}\}$ such that $m_1+2m_2+\dots+(k-1)m_{k-1}=k$.

Proposition 9. *Fix $k \geq 2$. Suppose that (H1) and (H2) hold, with the additional assumption that*

$$\sup_{x\in E, s\leq t}\mathtt{M}_s^{(\ell)}[f](x) < \infty, \qquad \ell \leq k-1, f\in B^+(E), t\geq 0. \tag{6.38}$$

Then,

$$\mathtt{M}_t^{(k)}[g](x) = (-1)^{k+1}\tilde{R}_k(x,t) + (-1)^k\int_0^t \mathtt{T}_s\left[\tilde{U}_k(\cdot,t-s)\right](x)\mathrm{d}s$$
$$- k\int_0^t \mathtt{T}_s\left[g\left(\mathtt{M}_{t-s}^{(k-1)}[g] + (-1)^{k-1}\tilde{R}_{k-1}(\cdot,t-s)\right)\right](x)\mathrm{d}s \tag{6.39}$$

where

$$\tilde{U}_k(x,t) = \tilde{K}_k(x,t) + \beta(x)\tilde{S}_k(x,t). \tag{6.40}$$

The proof of this Theorem is essentially the same as the proof of Theorem 8. Notice that the main difference between (6.15) and (6.39) is the last integral in the later. This time we define

$$\tilde{\mathtt{v}}_t^{(k)}[g](x) := \frac{\partial^k}{\partial\theta^k}\mathtt{V}_t[0,\theta g](x)\Big|_{\theta=0}, \qquad t \geq 0, g \in B^+(E), k \geq 1$$

as well as

$$\tilde{\mathtt{e}}_t[g](x) := \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\int_0^t \langle g, X_s\rangle \mathrm{d}s}\right], \qquad t \geq 0, f \in B^+(E),$$

and in that case, $\mathtt{V}_t[0,\theta g](x) = -\log \tilde{\mathtt{e}}_t[\theta g](x)$ and $\tilde{\mathtt{e}}_t[0](x) = 1$, so that

$$\tilde{\mathtt{e}}_t^{(k)}[\theta g](x) := \frac{\partial^k}{\partial\theta^k}\tilde{\mathtt{e}}_t[\theta g](x) = (-1)^k\mathbb{E}_{\delta_x}\left[\left(\int_0^t \langle g, X_s\rangle \,\mathrm{d}s\right)^k \mathrm{e}^{-\theta\int_0^t \langle g, X_s\rangle \mathrm{d}s}\right]$$

and

$$\tilde{\mathtt{e}}_t^{(k)}[\theta g](x)|_{\theta=0} = (-1)^k \mathtt{M}_t^{(k)}[g](x). \tag{6.41}$$

Now, if we differentiate the evolution equation (6.10) k times with $f = 0$ and θg at $\theta = 0$ we will have to consider the extra terms that come out from differentiating the last term inside the integral in (6.10). These will produce the last integral in (6.39).

6.4.1 Critical case

With this evolution equation we are ready to state and give a proof of the main results concerning the integrated moments of nonlocal superprocesses.

Theorem 10 (Critical, $\lambda = 0$)**.** *Suppose that (H1) holds along with (H2) for $k \geq 2$ and $\lambda = 0$. Define*

$$\Delta_t^{(k)} = \sup_{x\in E, f\in B_1^+(E)} \left| t^{-(2\ell-1)}\varphi(x)^{-1}\mathtt{M}_t^{\ell}[g](x) - 2^{-(\ell-1)}\ell!\,\langle g, \tilde{\varphi}\rangle^{\ell}\,\langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle^{\ell-1} L_\ell\right|,$$

where $L_1 = 1$ and L_k is defined through the recursion $L_k = (\sum_{\{k_1,k_2\}^+} L_{k_1}L_{k_2})/(2k-1)$ where $\{k_1,k_2\}^+$ is the set of positive integers k_1, k_2 such that $k_1 + k_2 = k$. Then, for all $\ell \leq k$

$$\sup_{t\geq 1}\Delta_t^{(\ell)} < \infty \text{ and } \lim_{t\to\infty}\Delta_t^{(\ell)} = 0. \tag{6.42}$$

Proof. We now proceed to prove the Theorem also by induction. First we consider the setting

$k = 1$. In that case,

$$\frac{1}{t}\mathtt{M}^{(1)}[g](x) = \frac{1}{t}\mathbb{E}_{\delta_x}\left[\int_0^t \langle g, X_s\rangle \mathrm{d}s\right] = \frac{1}{t}\int_0^t \mathtt{T}_s[g](x)\mathrm{d}s = \int_0^1 \mathtt{T}_{ut}[g](x)\mathrm{d}u.$$

Referring now to Theorem A.5 in the Appendix, we can take $F(x,s,t) = g(x)/\varphi(x)$, since $g \in B^+(E)$, the conditions of the theorem are trivially met and hence

$$\lim_{t\to\infty} \sup_{x\in E, g\in B_1^+(E)} \left|\frac{1}{t}\mathtt{M}^{(1)}[g](x) - \langle g, \tilde{\varphi}\rangle\right| = 0.$$

Now assume that the statement of the Theorem holds in the superprocess setting for all $\ell \le k$. Our aim is to prove that the result holds for $k+1$. Using Theorem 9 and a change of variables, we have that

$$\begin{aligned}\frac{1}{\varphi(x)t^{2(k+1)-1}}\mathtt{M}_t^{(k+1)}[g](x) =& \frac{(-1)^k}{\varphi(x)t^{2k+1}}\tilde{R}_{k+1}(x,t) + \frac{(-1)^{k+1}}{\varphi(x)t^{2k}}\int_0^1 \mathtt{T}_{ut}\left[\tilde{U}_{k+1}(\cdot, t(1-u))\right](x)\mathrm{d}u \\ &- \frac{k}{\varphi(x)t^{2k}}\int_0^1 \mathtt{T}_{ut}\left[g\mathtt{M}_{t(1-u)}^{(k)}[g] + (-1)^k\tilde{R}_k(\cdot, t(1-u))\right](x)\mathrm{d}u,\end{aligned} \tag{6.43}$$

Similarly to the proof of Theorem 1 will prove first that, for each $x \in E$,

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{\varphi(x)t^{2k+1}}\mathtt{M}^{(k+1)}[f](x) \\ &= \lim_{t\to\infty}\frac{1}{2\varphi(x)t^{2k}}\int_0^1 \mathtt{T}_{st}\left[\tilde{K}_{k+1}^{(2)}(\cdot, t(1-s)) + \beta(\cdot)\tilde{S}_{k+1}^{(2)}(\cdot, t(1-s))\right](x)\mathrm{d}s,\end{aligned} \tag{6.44}$$

where

$$\tilde{K}_{k+1}^{(2)}(x,t) := \sum_{\{k_1,k_2\}^+}\frac{(k+1)!}{k_1!k_2!}\psi''(x,0+)\mathtt{M}_t^{(k_1)}[g](x)\mathtt{M}_t^{(k_2)}[g](x) \tag{6.45}$$

and

$$\tilde{S}_{k+1}^{(2)}(x,t) = \int_{M(E)^\circ}\sum_{\{k_1,k_2\}^+}\frac{(k+1)!}{k_1!k_2!}\langle \mathtt{M}_t^{(k_1)}[g], \nu\rangle\langle \mathtt{M}_t^{(k_2)}[g], \nu\rangle\Gamma(x,\mathrm{d}\nu), \tag{6.46}$$

such that $\{k_1,k_2\}^+$ is defined to be the set of positive integers k_1, k_2 such that $k_1+k_2 = k+1$.

To this end, writing $c(m_1,\ldots,m_k)$ for the constants preceding the product summands in

(6.35), observe that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t^{2k+1}}\tilde{R}_{k+1}(x,t) &= \lim_{t\to\infty}\frac{(k+1)!}{t^{2k+1}}\sum_{\{m_1,\dots,m_k\}_{k+1}} c(m_1,\dots,m_k)\prod_{j=1}^{k}\left(\frac{(-1)^j\mathtt{M}_t^{(j)}[g](x)}{j!}\right)^{m_j}\\ &= (-1)^k(k+1)!\lim_{t\to\infty}\sum_{\{m_1,\dots,m_k\}_{k+1}}\frac{c(m_1,\dots,m_k)}{t^{m_1+\dots+m_k-1}}\prod_{j=1}^{k}\left(\frac{1}{j!}\frac{\mathtt{M}_t^{(j)}[g](x)}{t^{2j-1}}\right)^{m_j}\\ &= 0,\end{aligned}$$

where the final equality is due to the induction hypothesis and the fact that $m_1+\ldots+m_k > 1$, which follows from the fact that $m_1+2m_2+\ldots+\ldots+km_k = k+1$. Note, moreover that the induction hypothesis ensures that the limit is uniform in $x\in E$ and, in fact, that

$$\sup_{t\ge 0, x\in E}\frac{1}{t^{2\ell-1}}\tilde{R}_\ell(x,t) < \infty \text{ and } \lim_{t\to\infty}\sup_{x\in E}\frac{1}{t^{2\ell-1}}\tilde{R}_\ell(x,t) = 0 \qquad \ell = 1,\cdots,k+1. \tag{6.47}$$

We now deal with the last term in (6.43) and we prove that its limit vanishes as one discuses above. To this end, define

$$F[g](x,u,t) := \frac{1}{\varphi(x)t^{2k}}\left(g(x)\mathtt{M}_{t(1-u)}^{(k)}[g] + (-1)^k\tilde{R}_k(x,t(1-u))\right) \tag{6.48}$$

and notice that due to the induction hypothesis and (6.47) we get $\lim_{t\to\infty}F[f](x,u,t) = 0$. Using Theorem A.5 in the Appendix, we conclude that

$$\lim_{t\to\infty}\sup_{x\in E}\left|\frac{1}{\varphi(x)}\int_0^t \mathtt{T}_{ut}[\varphi F[g](\cdot,u,t)](x)\mathrm{d}u\right| = 0.$$

We now return to (6.43), to deal with the term involving $\tilde{U}_{k+1}$, which we recall is a linear combination of $\tilde{K}_{k+1}$ and $\tilde{S}_{k+1}$. As before, note that if any of the summands in either $\tilde{K}_{k+1}$ or $\tilde{S}_{k+1}$ have more than two of the m_j positive, the limit of that summand, when renormalised by $1/t^{2k}$, will be zero. The proof of this fact is the same as in the proof of Theorem 7 with $\tilde{U}$, $\tilde{K}$ and $\tilde{S}$ replacing U, K and S, respectively.

Now, following the same reasoning and calculations as in the proof of Theorem 1, if we define

$$F[g](x,u,t) := \frac{1}{2\varphi(x)t^{2k}}\left(\tilde{K}_{k+1}^{(2)}(x,t(1-u)) + \beta(x)\tilde{S}_2^{(2)}(x,t(1-u))\right). \tag{6.49}$$

Due to (1.25) and the induction hypothesis,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{2\varphi(x)t^{2k}}\tilde{K}^{(2)}_{k+1}(x,t(1-s))\\
&\qquad=\frac{(1-u)^{2(k+1)-2}}{2\varphi(x)}\sum_{\{k_1,k_2\}^+}\frac{(k+1)!}{k_1!k_2!}\psi''(x,0+)\lim_{t\to\infty}\frac{\mathtt{M}^{(k_1)}_{t(1-u)}\left[g\right](x)}{(t(1-u))^{2k_1-1}}\frac{\mathtt{M}^{(k_2)}_{t(1-u)}\left[g\right](x)}{(t(1-u))^{2k_2-1}}\\
&\qquad=(1-u)^{2k}\varphi(x)\sum_{\{k_1,k_2\}^+}(k+1)!2^{-k}\psi''(x,0+)\left\langle g,\tilde{\varphi}\right\rangle^{k+1}\left\langle \mathbb{V}\left[\varphi\right],\tilde{\varphi}\right\rangle^{k-1}L_{k_1}L_{k_2}\\
&\qquad=(1-u)^{2}\varphi(x)(k+1)!2^{-k}\psi''(x,0+)\left\langle g,\tilde{\varphi}\right\rangle^{k+1}\left\langle \mathbb{V}\left[\varphi\right],\tilde{\varphi}\right\rangle^{k-1}\sum_{\{k_1,k_2\}^+}L_{k_1}L_{k_2}.
\end{aligned}$$

To obtain the limit for $S^{(2)}_{k+1}$, we use (1.25), the induction hypothesis, dominated convergence and linearity to obtain

$$\begin{aligned}
&\lim_{t\to\infty}\frac{S^{(2)}_{k+1}(x,t(1-s))}{2\varphi(x)t^{k-1}}\\
&=\frac{(1-s)^{2k}}{2\varphi(x)}\int_{M(E)^\circ}\sum_{\{k_1,k_2\}^+}\frac{(k+1)!}{k_1!k_2!}\lim_{t\to\infty}\frac{\left\langle\mathtt{M}^{(k_1)}_{t(1-s)}\left[g\right],\nu\right\rangle}{(t(1-s))^{2k_1-1}}\frac{\left\langle\mathtt{M}^{(k_2)}_{t(1-s)}\left[g\right],\nu\right\rangle}{(t(1-s))^{2k_2-1}}\Gamma(x,\mathrm{d}\nu)\\
&=\frac{(1-s)^{2k}}{2^k\varphi(x)}\int_{M(E)^\circ}\sum_{\{k_1,k_2\}^+}(k+1)!\left\langle g,\tilde{\varphi}\right\rangle^{k+1}\left\langle \mathbb{V}\left[\varphi\right],\tilde{\varphi}\right\rangle^{k-1}\left\langle\varphi,\nu\right\rangle^2\Gamma(x,\mathrm{d}\nu)L_{k_1}L_{k_2}\\
&=\frac{(1-s)^{2k}}{2^k\varphi(x)}(k+1)!\left\langle g,\tilde{\varphi}\right\rangle^{k+1}\left\langle \mathbb{V}\left[\varphi\right],\tilde{\varphi}\right\rangle^{k-1}\int_{M(E)^\circ}\left\langle\varphi,\nu\right\rangle^2\Gamma(x,\mathrm{d}\nu)\sum_{\{k_1,k_2\}^+}L_{k_1}L_{k_2}.
\end{aligned}$$

Combining these two limits, then we get that

$$\begin{aligned}
F[g](x,s):=&\lim_{t\to\infty}F[g](x,u,t)\\
=&\frac{k(1-u)^{2k}}{\varphi(x)}\frac{(k+1)!}{2^k}\left\langle g,\tilde{\varphi}\right\rangle^{k+1}\left\langle \mathbb{V}\left[\varphi\right],\tilde{\varphi}\right\rangle^{k-1}\mathbb{V}\left[\varphi\right](x)\sum_{\{k_1,k_2\}^+}L_{k_1}L_{k_2}.
\end{aligned}$$

In order to complete the proof, we will use Theorem A.5 to deal with (6.44). By now the reader will be familiar with the arguments required to verify assumptions (A.5) and (A.6) and thus, we exclude the details. Hence, it follows that

$$\lim_{t\to\infty}\frac{\mathtt{M}^{(k+1)}\left[g\right](x)}{\varphi(x)t^{2k+1}} = \frac{(k+1)!}{2^k}\langle g,\tilde{\varphi}\rangle^{k+1}\langle\mathbb{V}\left[\varphi\right],\tilde{\varphi}\rangle^{k-1}\int_0^1(1-u)^{2k}\langle\mathbb{V}\left[\varphi\right],\tilde{\varphi}\rangle\,\mathrm{d}s\sum_{\{k_1,k_2\}^+}L_{k_1}L_{k_2}$$
$$= \frac{(k+1)!}{2^k}\langle g,\tilde{\varphi}\rangle^{k+1}\langle\mathbb{V}\left[\varphi\right],\tilde{\varphi}\rangle^{k}L_k,$$

where the limit is uniform in $x\in E$. Moreover, $\sup_{t\geq0,x\in E}\mathtt{M}^{(k+1)}\left[g\right](x)/\varphi(x)t^{2k-1}<\infty$.

□

6.4.2 Supercritical case

Theorem 11 (Supercritical, $\lambda>0$). *Suppose that (H1) holds along with (H2) for some $k\geq 2$ and $\lambda>0$. Redefine*

$$\Delta_t^{(\ell)} = \sup_{x\in E, f\in B_1^+(E)}\left|\mathrm{e}^{-\ell\lambda t}\mathtt{M}_t^{(\ell)}[g](x) - \ell!\langle g,\tilde{\varphi}\rangle^{\ell}L_\ell(x)\right|,$$

where L_k was defined in Theorem 8, albeit that $L_1(x)=\varphi(x)/\lambda$. Then, for all $\ell\leq k$ (6.42) holds.

Proof. Similarly to the moments case, the main difference now compared to the critical case is that all the terms in $\tilde{R}_k(x,t)$ will survive after the normalization $\mathrm{e}^{-\lambda kt}$ as the exponential term shares across the product. The case $k=1$ follows the same way as in the proof of Theorem 4. Now assume the result holds for all $\ell\leq k-1$.

Reflecting on proof of Theorem 8, we note that in this setting the starting point is almost identical except that the analogue of (6.31), which is derived from (6.39), is now the need to evaluate

$$\lim_{t\to\infty}\mathrm{e}^{-\lambda kt}\mathtt{M}_t^{(k)}\left[g\right](x) = (-1)^{k+1}\lim_{t\to\infty}\mathrm{e}^{-\lambda kt}\tilde{R}_k(x,t) + (-1)^k\lim_{t\to\infty}\mathrm{e}^{-\lambda kt}\int_0^t\mathtt{T}_s\left[\tilde{U}_k(\cdot,t-s)\right](x)\mathrm{d}s$$
$$- k\lim_{t\to\infty}\mathrm{e}^{-\lambda kt}\int_0^t\mathtt{T}_s\left[g\left(\mathtt{M}_{t-s}^{(k-1)}\left[g\right] + (-1)^{k-1}\tilde{R}_{k-1}(\cdot,t-s)\right)\right](x)\mathrm{d}s \tag{6.50}$$

The first two terms on the right-hand side of (6.50) can be handled in essentially the same way as in the proof of Theorem 8. The third one has limit zero, which can easily be seen similarly from earlier proofs, using the induction hypothesis. Hence combined with the first term on the right-hand side of (4.16), we recover the same recursion equation for L_k.

$\square$

6.4.3 Subcritical case

Theorem 12 (Subcritical, $\lambda < 0$)**.** *Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine*

$$\Delta_t^{(\ell)} = \sup_{x \in E, f \in B_1^+(E)} \left| \mathtt{M}_t^{(\ell)}[g](x) - L_\ell(x) \right|,$$

where $L_1(x) = \int_0^\infty \mathtt{T}_s[g](x)\mathrm{d}s$ and for $k \geq 2$, the constants L_k are defined recursively via

$$\begin{aligned} L_k(x) =& (-1)^{k+1}\mathbb{R}_k(x) + (-1)^k \int_0^\infty \mathtt{T}_s\left[\mathbb{U}_k\right](x)\mathrm{d}s \\ & - k\int_0^\infty \mathtt{T}_s\left[g\left(L_{k-1} + (-1)^{k-1}\mathbb{R}_{k-1}\right)\right](x)\mathrm{d}s, \end{aligned}$$

where

$$\begin{aligned} \mathbb{R}_k(x) = & \sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{k!}{m_1!\dots m_{k-1}!}(-1)^{m_1+\dots+m_{k-1}-1} \\ & \qquad (m_1+\dots+m_{k-1}-1)!\prod_{j=1}^{k-1}\left(\frac{(-1)^j L_j(x)}{j!}\right)^{m_j}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{U}_k(x) = & \sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{k!}{m_1!\dots m_{k-1}!} \\ & \left[\psi^{(m_1+\dots+m_{k-1})}(x,0+)\left(\mathtt{T}_t\left[f\right](x)\right)^{m_1} \prod_{j=2}^{k-1}\left(\frac{(-1)^{j+1}L_j(x) - \mathbb{R}_j(x)}{j!}\right)^{m_j}\right. \\ & \left. + \beta(x)\int_{M(E)^\circ}(-1)^{m_1+\dots+m_{k-1}}\langle L_1,\nu\rangle^{m_1}\prod_{j=2}^{k-1}\left(\frac{\langle(-1)^{j+1}L_j - \mathbb{R}_j,\nu\rangle}{j!}\right)^{m_j}\Gamma(x,\mathrm{d}\nu)\right], \end{aligned}$$

Then, for all $\ell \leq k$ (6.42) holds.

The proof of this result follows very similar calculations as in the proofs of Theorem 6, so in order to avoid repetition, we left the details to the reader.

Appendix A

We first state a Lemma that allows us to manipulate integral equations and is used repeatedly in this thesis to work effectively with evolution equations. This result follows similar arguments as Lemma 1.2, Chapter 4 in [13], and can be found in [25]. For the following result, denoting $(\xi, \mathbf{P})$ as a Markov process we define the associated expectation semigroup as

$$\mathtt{P}_t\left[g\right](x) = \mathbf{E}_x\left[g(\xi_t)\right], \quad t \geq 0,\, g \in B^+(E),\, x \in E.$$

Now suppose that $\beta : E \mapsto \mathbb{R}^+$ is the branching rate of the process and define

$$\mathtt{P}^\beta_t\left[g\right](x) = \mathbf{E}_x\left[\mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s} g(\xi_t)\right], \quad t \geq 0.$$

We will refer to the term $\exp\left(-\int_0^t \beta(\xi_s)\mathrm{d}s\right)$ as the *multiplicative potential.*
Suppose that the expectation semigroup $\mathtt{P}^\beta = (\mathtt{P}^\beta_t, t \geq 0)$ forms the basis of the evolution equation

$$\chi_t(x) = \mathtt{P}^\beta_t\left[g\right](x) + \int_0^t \mathtt{P}^\beta_s\left[h_{t-s}\right](x)\mathrm{d}s, \tag{A.1}$$

where $g \in B^+(E)$ and $h : [0,\infty) \times E \to [0,\infty)$ is such that $\sup_{s\leq t}|h_s| \in B^+(E)$ for all $t \geq 0$. Note that this assumptions imply that $\sup_{s\leq t}|\chi_s| \in B^+(E)$.

Lemma A.1. *Suppose that $|\beta| \in B^+(E)$, $g \in B^+(E)$ and $\sup_{s\leq t}|h_s| \in B^+(E)$, for all $t \geq 0$. Then if $(\chi_t, t \geq 0)$ is a solution to (A.1) then it also solves*

$$\chi_t(x) = \mathtt{P}_t\left[g\right](x) + \int_0^t \mathtt{P}_s\left[h_{t-s} - \beta\chi_{t-s}\right](x)\mathrm{d}s. \tag{A.2}$$

The converse is also true if $(\chi_t, t \geq 0)$ solves (A.2) with $\sup_{s\leq t}|\chi_s| \in B^+(E)$ for all $t \geq 0$.

In these calculations, the *multiplicative potential* in $\mathtt{P}^\beta$ is removed and appears instead as an *additive potential*. Note also that within the class of solutions $(\chi_t, t \geq 0)$ for which $\sup_{s\leq t}|\chi_s| \in B^+(E)$, for all $t \geq 0$, Grönwall's Lemma gives us that both (A.1) and (A.2) have unique solutions. This manipulation will be referred to as the *principle of transferring between multiplicative and additive potentials*, as it removes the term $\exp\left(-\int_0^t \beta(\xi_s)\mathrm{d}s\right)$ and adds an additive potential.

Proof of Lemma A.1. We start by writing

$$\Gamma_t := \mathrm{e}^{-\int_0^t \beta(\xi_s)\mathrm{d}s},$$

then we have that

$$\mathrm{d}\left(\frac{1}{\Gamma_t}\right) = \frac{\beta(\xi_t)}{\Gamma_t}\mathrm{d}t.$$

Then, for $t \geq 0$,

$$\frac{1}{\Gamma_t} - 1 = \int_0^t \frac{\beta(\xi_u)}{\Gamma_u}\mathrm{d}u. \tag{A.3}$$

Now assume that equation (A.1) holds. Using the Markov property we have that

$$\begin{aligned}
\mathtt{P}_s\left[\beta\mathtt{P}^\beta_{t-s}[g]\right](x) &= \mathbf{E}_x\left[\beta(\xi_s)\ \mathbf{E}_y\left[\Gamma_{t-s}g(\xi_{t-s})\right]\big|_{y=\xi_s}\right] \\
&= \mathbf{E}_x\left[\beta(\xi_s)\frac{\Gamma_t}{\Gamma_s}g(\xi_t)\right]
\end{aligned}$$

and hence using (A.1),

$$\begin{aligned}
\mathtt{P}_s\left[\beta\chi_{t-s}\right](x) &= \mathtt{P}_s\left[\beta\mathtt{P}^\beta_{t-s}[g]\right](x) + \int_0^{t-s}\mathtt{P}_s\left[\beta\mathtt{P}^\beta_u\left[h_{t-s-u}\right]\right](x)\mathrm{d}u \\
&= \mathbf{E}_x\left[\beta(\xi_s)\frac{\Gamma_t}{\Gamma_s}g(\xi_t)\right] + \int_0^{t-s}\mathbf{E}_x\left[\beta(\xi_s)\frac{\Gamma_{s+u}}{\Gamma_s}h_{t-s-u}(\xi_{s+u})\right]\mathrm{d}u \\
&= \mathbf{E}_x\left[\beta(\xi_s)\frac{\Gamma_t}{\Gamma_s}g(\xi_t)\right] + \int_s^t\mathbf{E}_x\left[\beta(\xi_s)\frac{\Gamma_u}{\Gamma_s}h_{t-u}(\xi_u)\right]\mathrm{d}u,
\end{aligned}$$

where we have used a change of variable for the integral in the last equality. Integrating the terms above, applying Fubinni's Theorem and using (A.3), we get

$$\begin{aligned}\int_0^t \mathtt{P}_s\left[\beta\chi_{t-s}\right](x)\mathrm{d}s &= \mathbf{E}_x\left[\Gamma_t g(\xi_t)\int_0^t \frac{\beta(\xi_s)}{\Gamma_s}\mathrm{d}s\right] + \int_0^t\int_s^t \mathbf{E}_x\left[\beta(\xi_s)\frac{\Gamma_u}{\Gamma_s}h_{t-u}(\xi_u)\right]\mathrm{d}u\ \mathrm{d}s\\ &= \mathbf{E}_x\left[\Gamma_t g(\xi_t)\int_0^t \frac{\beta(\xi_s)}{\Gamma_s}\mathrm{d}s\right] + \int_0^t \mathbf{E}_x\left[\Gamma_u h_{t-u}(\xi_u)\int_0^u \frac{\beta(\xi_s)}{\Gamma_s}\mathrm{d}s\right]\mathrm{d}u\\ &= \mathbf{E}_x\left[\Gamma_t g(\xi_t)\left(\frac{1}{\Gamma_t}-1\right)\right] + \int_0^t \mathbf{E}_x\left[\Gamma_u h_{t-u}(\xi_u)\left(\frac{1}{\Gamma_u}-1\right)\mathrm{d}s\right]\mathrm{d}u\end{aligned}$$

The definitions of $\mathtt{P}$ and $\mathtt{P}^\beta$ allow us to conclude that

$$\int_0^t \mathtt{P}_s\left[\beta\chi_{t-s}\right](x)\mathrm{d}s = \mathtt{P}_t\left[g\right](x) - \mathtt{P}^\beta_t\left[g\right](x) + \int_0^t \mathtt{P}_u\left[h_{t-u}\right](x)\mathrm{d}u - \int_0^t \mathtt{P}^\beta_u\left[h_{t-u}\right](x)\mathrm{d}u.$$

Rearranging terms, this tells us that

$$\mathtt{P}^\beta_t\left[g\right](x) + \int_0^t \mathtt{P}^\beta_u\left[h_{t-u}\right](x)\mathrm{d}u = \mathtt{P}_t\left[g\right](x) + \int_0^t \mathtt{P}_u\left[h_{t-u} - \beta\chi_{t-s}\right](x)\mathrm{d}s.$$

In other words, (A.1) implies (A.2). Reversing the arguments above, with the assumption that $\sup_{s\le t}|\chi_s| \in B^+(E)$, for all $t \ge 0$, we also see that (A.2) solves (A.1). $\square$

We now state one combinatorial result for independent, identically distributed random variables (iid). Then we present two fundamental combinatorial results for complex derivatives, the classical product Leibniz and Faà di Bruno's rules. In both cases, for a sufficiently smooth function g on $\mathbb{R}$, we will denote by $g^{(k)}$ by its k-th derivative.

Lemma A.2. *Suppose that $Y_1, \dots, Y_n$ are iid random variables which are equal in distribution to (Y, P). Then*

$$E\left[\left(\sum_{i=1}^n Y_i\right)^k\right] = \sum_{j=1}^k \binom{n}{j} \sum_{[k_1,\dots,k_j]_+} \binom{k}{k_1,\dots,k_j} \prod_{i=1}^j E\left[Y^{k_i}\right], \tag{A.4}$$

where the sum is over the set of all combinations of strictly positive $\{k_1, \dots, k_j\}$ such that $\sum_{i=1}^j k_i = k$

Lemma A.3 (Product Leibniz rule)**.** *Suppose $g_1, \dots, g_m$ are k-times continuously differentiable functions on $\mathbb{R}$, for $k \ge 1$. Then*

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\prod_{i=1}^m g_i(x)\right) = \sum_{k_1+\dots+k_m=k} \binom{k}{k_1,\dots,k_m} \prod_{\ell=1}^m g_\ell^{(k_\ell)}(x),$$

where the sum is taken over all non-negative integers $k_1, \cdots, k_m$ such that $\sum_{i=1}^m k_i = k$.

Lemma A.4 (Faà di Bruno rule). *Let f and g k-times continuously differentiable functions on $\mathbb{R}$. Then the k-th derivative is given by the following formula*

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} f(g(x)) = \sum_{\{m_1,\dots,m_k\}_k} \frac{k!}{m_1!\dots m_k!} f^{(m_1+\dots+m_k)}(g(x)) \prod_{j=1}^k \left(\frac{g^{(j)}(x)}{j!}\right)^{m_j},$$

where the sum goes over the set $\{m_1,\dots,m_k\}_k$ of non-negative integers such that

$$m_1 + 2m_2 + \ldots + km_k = k.$$

The last result of the appendix is a general ergodic limit theorem which is key to the moment convergence. We will only state the result in the critical case, since we will only apply it in the proofs of Theorems 1, 4, 7 and 10.

In order to state it, let us introduce a class of functions $\mathcal{C}$ on $E \times [0,1] \times [0,\infty)$ such that F belongs to class $\mathcal{C}$ if

$$F(x,s) := \lim_{t\to\infty} F(x,s,t), \qquad x \in E, s \in [0,1],$$

exists,

$$\sup_{x\in E, s\in[0,1]} |\varphi F(x,s)| < \infty, \tag{A.5}$$

and

$$\lim_{t\to\infty} \sup_{x\in E, s\in[0,1]} \varphi(x)|F(x,s) - F(x,s,t)| = 0. \tag{A.6}$$

Theorem A.5. *Assume (H1) holds, $\lambda = 0$ and that $F \in \mathcal{C}$. Define*

$$\Xi_t = \sup_{x\in E}\left|\frac{1}{\varphi(x)}\int_0^1 \mathtt{T}_{ut}[\varphi F(\cdot,u,t)](x)\mathrm{d}u - \int_0^1 \langle\tilde{\varphi}\varphi, F(\cdot,u)\rangle \mathrm{d}u\right|, \qquad t \geq 0.$$

Then

$$\sup_{t\geq 0} \Xi_t < \infty \text{ and } \lim_{t\to\infty} \Xi_t = 0. \tag{A.7}$$

Proof. We will show that

$$\lim_{t\to\infty}\sup_{x\in E}\left|\frac{1}{\varphi(x)}\mathtt{T}_{ut}[\varphi F(\cdot,u,t)](x)-\langle\tilde{\varphi}\varphi,F(\cdot,u)\rangle\right|=0,$$

since then

$$\begin{aligned}\lim_{t\to\infty}\sup_{x\in E}&\left|\int_0^1\frac{1}{\varphi(x)}\mathtt{T}_{ut}[\varphi F(\cdot,u,t)](x)\mathrm{d}u-\int_0^1\langle\tilde{\varphi}\varphi,F(\cdot,u)\rangle\mathrm{d}u\right|\\ &\le\int_0^1\lim_{t\to\infty}\sup_{x\in E}\left|\frac{1}{\varphi(x)}\mathtt{T}_{ut}[\varphi F(\cdot,u,t)](x)-\langle\tilde{\varphi}\varphi,F(\cdot,u)\rangle\right|\mathrm{d}u=0.\end{aligned}$$

First note that,

$$\begin{aligned}\left|\frac{1}{\varphi(x)}\mathtt{T}_{ut}[\varphi F(\cdot,u,t)](x)-\langle\tilde{\varphi}\varphi,F(\cdot,u)\rangle\right|&\le\frac{1}{\varphi(x)}\mathtt{T}_{ut}[|\varphi F(\cdot,u,t)-\varphi F(\cdot,u)|](x)\\ &\quad+\left|\frac{1}{\varphi(x)}\mathtt{T}_{ut}[\varphi F(\cdot,u)](x)-\langle\tilde{\varphi},\varphi F(\cdot,u)\rangle\right|\end{aligned}$$

Due to assumption (A.6), for t sufficiently large, the first term on the right-hand side above can be controlled by $\varphi^{-1}(x)\mathtt{T}_{ut}[\varepsilon](x)$. Combining this with the above inequality yields

$$\begin{aligned}\sup_{x\in E}&\left|\frac{1}{\varphi(x)}\mathtt{T}_{ut}[\varphi F(\cdot,u,t)](x)-\langle\tilde{\varphi}\varphi,F(\cdot,u)\rangle\right|\\ &\le\sup_{x\in E}\left|\varphi^{-1}(x)\mathtt{T}_{ut}[\varepsilon](x)-\langle\tilde{\varphi},\varepsilon\rangle\right|+\varepsilon\|\tilde{\varphi}\|_1\\ &\quad+\sup_{x\in E}\left|\frac{1}{\varphi(x)}\mathtt{T}_{ut}[\varphi F(\cdot,u)](x)-\langle\tilde{\varphi},\varphi F(\cdot,u)\rangle\right|.\end{aligned}\tag{A.8}$$

We note that (A.5) and the first (resp. second) statement of (1.10) in (H1), together with dominated convergence, immediately imply that the first (resp. second) statement in (A.7) holds. □

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