Erratum to: Asymptotic moments of spatial branching processes

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Abstract

We correct the statements of the *non-critical* convergence theorems in [2], principally correcting the recursive constants that appear in the limits.

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1 Summary of corrections

In this short note, we remark that there were erroneous limits used in the non-critical cases for the moment convergence and occupation moments in [2]. The source of the error was a misuse of the sense in which uniformity held in various convergence arguments. This affects the nature of the constants that appear in the limits. The results for the critical cases, which were the principal results, are correct with one minor adjustment in the statement which is that $\sup_{t\geq 0}\Delta_t^{(\ell)}<\infty$ should read $\sup_{t\geq c}\Delta_t^{(\ell)}<\infty$, for any $c>0$.

In what follows we assume the notation and hypotheses of [2] and provide the correct statements and brief corrections of the proofs for the non-critical setting. The theorem numbers correspond to the same theorem numbers in [2].

Theorem 2 (Supercritical, $\lambda>0$)**.** *Suppose that (H1) holds along with (H2) for some $k\geq 2$ and $\lambda>0$. Redefine*

$$\Delta_t^{(\ell)}=\sup_{x\in E, f\in B_1^+(E)}\left|\mathrm{e}^{-\ell\lambda t}\varphi(x)^{-1}\mathtt{T}_t^{(\ell)}[f](x)-\ell!\langle f,\tilde{\varphi}\rangle^\ell L_\ell(x)\right|,$$

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where $L_1(x) = 1$ and we define iteratively for $k \geq 2$

$$L_k(x) = \int_0^\infty \mathrm{e}^{-\lambda k s}\varphi(x)^{-1}\mathtt{T}_s\left[\beta\mathcal{E}_\cdot\Big[\sum_{[k_1,\ldots,k_N]_k^2}\prod_{\substack{j=1\\ j:k_j>0}}^{N}\varphi(x_j)L_{k_j}(x_j)\Big]\right](x)\mathrm{d}s,$$

with $[k_1,\ldots,k_N]_k^2$ is the set of all non-negative N-tuples $(k_1,\ldots,k_N)$ such that $\sum_{i=1}^N k_i = k$ and at least two of the k_i are strictly positive[1] *if $(X,\mathbb{P})$ is a branching Markov process. Alternatively, if $(X,\mathbb{P})$ is a superprocess, define iteratively for $k \geq 2$ with $L_1(x) = 1$ and $\mathfrak{I}_2(x) = \frac{1}{2}\varphi^{-1}(x)\int_0^\infty \mathrm{e}^{-2\lambda s}\mathtt{T}_s\left[\mathbb{V}\left[\varphi\right]\right](x)\mathrm{d}s$*

$$L_k(x) = \mathfrak{R}_k(x) + \mathfrak{I}_k(x),$$

where

$$(1)\quad \mathfrak{R}_k(x) = \sum_{\{m_1,\ldots,m_{k-1}\}_k}\frac{1}{m_1!\ldots m_{k-1}!}(m_1+\ldots+m_{k-1}-1)!\varphi(x)^{m_1+\ldots+m_{k-1}-1}\prod_{j=1}^{k-1}(-L_j(x))^{m_j}$$

and

$$\begin{aligned}\mathfrak{I}_k(x)&\\ = \int_0^\infty \mathrm{e}^{-\lambda k t}\varphi^{-1}(x)\mathtt{T}_s&\left[\sum_{\{m_1,\ldots,m_{k-1}\}_k}\frac{1}{m_1!\ldots m_{k-1}!}\left(\psi^{(m_1+\ldots+m_{k-1})}(\cdot,0+)(-\varphi(\cdot))^{m_1}\prod_{j=2}^{k-1}(-\varphi(\cdot)\mathfrak{I}_j(\cdot))^{m_j}+\right.\right.\\ &\left.\left.\beta(\cdot)\int_{M(E)^\circ}\langle\varphi,\nu\rangle^{m_1}\prod_{j=2}^{k-1}\langle\varphi\mathfrak{I}_j,\nu\rangle^{m_j}\,\Gamma(\cdot,\mathrm{d}\nu)\right)\right](x)\mathrm{d}s.\end{aligned}$$

Here the sums run over the set $\{m_1,\ldots,m_{k-1}\}_k$ of positive integers such that $m_1 + 2m_2 + \ldots + (k-1)m_{k-1} = k$. Then, for all $\ell \leq k$

$$(2)\qquad \sup_{t\geq 0}\Delta_t^{(\ell)} < \infty \text{ and } \lim_{t\to\infty}\Delta_t^{(\ell)} = 0.$$

Theorem 3 (Subcritical, $\lambda < 0$)**.** *Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine*

$$\Delta_t^{(k)} = \sup_{x\in E, f\in B_1^+(E)}\left|\varphi^{-1}\mathrm{e}^{-\lambda t}\mathtt{T}_t^{(k)}[f](x) - L_k\right|,$$

where we define iteratively $L_1 = \langle f,\tilde{\varphi}\rangle$ and for $k \geq 2$,

$$L_k = \langle f^k,\tilde{\varphi}\rangle + \int_0^\infty \mathrm{e}^{-\lambda s}\left\langle\beta\mathcal{E}_\cdot\Big[\sum_{[k_1,\ldots,k_N]_k^2}\binom{k}{k_1,\ldots,k_m}\prod_{\substack{j=1\\ j:k_j>0}}^{N}\mathtt{T}_s^{(k_j)}[f](x_j)\Big],\tilde{\varphi}\right\rangle\mathrm{d}s.$$

[1] Recall that we interpret $\sum_\emptyset = 0$ and $\prod_\emptyset = 1$

if $(X,\mathbb{P})$ is a branching Markov process. Alternatively, if $(X,\mathbb{P})$ is a superprocess,

$$L_k = \int_0^\infty \mathrm{e}^{-\lambda s} \left\langle \mathbb{V}_k\left[f\right], \tilde{\varphi}\right\rangle \mathrm{d}s$$

for $k \geq 2$, with

$$\begin{aligned}\mathbb{V}_k[f](x) = & \sum_{\{m_1,\ldots,m_{k-1}\}_k} \frac{k!}{m_1!\ldots m_{k-1}!} \\ & \times \Bigg[\psi^{(m_1+\ldots+m_{k-1})}(x,0+)(-\mathtt{T}_s\left[f\right](x))^{m_1} \prod_{j=2}^{k-1}\left(\frac{1}{j!}\left(-\mathtt{T}^{(j)}\left[f\right](x) + (-1)^{j+1}R_j(x,s)\right)\right)^{m_j} \\ & + \beta(x)\int_{M(E)^\circ} \left\langle \mathtt{T}_s\left[f\right], \nu\right\rangle^{m_1} \prod_{j=2}^{k-1}\left(\frac{1}{j!}\left\langle \mathtt{T}_s^{(j)}\left[f\right] + (-1)^j R_j(\cdot,s), \nu\right\rangle\right)^{m_j} \Gamma(x,\mathrm{d}\nu)\Bigg].\end{aligned}$$

Here the sums run over the set $\{m_1,\ldots,m_{k-1}\}_k$ of non-negative integers such that $m_1 + 2m_2 + \ldots + (k-1)m_{k-1} = k$. Then, for all $\ell \leq k$

$$\sup_{t\geq 0} \Delta_t^{(\ell)} < \infty \textit{ and } \lim_{t\to\infty} \Delta_t^{(\ell)} = 0. \tag{3}$$

Theorem 5 (Supercritical, $\lambda > 0$)**.** *Let $(X,\mathbb{P})$ be either a branching Markov process or a superprocess. Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda > 0$. Redefine*

$$\Delta_t^{(\ell)} = \sup_{x\in E, f\in B_1^+(E)} \left|\mathrm{e}^{-\ell\lambda t}\varphi(x)^{-1}\mathtt{M}_t^{(\ell)}[g](x) - \ell!\langle g,\tilde{\varphi}\rangle^\ell L_\ell(x)\right|,$$

where $L_k(x)$ was defined in Theorem 2 (both for branching Markov processes and superprocesses), albeit that $L_1(x) = 1/\lambda$. Then, for all $\ell \leq k$

$$\sup_{t\geq 0} \Delta_t^{(\ell)} < \infty \textit{ and } \lim_{t\to\infty} \Delta_t^{(\ell)} = 0. \tag{4}$$

Theorem 6 (Subcritical, $\lambda < 0$)**.** *Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine*

$$\Delta_t^{(\ell)} = \sup_{x\in E, f\in B_1^+(E)} \left|\varphi(x)^{-1}\mathtt{M}_t^{(\ell)}[g](x) - L_\ell(x)\right|,$$

where $L_1(x) = \int_0^\infty \varphi(x)^{-1}\mathtt{T}_s[g](x)\mathrm{d}s$ and for $k \geq 2$, the $L_k(x)$ are defined recursively via

$$\begin{aligned}L_k(x) = & \int_0^\infty \varphi(x)^{-1}\mathtt{T}_s\left[\beta\mathcal{E}_\cdot\left[\sum_{[k_1,\ldots,k_N]_k^2}\binom{k}{k_1,\ldots,k_N}\prod_{\substack{j=1\\ j:k_j>0}}^N \varphi(x_j)L_{k_j}(x_j)\right]\right](x)\,\mathrm{d}s \\ & - k\int_0^\infty \varphi(x)^{-1}\mathtt{T}_s\left[g\varphi L_{k-1}\right](x)\,\mathrm{d}s,\end{aligned}$$

if X is a branching Markov process. Alternatively, if X is a superprocess,

$$L_k(x) =(-1)^{k+1}\mathbb{R}_k(x) + (-1)^k \int_0^\infty \varphi(x)^{-1}\mathtt{T}_s\left[\mathbb{U}_k\right](x)\mathrm{d}s - k\int_0^\infty \varphi(x)^{-1}\mathtt{T}_s\left[g\varphi\left(L_{k-1}+(-1)^{k-1}\mathbb{R}_{k-1}\right)\right](x)\mathrm{d}s,$$

where

$$\mathbb{R}_k(x) = \varphi(x)^{-1}\sum_{\{m_1,\ldots,m_{k-1}\}_k}\frac{k!}{m_1!\ldots m_{k-1}!}(-1)^{m_1+\ldots+m_{k-1}-1} (m_1+\ldots+m_{k-1}-1)!\prod_{j=1}^{k-1}\left(\frac{(-1)^j\varphi(x)L_j(x)}{j!}\right)^{m_j},$$

and

$$\mathbb{U}_k(x) = \sum_{\{m_1,\ldots,m_{k-1}\}_k}\frac{k!}{m_1!\ldots m_{k-1}!} \Bigg[\psi^{(m_1+\ldots+m_{k-1})}(x,0+)\left(\varphi(x)L_1(x)\right)^{m_1}\prod_{j=2}^{k-1}\left(\frac{(-1)^{j+1}\varphi(x)L_j(x)-\varphi(x)\mathbb{R}_j(x)}{j!}\right)^{m_j} + \beta(x)\int_{M(E)^\circ}(-1)^{m_1+\ldots+m_{k-1}}\langle\varphi L_1,\nu\rangle^{m_1}\prod_{j=2}^{k-1}\left(\frac{\langle(-1)^{j+1}\varphi L_j-\varphi\mathbb{R}_j,\nu\rangle}{j!}\right)^{m_j}\Gamma(x,\mathrm{d}\nu)\Bigg].$$

Then, for all $\ell \le k$

$$\sup_{t\ge 0}\Delta_t^{(\ell)} < \infty \text{ and } \lim_{t\to\infty}\Delta_t^{(\ell)} = 0. \tag{5}$$

2 Summary of proofs

We give a brief proof of Theorems 2 and 3 to give a sense of the corrected reasoning. The proofs of Theorems 5 and 6 are left out given that the corrected reasoning uses the same logic. The reader is referblack to [1] for more details.

Proof of Theorem 2. Suppose for induction that the result is true for all ℓ-th integer moments with $1 \le \ell \le k-1$. From the evolution equation in Proposition 1 of [2], noting that $\sum_{j=1}^N k_j = k$, when the limit exists, we have

$$\lim_{t\to\infty}\mathrm{e}^{-\lambda kt}\int_0^t \varphi(x)^{-1}\mathtt{T}_s\left[\beta\mathcal{E}.\left[\sum_{[k_1,\ldots,k_N]_k^2}\binom{k}{k_1,\ldots,k_N}\prod_{j=1}^N \mathtt{T}_{t-s}^{(k_j)}[f](x_j)\right]\right](x)\mathrm{d}s$$

$$= \lim_{t\to\infty} t\int_0^1 \mathrm{e}^{-\lambda(k-1)ut}\mathrm{e}^{-\lambda ut}\varphi(x)^{-1}\mathtt{T}_{ut}\Big[H[f](x,u,t)\Big](x)\mathrm{d}u, \tag{6}$$

where

$$H[f](x,u,t) := \beta(x)\mathcal{E}_x\left[\sum_{[k_1,\dots,k_N]^2_k}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathrm{e}^{-\lambda k_j t(1-u)}\mathtt{T}^{(k_j)}_{t(1-u)}[f](x_j)\right].$$

It is easy to see that, pointwise in $x\in E$ and $u\in[0,1]$, using the induction hypothesis and (H2),

$$H[f](x) := \lim_{t\to\infty} H[f](x,u,t) = k!\langle f,\tilde{\varphi}\rangle^k\beta(x)\mathcal{E}_x\left[\sum_{[k_1,\dots,k_N]^2_k}\prod_{\substack{j=1\\ j:k_j>0}}^N \varphi(x_j)L_{k_j}(x_j)\right],$$

where we have again used the fact that the k_js sum to k to extract the $\langle f,\tilde{\varphi}\rangle^k$ term.

Using the expressions for $H[f](x,u,t)$ and $H[f](x)$ together with the definition of $L_k(x)$, we have, for any $\epsilon>0$, as $t\to\infty$,

$$\begin{aligned}&\sup_{x\in E, f\in B_1^+(E)} |\mathrm{e}^{-k\lambda t}\varphi^{-1}\mathtt{T}^{(k)}_t[f] - k!\langle f,\tilde{\varphi}\rangle^k L_k|\\ &\le t\int_0^1 \mathrm{e}^{-\lambda(k-1)ut}\sup_{x\in E, f\in B_1^+(E)}\left|\mathrm{e}^{-\lambda ut}\varphi^{-1}\mathtt{T}_{ut}\left[H[f](\cdot,u,t)-H[f]\right]\right|\mathrm{d}u+\epsilon,\end{aligned} \tag{7}$$

where ϵ is an upper estimate for

$$\sup_{x\in E, f\in B_1^+(E)} k!\langle f,\tilde{\varphi}\rangle^k\int_t^\infty \mathrm{e}^{-\lambda ks}\varphi(x)^{-1}\mathtt{T}_s\left[\beta\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]^2_k}\prod_{\substack{j=1\\ j:k_j>0}}^N \varphi(x_j)L_{k_j}(x_j)\right]\right](x)\mathrm{d}s. \tag{8}$$

Note, convergence to zero as $t\to\infty$ in (8) follows thanks to the induction hypothesis (ensuring that $L_{k_j}(x)$ is uniformly bounded), (H2) and the uniform boundedness of β.

The induction hypothesis, (H2) and dominated convergence again ensure that

$$\lim_{t\to\infty}\sup_{x\in E, f\in B_1^+(E), u\in[0,\varepsilon]} |H[f](\cdot,u,t)-H[f]| = 0. \tag{9}$$

As such, in (7), we can split the integral on the right-hand side over $[0,\varepsilon]$ and $(\varepsilon,1]$, for $\varepsilon\in(0,1)$. Using (9), we can ensure that, for any arbitrarily small $\varepsilon'>0$, making use of the boundedness in (H1), there is a global constant $C>0$ such that, for all t sufficiently large,

$$\begin{aligned}t\int_0^\varepsilon &\mathrm{e}^{-\lambda(k-1)ut}\sup_{x\in E, f\in B_1^+(E)}\left|\mathrm{e}^{-\lambda ut}\varphi^{-1}\mathtt{T}_{ut}\left[H[f](\cdot,u,t)-H[f]\right]\right|\mathrm{d}u\\ &\le \varepsilon' Ct\int_0^\varepsilon \mathrm{e}^{-\lambda(k-1)ut}\mathrm{d}u\\ &= \frac{\varepsilon' C}{\lambda(k-1)}(1-\mathrm{e}^{-\lambda(k-1)\varepsilon t}).\end{aligned} \tag{10}$$

On the other hand, we can also control the integral over $(\varepsilon, 1]$, again appealing to (H1) and (H2) to ensure that

$$\sup_{x\in E, f\in B_1^+(E), u\in(\varepsilon,1]} \left|\mathrm{e}^{-\lambda ut}\varphi^{-1}\mathtt{T}_{ut}\left[H[f](\cdot,u,t)-H[f]\right]\right| < \infty.$$

We can again work with a (different) global constant $C>0$ such that

$$\begin{aligned} t\int_\varepsilon^1 \mathrm{e}^{-\lambda(k-1)ut} &\sup_{x\in E, f\in B_1^+(E)} \left|\mathrm{e}^{-\lambda ut}\varphi^{-1}\mathtt{T}_{ut}\left[H[f](\cdot,u,t)-H[f]\right]\right| \mathrm{d}u \\ &\le Ct\int_\varepsilon^1 \mathrm{e}^{-\lambda(k-1)ut}\mathrm{d}u \\ &= \frac{C}{\lambda(k-1)}(\mathrm{e}^{-\lambda(k-1)\varepsilon t}-\mathrm{e}^{-\lambda(k-1)t}). \end{aligned} \tag{11}$$

In conclusion, using (10) and (11), we can take limits as $t\to\infty$ in (7) and the statement of the theorem follows for branching Markov processes.

The proof in the superprocess setting starts the same way as in [2] up to equation (90) therein, noting that the term $R_k(x,t)$ in the moment evolution equation

$$\mathtt{T}_t^{(k)}[f](x) = (-1)^{k+1}R_k(x,t) + (-1)^k\int_0^t \mathtt{T}_s\left[U_k(\cdot,t-s)\right](x)\mathrm{d}s, \tag{12}$$

from equation (77) of [2] can be compensated in the limit using $\mathfrak{R}_k(x,t)$ defined in (1) above. The remainder of the proof deals with the compensation of the integral term in (12).

We have

$$\begin{aligned} \lim_{t\to\infty} \mathrm{e}^{-\lambda kt}(-1)^k &\int_0^t \varphi(x)^{-1}\mathtt{T}_s\left[U_k(\cdot,t-s)\right](x)\mathrm{d}s \\ &= \lim_{t\to\infty} t\int_0^1 \mathrm{e}^{-\lambda(k-1)ut}\mathrm{e}^{-\lambda ut}\varphi(x)^{-1}\mathtt{T}_{ut}\Big[H[f](x,u,t)\Big](x)\mathrm{d}u, \end{aligned}$$

where $H[f](x,u,t)$ as

$$H[f](x,u,t) = (-1)^k\mathrm{e}^{-\lambda kt(1-u)}U_k(x,t(1-u)),$$

that is,

$$\begin{aligned} H[f](x,u,t) := \sum_{\{m_1,\dots,m_{k-1}\}} \frac{k!}{m_1!\dots m_{k-1}!} &\Bigg[\psi^{(m_1+\dots+m_{k-1})}(x,0+)(-\mathrm{e}^{-\lambda t(1-u)}\mathtt{T}_{t(1-u)}[f](x))^{m_1} \\ &\prod_{j=2}^{k-1}\left(-\frac{\mathrm{e}^{-\lambda jt(1-u)}}{j!}\left(\mathtt{T}_{t(1-u)}^{(j)}[f](x)+(-1)^jR_j(x,t(1-u))\right)\right)^{m_j} \\ &+\beta(x)\int_{M(E)^\circ}\left\langle \mathrm{e}^{-\lambda t(1-u)}\mathtt{T}_{t(1-u)}[f],\nu\right\rangle^{m_1} \\ &\prod_{j=2}^{k-1}\left\langle\frac{\mathrm{e}^{-\lambda jt(1-u)}}{j!}(\mathtt{T}_{t(1-u)}^{(j)}[f]+(-1)^jR_j(\cdot,t(1-u))),\nu\right\rangle^{m_j}\Gamma(x,\mathrm{d}\nu)\Bigg]. \end{aligned}$$

The induction hypothesis and (H1) allow us to get

$$
\begin{aligned}
&H\left[f\right](x)\\
&:= \lim_{t\to\infty} H(x,u,t)\\
&= \sum_{\{m_1,\dots,m_{k-1}\}_k} \frac{1}{m_1!\dots m_{k-1}!}\Bigg(\psi^{(m_1+\dots+m_{k-1})}(x,0+)(-\varphi(x))^{m_1}\prod_{j=2}^{k-1}(-\varphi(x)\mathfrak{I}_j(x))^{m_j}\\
&\qquad\qquad + \beta(x)\int_{M(E)^\circ}\langle\varphi,\nu\rangle^{m_1}\prod_{j=2}^{k-1}\langle\varphi\mathfrak{I}_j,\nu\rangle^{m_j}\,\Gamma(x,\mathrm{d}\nu)\Bigg).
\end{aligned}
$$

Using the same arguments used above from (10) onwards, we get the desiblack result. □

Proof of Theorem 3. First note that since we only compensate by $\mathrm{e}^{-\lambda t}$, the term $\mathtt{T}_t[f^k](x)$ that appears in equation (41) of [2] does not vanish after the normalisation. Due to assumption (H1), we have

$$\lim_{t\to\infty}\varphi^{-1}(x)\mathrm{e}^{-\lambda t}\mathtt{T}_t[f^k](x) = \langle f^k,\tilde\varphi\rangle.$$

Next we turn to the integral term in (41) of [2]. Define $[k_1,\dots,k_N]_k^{(n)}$, for $2\le n\le k$ to be the set of tuples $(k_1,\dots,k_N)$ with exactly n positive terms and whose sum is equal to k. Similar calculations to those given above yield

$$
\begin{aligned}
&\frac{\mathrm{e}^{-\lambda t}}{\varphi(x)}\int_0^t \mathtt{T}_s\left[\beta\mathcal{E}_x\left[\sum_{[k_1,\dots,k_N]_k^2}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathtt{T}_{t-s}^{(k_j)}[f](x_j)\right]\right](x)\mathrm{d}s\\
&= t\int_0^1\sum_{n=2}^k \mathrm{e}^{\lambda(n-1)ut}\frac{\mathrm{e}^{-\lambda(1-u)t}}{\varphi(x)}\\
&\qquad\times\mathtt{T}_{(1-u)t}\left[\beta\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]_k^{(n)}}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}^{(k_j)}[f](x_j)\right]\right](x)\mathrm{d}u. \qquad (13)
\end{aligned}
$$

Now suppose for induction that the result holds for all ℓ-th integer moments with $1\le \ell\le k-1$. Roughly speaking the argument can be completed by noting that the integral in the definition of L_k can be written as

$$\int_0^\infty\sum_{n=2}^k \mathrm{e}^{\lambda(n-1)s}\left\langle\beta\mathcal{E}_\cdot\left[\sum_{[k_1,\dots,k_N]_k^{(n)}}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathrm{e}^{-\lambda s}\mathtt{T}_s^{(k_j)}[f](x_j),\tilde\varphi\right\rangle\mathrm{d}s, \qquad (14)$$

which is convergent by appealing to (H2), the fact that $\beta\in B^+(E)$ and the induction hypothesis. As a convergent integral, it can be truncated at $t>0$ and the residual of the integral over (t,∞) can be made arbitrarily small by taking t sufficiently large. By changing variables in (14) when the integral is truncated at arbitrarily large t, so it is of a similar form to that of (13), we can subtract it from (13) to get

$$t\int_0^1\sum_{n=2}^k \mathrm{e}^{\lambda(n-1)ut}\left(\frac{\mathrm{e}^{-\lambda(1-u)t}}{\varphi(x)}\mathtt{T}_{(1-u)t}[H_{ut}^{(n)}] - \langle H_{ut}^{(n)},\tilde\varphi\rangle\right)\mathrm{d}u,$$

where

$$H^{(n)}_{ut}(x) = \beta\mathcal{E}_x\left[\sum_{[k_1,\dots,k_N]^{(n)}_k}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^{N}\mathrm{e}^{-\lambda ut}\mathtt{T}^{(k_j)}_{ut}[f](x_j)\right].$$

One proceeds to split the integral of the difference over $[0,1]$ into two integrals, one over $[0,1-\varepsilon]$ and one over $(1-\varepsilon,1]$. For the aforesaid integral over $[0,1-\varepsilon]$, we can control the behaviour of $\varphi^{-1}\mathrm{e}^{-\lambda(1-u)t}\mathtt{T}_{(1-u)t}[H^{(n)}_{ut}] - \langle H^{(n)}_{ut},\tilde{\varphi}\rangle$ as $t\to\infty$, making it arbitrarily small, by appealing to uniform dominated control of its argument in square brackets thanks to (H1). The integral over $[0,1-\varepsilon]$ can thus be bounded, as $t\to\infty$, by $t(1-\mathrm{e}^{\lambda(n-1)(1-\varepsilon)})/|\lambda|(n-1)$.

For the integral over $(1-\varepsilon,1]$, we can appeal to the uniformity in (H1) and (H2) to control the entire term $\mathrm{e}^{-\lambda(1-u)t}\mathtt{T}_{(1-u)t}[H^{(n)}_{ut}]$ (over time and its argument in the square brackets) by a global constant. Up to a multiplicative constant, the magnitude of the integral is thus of order

$$t\int_{1-\varepsilon}^{1}\mathrm{e}^{\lambda(n-1)ut}\mathrm{d}u = \frac{1}{|\lambda|(n-1)}(\mathrm{e}^{\lambda(n-1)(1-\varepsilon)t} - \mathrm{e}^{\lambda(n-1)t}),$$

which tends to zero as $t\to\infty$.

In the superprocess setting, as in the original proof, the exponential scaling kills the term $R_k(x,t)$ in (12). For the integral term in (12), define $H^{(m_1,\dots,m_{k-1})}_{ut}$ by

$$H^{(m_1,\dots,m_{k-1})}_{ut}(x) :=$$

$$\psi^{(m_1+\ldots+m_{k-1})}(x,0+)(-\mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}\left[f\right](x))^{m_1}\quad\prod_{j=2}^{k-1}\left(-\frac{\mathrm{e}^{-\lambda ut}}{j!}\left(\mathtt{T}^{(j)}_{ut}\left[f\right](x)+(-1)^jR_j(x,ut)\right)\right)^{m_j}$$

$$+\beta(x)\int_{M(E)^\circ}\left\langle\mathrm{e}^{-\lambda ut}\mathtt{T}_{ut}\left[f\right],\nu\right\rangle^{m_1}\prod_{j=2}^{k-1}\left\langle\frac{\mathrm{e}^{-\lambda ut}}{j!}(\mathtt{T}^{(j)}_{ut}\left[f\right]+(-1)^jR_j(\cdot,ut)),\nu\right\rangle^{m_j}\Gamma(x,\mathrm{d}\nu)$$

and noting that L_k can be written as

$$\int_0^\infty\sum_{\{m_1,\dots,m_{k-1}\}^k}\frac{k!}{m_1!\dots m_{k-1}!}\mathrm{e}^{\lambda(m_1+\ldots+m_{k-1}-1)s}\left\langle H^{(m_1,\dots,m_{k-1})}_s\left[f\right],\tilde{\varphi}\right\rangle\mathrm{d}s, \tag{15}$$

which is also convergent by appealing to (H2). The rest of the proof follows similar arguments to that of the particle system. That is, one splits the integral (15) at t and uses the integral over $[0,t]$ to compensate the integral component of (12), changing variable so that it becomes an integral over $[0,1]$ and handling things as with the particle system. The remaining integral from (t,∞) can be argued away as arbitrarily small because of the convergence of (15). □

3 Concluding remarks

Fundamentally, the corrected results offer the same rates of convergence and simply the constants take a different iterative structure. In Theorems 5 and 6, the uniformity in the convergence no longer accommodates for dividing by φ in the statement of the results. Finally, the corrections also remove the discrepancy between when there is dependency on x or not in the constants in the case of branching Markov processes and superprocesses.

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