Stable processes through the theory of self-similar Markov processes

Andreas E. Kyprianou
University of Bath

Which are our favourite stochastic processes?

- Markov chains ✓
- Diffusions → Brownian motion ✓
- Cts-time Markov processes with jumps → Lévy processes ✓
- Self-similar Markov processes ↑

- This talk: a gentle guided tour through some of the interaction "Self-similar Markov processes → Stable processes".

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Lévy processes

- Stick to one-dimension
- A Lévy process is an $\mathbb{R}$-valued random trajectory $\{X_t : t \geq 0\}$ issued from the origin with paths that are right-continuous and left limits and which has stationary and independent increments.
- More formally stationary and independent increments means:
  - for $0 \leq s \leq t < \infty$, $X_t - X_s = X_{t-s}$
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- It can be shown that this means the entire process is characterised by its position at time t (in fact it suffices to characterise its position at time 1)

$$\mathbb{E}[e^{i\theta X_t}] = e^{-\Psi(\theta)t}$$

for some appropriate function Ψ (the characteristic exponent).

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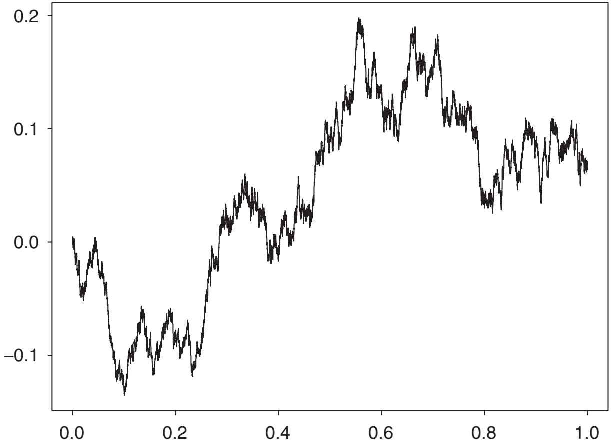
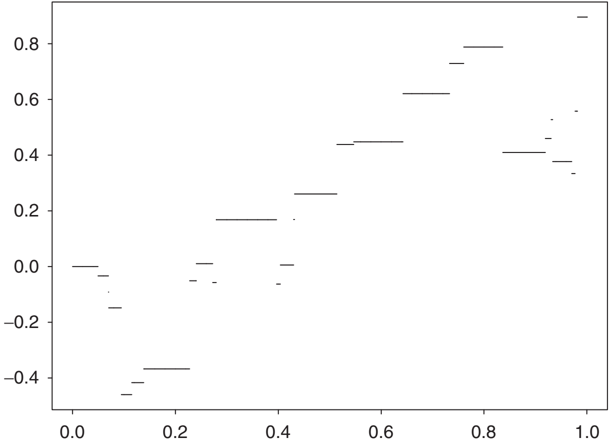
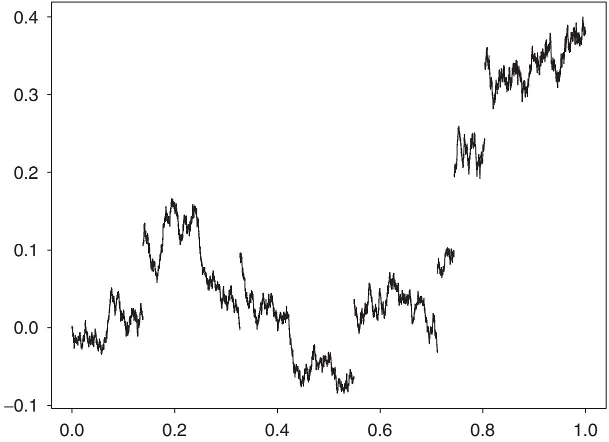
Brownian motion



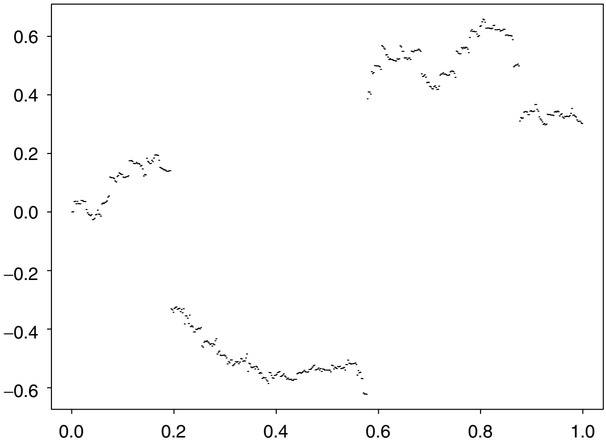
Compound Poisson process



Brownian motion + compound Poisson process



Unbounded variation paths



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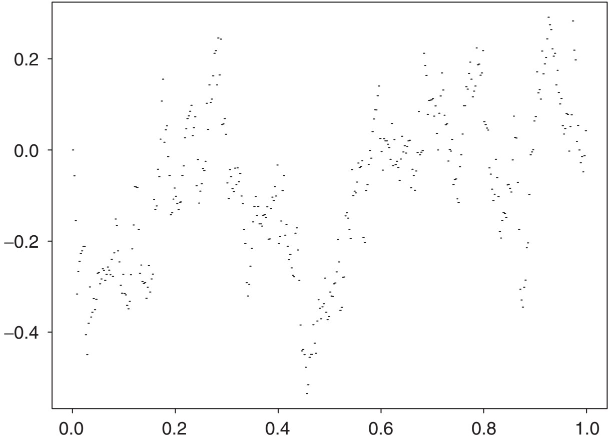

Space exploration: some successes and dissatisfaction

- Fundamentally we want to understand how Lévy processes explore space.
- 25 years of research has been very successful in giving an (relatively) complete theoretical description .....
- ......with the caveat that the database of tractable examples for the aforesaid theory is uncomfortably small (relative to Markov chains and diffusions).

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Example 1:

$$\mathbb{P}(\text{Process first exceeds level } x \text{ by an amount y}) = \int_{[0,x)} U(\mathrm{d}z)\bar{\nu}(z-x+y)$$

where

$$\Psi(\theta) = \kappa^{+}(-\mathrm{i}\theta)\kappa^{-}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R},$$

$$\kappa^{+}(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\nu(\mathrm{d}x), \qquad \lambda \geq 0,$$

$$\bar{\nu}(x) = \nu(x,\infty) \qquad \text{and} \qquad \int_{[0,\infty)} \mathrm{e}^{-\lambda x} U(\mathrm{d}x) = \frac{1}{\kappa^{+}(\lambda)}$$

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Example 2:

Under appropriate assumptions,

$$\mathbb{P}(\text{Process ever hits a point } x) = \frac{u(x)}{u(0)}, \qquad x \in \mathbb{R},$$

where

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta x} u(x) \mathrm{d}x = \frac{1}{\Psi(\theta)}, \qquad \theta \in \mathbb{R}.$$

Self-similar Markov processes on $\mathbb{R}$

α-ssMp

$\mathbb{R}$-valued Markov process,
equipped with initial measures P_x, $x \in \mathbb{R}\backslash\{0\}$,
with 0 an absorbing state,
satisfying the scaling property

$$\left(cX_{c^{-\alpha}t}\right)_{t\geq 0}\Big|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \qquad x, c > 0$$

Space-time changes and modulation

It turns out that up to first hitting of the origin every ssMp can be characterised using radial distance from the origin and positive or negative orientation as follows:

$$X_t = |x| \exp\left\{\xi_{\varphi(|x|^{-\alpha} t)}\right\} J_{\varphi(|x|^{-\alpha} t)}, \qquad t \geq 0, x \neq 0,$$

where $(\xi, J) \in (0, \infty) \times \{1, -1\}$ is a so-called Markov modulated Lévy process and

$$\varphi(t) = \inf\left\{s > 0 : \int_0^s \mathrm{e}^{\alpha \xi_u} \mathrm{d}u > t\right\}.$$

(ξ, J):

- $J = \{J_t : t \geq 0\}$ is a Markov chain on $\{1, 2\}$ with intensity matrix Q.
- When $J_t = i$, ξ moves as a Lévy process of type i. "$\mathrm{d}\xi_t = \mathrm{d}\xi_t^{(i)}$"
- When J makes a jump at time t, e.g. $1 \to 2$, then ξ experiences an additional jump $\Delta\xi_t$ which is an i.i.d. copy of some pre specified r.v. $U_{1,2}$.

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$$\varphi(t) = \inf\left\{s > 0 : \int_0^s e^{\alpha \xi_u} du > t\right\}.$$

(ξ, J): Markov modulated Lévy processes can also be characterised by a "characteristic exponetnt".

$$\mathbb{E}_i[e^{i\theta X_t}; J_t = j] = (\exp\{-\boldsymbol{\Psi}(\theta)t\})_{i,j}$$

where

$$\boldsymbol{\Psi}(\theta) = \begin{pmatrix} \Psi_1(\theta) & 0 \\ 0 & \Psi_2(\theta) \end{pmatrix} - Q \circ \begin{pmatrix} 1 & \mathbb{E}(e^{i\theta U_{1,2}}) \\ \mathbb{E}(e^{i\theta U_{2,1}}) & 1 \end{pmatrix}$$

X, $|X|$ and ξ

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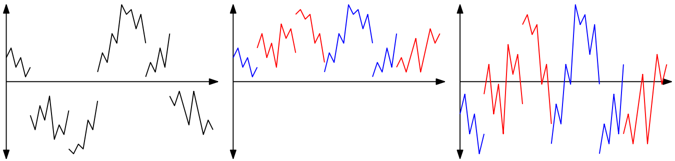

Space-time changes and modulation

If the Markov chain J has an absorbing state at -1 or never jumps to -1, then the ssMp is a "positive self-similar Markov process" (pssMp)

$$X_t = x \exp\left\{\xi_{\varphi(x^{-\alpha}t)}\right\}, \qquad t \geq 0, x\, 0,$$

where ξ is a Lévy process.

Positive feedback

- There is one class of Lévy processes which has always been considered to be "the next best thing after Brownian motion": the (α,ρ)-stable process.
- $\Psi(\theta) = |\theta|^\alpha \left(e^{\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{\{\theta>0\}} + e^{-\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{\{\theta<0\}} \right), \qquad \theta \in \mathbb{R},$
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- In fact stable processes are also self-similar Markov processes:

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- So α is the index of self-similarity, but $\rho \in [0,1]$ is a symmetry index. When $\rho = 1/2$, $\Psi(\theta) = |\theta|^\alpha$ so $-X =^d X$.
- When $\rho \in (0,1)$ (keep away from complete asymmetry!) and $\alpha \in (0,2)$ then the underlying Markov modulated Lévy process has exponent

$$\boldsymbol{\Psi}(\theta) = \begin{pmatrix} -\frac{\Gamma(\alpha - i\theta)\Gamma(1+i\theta)}{\Gamma(\alpha\hat{\rho} - i\theta)\Gamma(1-\alpha\hat{\rho}+i\theta)} & \frac{\Gamma(\alpha-i\theta)\Gamma(1+i\theta)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ \\ \frac{\Gamma(\alpha-i\theta)\Gamma(1+i\theta)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\frac{\Gamma(\alpha-i\theta))\Gamma(1+i\theta)}{\Gamma(\alpha\rho-i\theta)\Gamma(1-\alpha\rho+i\theta)} \end{pmatrix}.$$

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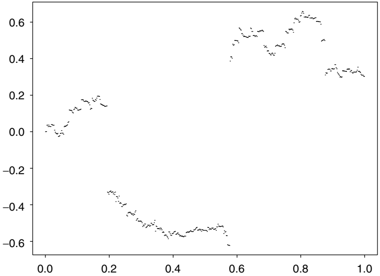

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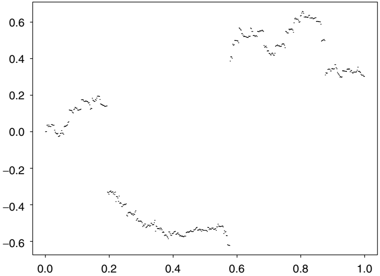

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$$\Psi(\theta) = \frac{\Gamma(\alpha\rho - \mathrm{i}\theta)}{\Gamma(-\mathrm{i}\theta)} \times \frac{\Gamma(1 - \alpha\rho + \mathrm{i}\theta)}{\Gamma(1 - \alpha + \mathrm{i}\theta)}, \qquad \theta \in \mathbb{R}.$$

Space exploration: some successes and dissatisfaction

$$\mathbb{P}(\text{Process first exceeds level } x \text{ by an amount y}) = \int_{[0,x)} U(\mathrm{d}z)\bar{\nu}(z-x+y)$$

where

$$\Psi(\theta) = \kappa^+(-\mathrm{i}\theta)\kappa^-(\mathrm{i}\theta), \qquad \theta \in \mathbb{R},$$

$$\kappa^+(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\nu(\mathrm{d}x), \qquad \lambda \geq 0,$$

$$\bar{\nu}(x) = \nu(x,\infty) \qquad \text{and} \qquad \int_{[0,\infty)} \mathrm{e}^{-\lambda x} U(\mathrm{d}x) = \frac{1}{\kappa^+(\lambda)}$$

$$\mathbb{P}(\text{Process ever hits a point } x) = \frac{u(x)}{u(0)}, \qquad x \in \mathbb{R},$$

where

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta x} u(x)\mathrm{d}x = \frac{1}{\Psi(\theta)}, \qquad \theta \in \mathbb{R}.$$

Positive feedback

- $\alpha \in (0,1)$

$$\begin{aligned}&\mathbb{P}(\text{Stable process first enters } [0,1] \text{ in } \mathrm{d}y)\\ &\quad= \mathbb{P}(\xi \text{ first enters } (-\infty,0] \text{ in } \mathrm{d}(\log y))\\ &\quad= \frac{\sin(\pi\alpha\hat{\rho})}{\pi} x^{\alpha\rho} y^{-\alpha\rho}(x-1)^{\alpha\hat{\rho}}(1-y)^{-\alpha\hat{\rho}}(x-y)^{-1}\mathrm{d}y\end{aligned}$$

- $\alpha \in (1,2)$

$$\begin{aligned}&\mathbb{P}(\text{Stable process hits 1 before 0 when starting from } x > 0)\\ &= \mathbb{P}(\xi \text{ ever hits 0 when starting from } \log x)\\ &= \frac{\sin(\pi\rho\alpha) - |x-1|^{\alpha-1}[\mathbf{1}_{(x>1)}\sin(\pi\hat{\rho}\alpha) + \mathbf{1}_{(0<x<1)}\sin(\pi\rho\alpha)] + x^{\alpha-1}\sin(\pi\hat{\rho}\alpha)}{(\sin(\pi\rho\alpha) + \sin(\pi\hat{\rho}\alpha))}.\end{aligned}$$

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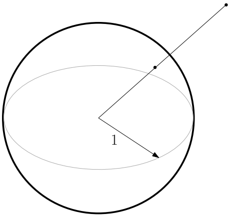
A bigger picture

A d-dimensional ssMp can be characterised using radial distance from the origin and angular orientation in $\mathbb{S}_{d-1}$ (think generalised Polar coordinates) as follows:

$$X_t = |x| \exp\left\{\xi_{\varphi(|x|^{-\alpha} t)}\right\} \Theta_{\varphi(|x|^{-\alpha} t)}, \qquad t \geq 0, x \neq 0,$$

where $(\xi, \Theta) \in (0, \infty) \times \mathbb{S}_{d-1}$ is a so-called Markov modulated Lévy process and

$$\varphi(t) = \inf\left\{ s > 0 : \int_0^s \mathrm{e}^{\alpha \xi_u} \mathrm{d}u > t \right\}.$$



A bigger picture

- A d-dimensional isotropic stable Lévy process is also a ssMp:

$$\mathbf{E}[e^{i\theta\cdot X_t}] = \exp\{-|\theta|^\alpha t\}, \qquad t \geq 0, \theta \in \mathbb{R}^d,$$

  necessarily $\alpha \in (0, 2]$.

- The radial distance of such a process from the origin, $|X_t|$, $t \geq 0$, is a pssMp. Its underlying Lévy process has characteristic exponent

$$\Psi(\theta) = \frac{\Gamma(\frac{1}{2}(-i\theta + \alpha))}{\Gamma(-\frac{1}{2}i\theta)} \frac{\Gamma(\frac{1}{2}(i\theta + d))}{\Gamma(\frac{1}{2}(i\theta + d - \alpha))}, \qquad \theta \in \mathbb{R}.$$

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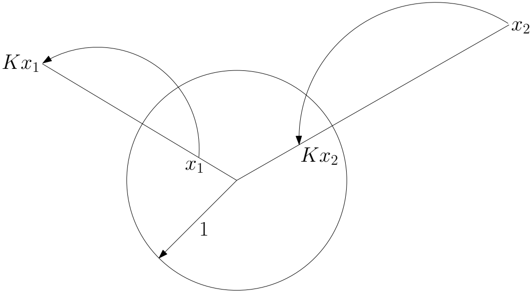
Riesz-Bogdan-Zak transform

The inversion of $\mathbb{R}^d$ through the unit sphere:

$$Kx = \frac{x}{|x|^2}, \qquad x \in \mathbb{R}^d.$$



Riesz-Bogdan-Zak transform

Riesz-Bogdan-Zak transform

Suppose that X is a d-dimensional isotropic stable process with $d \geq 2$. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}^d \backslash \{0\}$, $\{KX_{\eta(t)} : t \geq 0\}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^h_{Kx})$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^h_x}{\mathrm{d}\mathbb{P}_x}\right|_{\sigma(X_s : s \leq t)} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0,$$

In fact it can be shown that $(X, \mathbb{P}^h_x)$, $x \neq 0$ corresponds to the law of X conditioned to be continuously absorbed at the origin, that is: for $A \in \sigma(X_s : s \leq t)$, $x \neq 0$,

$$\mathbb{P}^h_x(A,\, t < \tau^{\{0\}}) = \lim_{a \to 0} \mathbb{P}_x(A,\, t < \tau^{\{0\}} | \tau^{B(0,a)} < \infty),$$

where $\tau^{B(0,a)} = \inf\{t > 0 : |X_t| < a\}$ and $\tau^{\{0\}} = \inf\{t > 0 : X_t = 0\}$.

Stable SDEs entering at $\pm\infty$

- Consider the simple SDE

$$dZ_t = \sigma(Z_{t-})\,dX_t, \qquad t \geq 0,$$

where X is a two-sided jumping 1-d stable process with index $\alpha \in (1,2)$.

- The weak solution of this SDE is equal in law to $(X_{\tau_t} : t \geq 0)$ where

$$\tau_t = \inf\{s > 0 : \int_0^s \sigma(X_s)^{-\alpha} ds > t\}, \qquad t \geq 0.$$

- Can the SDE solution enter simultaneously at $\pm\infty$?
- Apply Riesz-Bogdan-Zak transform, compounding time changes, to discover (with quite a bit of work) that this can happen if and only if

$$\int_{|x|>1} \sigma(x)^{-\alpha} |x|^{\alpha-1}\, dx < \infty.$$

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For the future

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- And to some extent Lévy processes and their subtle path properties
- Stable processes benefit from the theory of self-similarly to provide explicit answers for questions relating to path behaviour, promising some robustness in the arguments
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Thank you!