Self-similar Markov processes

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- Diffusions → Brownian motion ✔
- Cts-time Markov processes with jumps → Lévy processes ✔
- Self-similar Markov processes ↑
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Lévy processes

- **Stick to one-dimension**
  - A Lévy process is an $\mathbb{R}$-valued random trajectory \( \{X_t : t \geq 0\} \) issued from the origin with paths that are right-continuous and left limits and which has stationary and independent increments.
  - More formally stationary and independent increments means:
    - for \( 0 \leq s \leq t < \infty \), \( X_t - X_s = X_{t-s} \)
    - for \( 0 \leq s \leq t < \infty \), \( X_t - X_s = X_{t-s} \).
  - It can be shown that this means the entire process is characterised by its position at time 1
    \[
    \mathbb{E}[e^{i\theta X_t}] = e^{-\psi(\theta)t}
    \]
    for some appropriate function $\Psi$ (the characteristic exponent).
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Brownian motion
Compound Poisson process
Brownian motion + compound Poisson process
Bounded variation paths
Space exploration: some successes and dissatisfaction

Fundamentally we want to understand how Lévy processes explore space.

25 years of research has been very successful in giving an (relatively) complete theoretical description .....  

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Example 1:

\[ 
\mathbb{P}(\text{Process first exceeds level } x \text{ by an amount } y) = \int_{[0,x)} U(dz)\bar{\nu}(z-x+y) 
\]

where

\[ 
\Psi(\theta) = \kappa^+(\theta) - i\theta \kappa^-(\theta), \quad \theta \in \mathbb{R},
\]

\[ 
\kappa^+(\lambda) = q + \delta \lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\nu(dx), \quad \lambda \geq 0,
\]

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\bar{\nu}(x) = \nu(x,\infty) \quad \text{and} \quad \int_{[0,\infty)} e^{-\lambda x} U(dx) = \frac{1}{\kappa^+(\lambda)}
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Example 1:

\[ P(\text{Process first exceeds level } x \text{ by an amount } y) = \int_{[0,x)} U(dz)\tilde{\nu}(z-x+y) \]

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Example 2:

Under appropriate assumptions,

\[ \mathbb{P}(\text{Process ever hits a point } x) = \frac{u(x)}{u(0)}, \quad x \in \mathbb{R}, \]

where

\[ \int_{\mathbb{R}} e^{i\theta x} u(x) dx = \frac{1}{\Psi(\theta)}, \quad \theta \in \mathbb{R}. \]
Self-similar Markov processes on $\mathbb{R}$

$\alpha$-ssMp

$\mathbb{R}$-valued Markov process, equipped with initial measures $P_x$, $x \in \mathbb{R}\setminus\{0\}$, with $0$ an absorbing state, satisfying the scaling property

$$(cX_{c^{-\alpha}t})_{t \geq 0}|_{P_x} \overset{d}{=} X|_{P_{cx}}, \quad x, c > 0$$
Space-time changes and modulation

It turns out that every \( \mathbb{R} \)-valued ssMp can be characterised using polar coordinates in \( \mathbb{C} \) (think \( re^{i\theta} \)) as follows:

\[
X_t = |x| \exp \left\{ \xi \varphi(|x|-\alpha t) + i\pi (J\varphi(|x|-\alpha t) + 1) \right\}, \quad t \geq 0, x \neq 0,
\]

where \((\xi, J)\) is a so-called Markov modulated Lévy process and

\[
\varphi(t) = \inf \left\{ s > 0 : \int_0^s e^{\alpha \xi_u} du > t \right\}.
\]

\((\xi, J):\)

- \( J = \{J_t : t \geq 0\} \) is a Markov chain on \( \{1, 2\} \) with intensity matrix \( Q \).
- When \( J_t = i \), \( \xi \) moves as a Lévy process of type \( i \). “\( d\xi_t = d\xi_t^{(i)} \)”
- When \( J \) makes a jump at time \( t \), e.g. \( 1 \rightarrow 2 \), then \( \xi \) experiences an additional jump \( \Delta \xi_t \) which is an i.i.d. copy of some pre specified r.v. \( U_{1,2} \).
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$(\xi, J)$: Markov modulated Lévy processes can also be characterised by a “characteristic exponentnt”.

$$\mathbb{E}_i[e^{i\theta X_t}; J_t = j] = (\exp\{-\Psi(\theta)t\})_{i,j}$$

where

$$\Psi(\theta) = \begin{pmatrix} \psi_1(\theta) & 0 \\ 0 & \psi_2(\theta) \end{pmatrix} - Q \circ \begin{pmatrix} 1 & \mathbb{E}(e^{i\theta U_{1,2}}) \\ \mathbb{E}(e^{i\theta U_{2,1}}) & 1 \end{pmatrix}$$
If the Markov chain has an absorbing state, then the ssMp is in effect a “positive self-similar Markov process” (pssMp)

$$X_t = |x| \exp \left\{ \xi \varphi(|x|^{-\alpha} t) \right\}, \quad t \geq 0, x \neq 0,$$

where $\xi$ is a Lévy process.
Positive feedback

- There is one class of Lévy processes which has always been considered to be “the next best thing after Brownian motion”: the \((\alpha, \rho)\)-stable process.

\[
\Psi(\theta) = |\theta|^\alpha \left( e^{\pi i \alpha (\frac{1}{2} - \rho)} 1_{\{\theta > 0\}} + e^{-\pi i \alpha (\frac{1}{2} - \rho)} 1_{\{\theta < 0\}} \right), \quad \theta \in \mathbb{R},
\]

Only allowed to take \(\alpha \in (0, 2],\ \rho \in [0, 1]\).

- In fact stable processes are also self-similar Markov processes:

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\mathbb{E}[e^{i\theta X_t}] = e^{-\Psi(\theta)t} \quad \text{and} \quad \mathbb{E}[e^{i\theta cX_{c^{-\alpha}t}}] = e^{-\Psi(\theta)t} \quad \text{for all} \ c > 0.
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- When \(\rho \in (0, 1)\) (keep away from complete asymmetry!) and \(\alpha \in (0, 2)\) then the underlying Markov modulated Lévy process has exponent

\[
\Psi(\theta) = \left( \begin{array}{cc}
\frac{\Gamma(\alpha - i\theta) \Gamma(1+i\theta)}{\Gamma(\alpha \rho - i\theta) \Gamma(1 - \alpha \rho + i\theta)} & \frac{\Gamma(\alpha - i\theta) \Gamma(1+i\theta)}{\Gamma(\alpha \rho) \Gamma(1 - \alpha \rho)} \\
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Only allowed to take \(α \in (0, 2], ρ \in [0, 1]\).

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- Take the trajectory of an \((\alpha, \rho), \alpha \in (0, 1), \rho \in (0, 1)\) and imagine cutting out all the negative parts of its trajectory and then shunting up the remaining bits of path

- The resulting object is a positive self-similar Markov process. The underlying Lévy process, \(\xi\), has exponent

\[
\psi(\theta) = \frac{\Gamma(\alpha \rho - i \theta)}{\Gamma(-i \theta)} \times \frac{\Gamma(1 - \alpha \rho + i \theta)}{\Gamma(1 - \alpha + i \theta)}, \quad \theta \in \mathbb{R}.
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- Take the trajectory of an \((\alpha, \rho), \alpha \in (0, 1), \rho \in (0, 1)\) and imagine cutting out all the negative parts of its trajectory and then shunting up the remaining bits of path.

![Graph showing the trajectory](image)

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\( \alpha \in (0, 1) \)

\[ \mathbb{P}(\text{Stable process first enters } [0, 1] \text{ in } \mathrm{d}y) = \mathbb{P}(\xi \text{ first enters } (-\infty, 0] \text{ in } \mathrm{d} (\log y)) = \frac{\sin(\pi \alpha \hat{\rho})}{\pi} x^{\alpha \rho} y^{-\alpha \rho} (x - 1)^{\alpha \hat{\rho}} (1 - y)^{-\alpha \hat{\rho}} (x - y)^{-1} \mathrm{d}y \]

\( \alpha \in (1, 2) \)

\[ \mathbb{P}(\text{Stable process hits 1 before 0 when starting from } x > 0) = \mathbb{P}(\xi \text{ ever hits 0 when starting from } \log x) = \frac{\sin(\pi \rho \alpha) - |x - 1|^{\alpha - 1} \left[ \mathbf{1}_{(x > 1)} \sin(\pi \hat{\rho} \alpha) + \mathbf{1}_{(0 < x < 1)} \sin(\pi \rho \alpha) \right] + x^{\alpha - 1} \sin(\pi \hat{\rho} \alpha)}{(\sin(\pi \rho \alpha) + \sin(\pi \hat{\rho} \alpha))}. \]
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\[
\mathbb{P}(\text{Stable process first enters } [0, 1] \text{ in } dy) = \mathbb{P}(\xi \text{ first enters } (-\infty, 0] \text{ in } d(\log y)) = \sin(\pi \hat{\rho} \alpha) = \frac{\sin(\pi \alpha \hat{\rho})}{\pi} x^{\alpha \rho} y^{-\alpha \rho} (x - 1)^{\alpha \hat{\rho}} (1 - y)^{-\alpha \hat{\rho}} (x - y)^{-1} dy
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- \( \alpha \in (1, 2) \)

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\mathbb{P}(\text{Stable process hits 1 before 0 when starting from } x > 0) = \mathbb{P}(\xi \text{ ever hits 0 when starting from } \log x) = \sin(\pi \rho \alpha) - |x - 1|^{\alpha - 1} [\mathbf{1}_{(x > 1)} \sin(\pi \hat{\rho} \alpha) + \mathbf{1}_{(0 < x < 1)} \sin(\pi \rho \alpha)] + x^{\alpha - 1} \sin(\pi \rho \alpha) (\sin(\pi \rho \alpha) + \sin(\pi \hat{\rho} \alpha))^{-1}
\]
It’s possible to extend the notion of both Lévy processes and ssMp to higher dimensions.

For example, a $d$-dimensional isotropic stable Lévy process is also a ssMp:

$$E[e^{i\theta \cdot X_t}] = \exp\{-|\theta|^\alpha t\}, \quad t \geq 0, \theta \in \mathbb{R}^d,$$

necessarily $\alpha \in (0, 2]$.

The radial distance of such a process from the origin, $|X_t|$, $t \geq 0$, is a pssMp. Its underlying Lévy process has characteristic exponent

$$\Psi(\theta) = \frac{\Gamma\left(\frac{1}{2}(-i\theta + \alpha)\right)}{\Gamma\left(-\frac{1}{2}i\theta\right)} \frac{\Gamma\left(\frac{1}{2}(i\theta + d)\right)}{\Gamma\left(\frac{1}{2}(i\theta + d - \alpha)\right)}, \quad \theta \in \mathbb{R}.$$
A bigger picture

- It’s possible to extend the notion of both Lévy processes and ssMp to higher dimensions.
- For example, a $d$-dimensional isotropic stable Lévy process is also a ssMp:
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  E[e^{i\theta \cdot X_t}] = \exp\{-|\theta|^\alpha t\}, \quad t \geq 0, \theta \in \mathbb{R}^d,
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The Kelvin transform is an inversion of $\mathbb{R}^d$ through the unit sphere:

$$Kx = \frac{x}{|x|^2}, \quad x \in \mathbb{R}^d.$$
Suppose that $X$ is a $d$-dimensional isotropic stable process with $d \geq 2$. Define

$$\eta(t) = \inf \{ s > 0 : \int_0^s |X_u|^{-2\alpha} \, du > t \}, \quad t \geq 0.$$  

Then, for all $x \in \mathbb{R}^d \setminus \{0\}$, $\{KX_{\eta(t)} : t \geq 0\}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}_{Kx}^h)$, where

$$\frac{d\mathbb{P}_x^h}{d\mathbb{P}_x} \bigg|_{\sigma(X_s : s \leq t)} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \quad t \geq 0,$$
Bogdan-Zak transform

The Kelvin transform maps the ball \( \{ x \in \mathbb{R}^d : |x - 1/2| \leq 1/2 \} \) to a half-space.
Bogdan-Zak transform

- How does an isotropic stable process first enter/exit a ball? → how does an isotropic stable process cross a hyperplane → use rotational symmetry to the orthogonal projection

- Where does an isotropic stable process first hit the surface of a sphere? → where does an isotropic stable process hit a hyperplane → use rotational symmetry to the orthogonal projection
How does an isotropic stable process first enter/exit a ball? \(\rightarrow\) how does an isotropic stable process cross a hyperplane \(\rightarrow\) use rotational symmetry to the orthogonal projection

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