Stochastic Analysis of the Neutron Transport Equation

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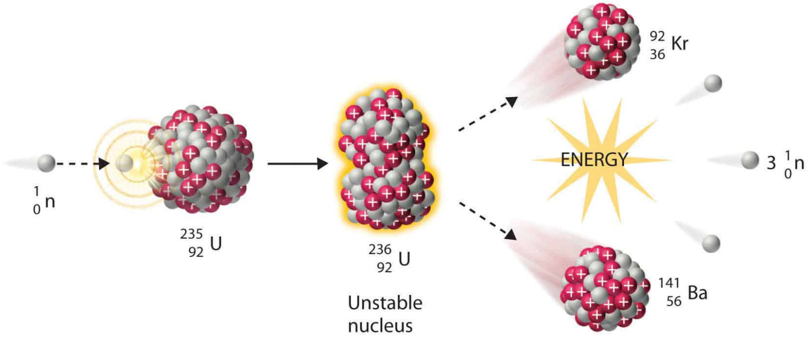
NEUTRON FLUX

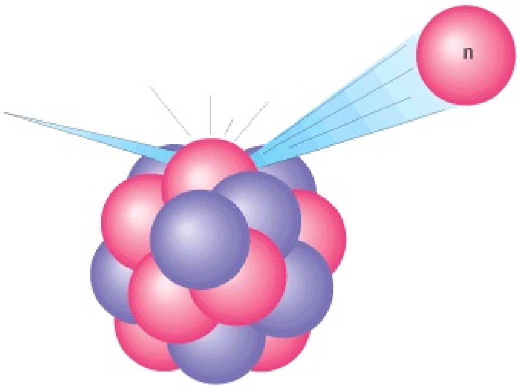
- **Neutron flux** is a measure of the intensity of neutron radiation, determined by the rate of flow of neutrons; measured in (# neutrons)/cm^2/s.
- We want to describe neutron flux as a function of spatial position and time in complex domains:

$$\Psi(r, \upsilon, t), \qquad r \in D \subseteq \mathbb{R}^d, \upsilon \in V := \{\upsilon \in \mathbb{R}^d : \upsilon_{\texttt{min}} \leq |\upsilon| \leq \upsilon_{\texttt{max}}\},$$

for $0 < \upsilon_{\texttt{min}} < \upsilon_{\texttt{max}} < \infty$.

Neutron Fission

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NEUTRON TRANSPORT EQUATION

Neutron flux is thus identified as $\Psi_g : D \times V \to [0, \infty)$, which solves the integro-differential equation

$$\frac{\partial \Psi_g}{\partial t}(t,r,\upsilon) + \upsilon \cdot \nabla \Psi_g(t,r,\upsilon) + \sigma(r,\upsilon)\Psi_g(t,r,\upsilon)$$
$$= Q(r,\upsilon,t) + \int_V \Psi_g(r,\upsilon',t)\sigma_{\mathtt{s}}(r,\upsilon')\pi_{\mathtt{s}}(r,\upsilon',\upsilon)\mathrm{d}\upsilon' + \int_V \Psi_g(r,\upsilon',t)\sigma_{\mathtt{f}}(r,\upsilon')\pi_{\mathtt{f}}(r,\upsilon',\upsilon)\mathrm{d}\upsilon',$$

where the different components are measurable in their dependency on (r, υ) and are explained as follows:

$\sigma_{\mathtt{s}}(r,\upsilon')$: the rate at which scattering occurs from incoming velocity υ',

$\sigma_{\mathtt{f}}(r,\upsilon')$: the rate at which fission occurs from incoming velocity υ',

$\sigma(r,\upsilon)$: the sum of the rates $\sigma_{\mathtt{f}} + \sigma_{\mathtt{s}}$ and is known as the cross section,

$\pi_{\mathtt{s}}(r,\upsilon',\upsilon)\mathrm{d}\upsilon'$: the scattering yield at velocity υ from incoming velocity υ', satisfying $\pi_{\mathtt{s}}(r,\upsilon,V) = 1$, and

$\pi_{\mathtt{f}}(r,\upsilon',\upsilon)\mathrm{d}\upsilon'$: the neutron yield at velocity υ from fission with incoming velocity υ', satisfying $\pi_{\mathtt{f}}(r,\upsilon,V) < \infty$ and

$Q(r,\upsilon,t)$: non-negative source term. (Immediately remove the source term $Q = 0$)

We will assume that all quantities are uniformly bounded away from zero and infinity.

Boundary conditions

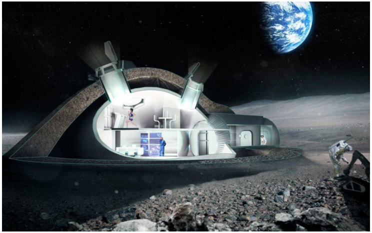
- Boundary conditions which represent 'fission containment'

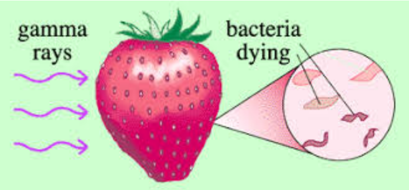
$$\begin{cases} \Psi_g(0,r,\upsilon) = g(r,\upsilon) & \text{for } r \in D, \upsilon \in V, \text{ (initial condition)} \\ \Psi_g(t,r,\upsilon) = g(r,\upsilon) = 0 & \text{for } r \in \partial D \text{ if } \upsilon \cdot \mathbf{n}_r < 0, \text{ (neutron annihilation)} \end{cases}$$

- $\mathbf{n}_r$ is the outward facing normal of D at $r \in \partial D$
- $g : D \times V \to [0, \infty)$ is a bounded, measurable function which we will later assume has some additional properties.

Don't Panic!

YOUR NUCLEAR FUTURE





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$$= \int_V \Psi_g(r,\upsilon',t)\sigma_{\mathtt{s}}(r,\upsilon')\pi_{\mathtt{s}}(r,\upsilon',\upsilon)\mathrm{d}\upsilon' + \int_V \Psi_g(r,\upsilon',t)\sigma_{\mathtt{f}}(r,\upsilon')\pi_{\mathtt{f}}(r,\upsilon',\upsilon)\mathrm{d}\upsilon',$$

- With some rearrangements, the components of NTE separate into transport, scattering and fission. Specifically,

$$\begin{cases} \mathtt{T}g(r,\upsilon) & := -\upsilon \cdot \nabla g(r,\upsilon) - \sigma(r,\upsilon)g(r,\upsilon) & \text{(forwards transport)} \\ \mathtt{S}g(r,\upsilon) & := \int_V g(r,\upsilon')\sigma_{\mathtt{s}}(r,\upsilon')\pi_{\mathtt{s}}(r,\upsilon',\upsilon)\mathrm{d}\upsilon' & \text{(forwards scattering)} \\ \mathtt{F}g(r,\upsilon) & := \int_V g(r,\upsilon')\sigma_{\mathtt{g}}(r,\upsilon')\pi_{\mathtt{g}}(r,\upsilon',\upsilon)\mathrm{d}\upsilon' & \text{(forwards fission)} \end{cases}$$

- More natural to look for solutions as time-varying in $L^2(D \times V)$ so that, for $f \in L^2(D \times V)$,

$$\frac{\partial}{\partial t}\langle f, \Psi_g(t,\cdot,\cdot)\rangle = \langle f, (\mathtt{T}+\mathtt{S}+\mathtt{F})\Psi_g(t,\cdot,\cdot)\rangle$$

Abstract Cauchy problem - taking the problem into the domain of c_0-semigroups

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Abstract Cauchy problem - taking the problem into the domain of c_0-semigroups

Abstract Cauchy problem

- Written more simply with everything in understood in the $L^2(D \times V)$ space

$$\frac{\partial}{\partial t}\Psi_g(t,\cdot,\cdot) = (\mathtt{T}+\mathtt{S}+\mathtt{F})\Psi_g(t,\cdot,\cdot)$$

- c_0-semigroup allows us to see the solution to this problem as the orbit in L^2 space:

$$\Psi_g(t,r,\upsilon) = \mathrm{e}^{(\mathtt{T}+\mathtt{S}+\mathtt{F})t}g(r,\upsilon), \qquad t \geq 0,$$

  where $\mathrm{e}^{(\mathtt{T}+\mathtt{S}+\mathtt{F})t} = \sum_{k=0}^{\infty}(\mathtt{T}+\mathtt{S}+\mathtt{F})^k t^k/k!$
- More generally can replace $L^2(D \times V)$ by $L^p(D \times V)$ for $p \in (1,\infty)$.
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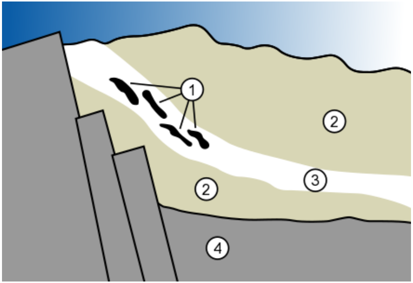
Steady-state reactors

- What constitutes a nuclear reactor?



- Heuristically we want to find an eigenvalue $\lambda \in \mathbb{R}$, positive eigenfunction pair $h : D \times V \to [0, \infty)$ and $\tilde{h}$ on $D \times V$ such that, ideally with $\lambda = 0$

$$\text{Forwards}: \quad \lambda\langle h, f\rangle = \langle h, (\mathtt{T} + \mathtt{S} + \mathtt{F})f\rangle \quad \text{and } \lambda\langle g, \tilde{h}\rangle = \langle g, (\mathtt{T} + \mathtt{S} + \mathtt{F})\tilde{h}\rangle$$

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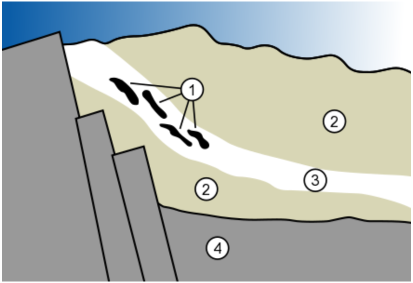

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Steady-state reactors

- The eigenfunction h is called an *importance map* and gives the first order neutron flux (radioactivity) profile
- Roughly speaking, now as an Abstract Cauchy Problem on $L^2(D \times V)$,

$$\frac{\partial \psi_h}{\partial t} = (\mathtt{T} + \mathtt{S} + \mathtt{F})\psi_h, \qquad \psi_h = h \text{ at } t = 0 \text{ and } \psi_h = 0 \text{ for } r \in \partial D, \upsilon \cdot \mathbf{n}_r > 0$$

  the solution can be thought of as

$$\psi_h(t, r, \upsilon) = \mathrm{e}^{(\mathtt{T}+\mathtt{S}+\mathtt{F})t} h(r, \upsilon) := \sum_{k \geq 0} \frac{t^k}{k!} (\mathtt{T} + \mathtt{S} + \mathtt{F})^k h(r, \upsilon)$$

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Perron-Frobenius

- T is a nasty (unbounded) operator making it harder than usual to find eigenfunctions, S and F are nice (bounded) operators whose spectral analysis is easier to handle.
- In looking for λ, h as a lead eigen pair we need

$$(\mathtt{T} + \mathtt{S} + \mathtt{F})h = \lambda h \quad \Longrightarrow \quad (\mathtt{T} - \lambda I)^{-1}(\mathtt{S} + \mathtt{F})h = h$$

- Fix μ, use either operator perturbation methods or Krein-Rutman Theorem to deduce that (as a linear operator on an L^2 space),

$$(\mathtt{T} - \mu I)^{-1}(\mathtt{S} + \mathtt{F})$$

  has a spectral radius r_μ and positive eigenfunction h_μ
- Verify that r_μ varies continuously with μ on a range $(0, r^*)$, where $r^* > 1$.
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Projecting onto the lead eigenvalue, $\exists \varepsilon > 0$:

$$e^{-\lambda_* t}\psi_g(t,r,\upsilon) \sim h(r,\upsilon)\langle \tilde{h}, g\rangle + O(e^{-\varepsilon t})$$

Theorem

Let D be convex. Assume that $\sigma_{\mathtt{f}}(r,\upsilon)\pi_{\mathtt{f}}(r,\upsilon,\upsilon')$ and $\sigma_{\mathtt{s}}(r,\upsilon)\pi_{\mathtt{s}}(r,\upsilon,\upsilon')$ are piece-wise continuous and uniformly bounded from above and below on $D \times V \times V$. Then,

(i) *the neutron transport operator $(\mathtt{T} + \mathtt{S} + \mathtt{F})$ has a leading eigenvalue $\lambda_* \in \mathbb{R}$, which is simple and isolated and which has a corresponding positive right and left eigenfunctions in $L_2(D \times V)$, h and $\tilde{h}$ respectively, and*

(ii) *there exists an $\varepsilon > 0$ such that, as $t \to \infty$,*

$$||e^{-\lambda_* t}\psi_g(t,\cdot,\cdot) - \langle \tilde{h}, g\rangle h||_2 = O(e^{-\varepsilon t}), \tag{1}$$

for all $g \in L_2(D \times V)$.

The sign of λ_* dictates the criticality of the system:

- $\lambda_* < 0$: subcritical and fission dies out
- $\lambda_* = 0$: critical, i.e. a nuclear reactor
- $\lambda_* > 0$: supercritical (not quite a bomb, that would be non-existence of λ_*)

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(i) *the neutron transport operator $(\mathtt{T} + \mathtt{S} + \mathtt{F})$ has a leading eigenvalue $\lambda_* \in \mathbb{R}$, which is simple and isolated and which has a corresponding positive right and left eigenfunctions in $L_2(D \times V)$, h and $\tilde{h}$ respectively, and*

(ii) *there exists an $\varepsilon > 0$ such that, as $t \to \infty$,*

$$||e^{-\lambda_* t}\psi_g(t,\cdot,\cdot) - \langle \tilde{h}, g\rangle h||_2 = O(e^{-\varepsilon t}), \tag{1}$$

for all $g \in L_2(D \times V)$.

The sign of λ_* dictates the criticality of the system:

- $\lambda_* < 0$: subcritical and fission dies out
- $\lambda_* = 0$: critical, i.e. a nuclear reactor
- $\lambda_* > 0$: supercritical (not quite a bomb, that would be non-existence of λ_*)

Perron-Frobenius

Projecting onto the lead eigenvalue, $\exists \varepsilon > 0$:

$$e^{-\lambda_* t}\psi_g(t,r,\upsilon) \sim h(r,\upsilon)\langle \tilde{h}, g\rangle + O(e^{-\varepsilon t})$$

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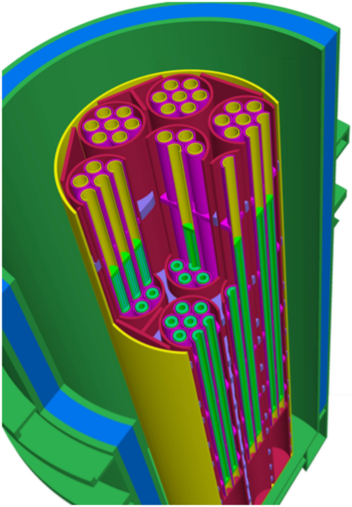
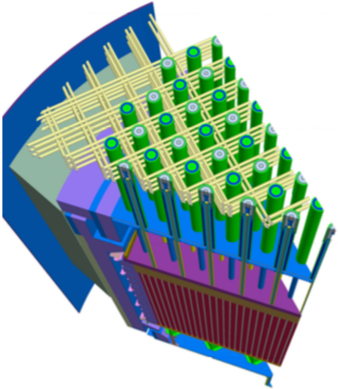
Over what domains do we need eigenfunctions of the NTE?





- Note that, for $f, g \in L^2(D \times V)$, with f respecting the boundary condition $g(r, \upsilon) = 0$ for $r \in \partial D$ if $\upsilon \cdot \mathbf{n}_r < 0$, we can verify with a simple integration by parts that

$$\langle f, \upsilon \cdot \nabla g \rangle = \int_{\partial D \times V} (\upsilon \cdot \upsilon') f(r, \upsilon') g(r, \upsilon') \mathrm{d}r \mathrm{d}\upsilon' - \langle \upsilon \cdot \nabla f, g \rangle = -\langle \upsilon \cdot \nabla f, g \rangle$$

providing we insist that f respects the boundary $f(r, \upsilon) = 0$ for $r \in \partial D$ if $\upsilon \cdot \mathbf{n}_r > 0$.

- Moreover, Fubini's theorem also tells us that, for example, with $f, g \in L^2(D \times V)$,

$$\langle f, \int_V g(\cdot, \upsilon') \sigma_{\mathrm{s}}(\cdot, \upsilon') \pi_{\mathrm{s}}(\cdot, \upsilon', \cdot) \mathrm{d}\upsilon' \rangle = \langle \sigma_{\mathrm{s}}(\cdot, \cdot) \int_V f(\cdot, \upsilon) \pi_{\mathrm{s}}(\cdot, \cdot, \upsilon) \mathrm{d}\upsilon, g \rangle.$$

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(Forward →Backwards) Neutron Transport Equation

- Hence, with similar computations, this tells us that, for $f, g \in L^2(D \times V)$,

$$\langle f, (\mathtt{T} + \mathtt{S} + \mathtt{F})g \rangle = \langle (\mathcal{T} + \mathcal{S} + \mathcal{F})f, g \rangle,$$

where

$$\begin{cases} \mathcal{T}f(r,\upsilon) & := \upsilon \cdot \nabla f(r,\upsilon) & \text{(backwards transport)} \\ \mathcal{S}f(r,\upsilon) & := \sigma_{\mathtt{s}}(r,\upsilon) \int_V f(r,\upsilon')\pi_{\mathtt{s}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon' - \sigma_{\mathtt{s}}(r,\upsilon)f(r,\upsilon) & \text{(backwards scattering)} \\ \mathcal{F}f(r,\upsilon) & := \sigma_{\mathtt{f}}(r,\upsilon) \int_V f(r,\upsilon')\pi_{\mathtt{f}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon' - \sigma_{\mathtt{f}}(r,\upsilon)f(r,\upsilon) & \text{(backwards fission)} \end{cases}$$

- This leads us to the so called *backwards neutron transport equation* (which is also known as the *adjoint neutron transport equation*) given by the Abstract Cauchy Problem on $L^2(D \times V)$,

$$\frac{\partial \psi}{\partial t}(t,\cdot,\cdot) = (\mathcal{T} + \mathcal{S} + \mathcal{F})\psi(t,\cdot,\cdot)$$

with additional boundary conditions

$$\begin{cases} \psi(0,r,\upsilon) = g(r,\upsilon) & \text{for } r \in D, \upsilon \in V, \\ \psi(t,r,\upsilon) = 0 & \text{for } r \in \partial D \text{ if } \upsilon \cdot \mathbf{n}_r > 0. \end{cases}$$

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Underlying stochastics (leading to Monte-Carlo)

- Backwards equation lends itself well to stochastic representation in the L_2 sense,

$$\frac{\partial \psi}{\partial t}(t,r,\upsilon) = \upsilon \cdot \nabla \psi(t,r,\upsilon) - \sigma(r,\upsilon)\psi(t,r,\upsilon)$$
$$+ \sigma_{\mathtt{s}}(r,\upsilon)\int_V \psi(r,\upsilon',t)\pi_{\mathtt{s}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon' + \sigma_{\mathtt{f}}(r,\upsilon)\int_V \psi(r,\upsilon',t)\pi_{\mathtt{f}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon',$$

- The physical process of fission is a Markov-additive branching process (*neutron branching process*).
- Represented by a configuration of physical location and velocity of particles in $D \times V$, say $\{(r_i(t), \upsilon_i(t)) : i = 1, \dots, N_t\}$, where N_t is the number of particles alive at time $t \geq 0$.
- Represent as a process in the space of the atomic measures

$$X_t(A) = \sum_{i=1}^{N_t} \delta_{(r_i(t),\upsilon_i(t))}(A), \qquad A \in \mathcal{B}(D \times V),\ t \geq 0,$$

where δ is the Dirac measure, define on $\mathcal{B}(D \times V)$, the Borel subsets of D.
- Then the stochastic representation of the backwards NTE is nothing more than

$$\phi_t[g](r,\upsilon) = \mathbb{E}_{\delta_{(r,\upsilon)}}[\langle g, X_t\rangle] = \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\sum_{i=1}^{N_t} g(r_i(t),\upsilon_i(t))\right], \qquad t \geq 0.$$

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Neutron branching process

- A particle position at r with velocity υ (configruration (r, υ)) will continue to move along the trajectory $r + \upsilon t$, until one of the following things happens.
- The particles that leave the physical domain D are killed.
- For a neutron with configuration (r, υ), if $T_{\mathtt{s}}$ is the random time that scattering may occur, then

$$\Pr(T_{\mathtt{s}} > t) = \exp\left\{-\int_0^t \sigma_{\mathtt{s}}(r + \upsilon t, \upsilon))\mathrm{d}s\right\}.$$

- When scattering occurs at space-velocity (r, υ), the new velocity is selected independently with probability $\pi_{\mathtt{s}}(r, \upsilon, \upsilon')\mathrm{d}\upsilon'$.
- For a neutron with configuration (r, υ), if $T_{\mathtt{f}}$ is the random time that scattering may occur, then independently of any other physical event that may affect the neutron,

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Mild equation

- Define for $g \in L^+_\infty(D \times V)$, the (physical process) expectation semigroup

$$\phi_t[g](r,\upsilon) := \mathbb{E}_{\delta_{(r,\upsilon)}}[\langle g, X_t\rangle], \qquad t \geq 0, r \in D, \upsilon \in V,$$

and the advection semigroup

$$\mathtt{U}_t[g](r,\upsilon) = g(r+\upsilon t, \upsilon)\mathbf{1}_{\{t<\kappa^D_{r,\upsilon}\}}, \qquad t \geq 0.$$

where $\kappa^D_{r,\upsilon} := \inf\{t > 0 : r + \upsilon t \notin D\}$.

Lemma

When $g \in L^+_\infty(D \times V)$, the space of non-negative functions in $L^+_\infty(D \times V)$, the expectation semigroup $(\phi_t[g], t \geq 0)$ is the unique bounded solution to the mild equation

$$\phi_t[g] = \mathtt{U}_t[g] + \int_0^t \mathtt{U}_s[(\mathcal{S}+\mathcal{F})\phi_{t-s}[g]]\mathrm{d}s, \qquad t \geq 0.$$

Lemma

The mild solution $(\phi_t, t \geq 0)$, is dual to $(\psi(t,\cdot,\cdot), t \geq 0)$ on $L_2(D \times V)$, i.e.

$$\langle f, \phi_t[g]\rangle = \langle \psi_f(t,\cdot,\cdot), g\rangle$$

for all $f, g \in L_2(D \times V)$

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When $g \in L^+_\infty(D \times V)$, the space of non-negative functions in $L^+_\infty(D \times V)$, the expectation semigroup $(\phi_t[g], t \geq 0)$ is the unique bounded solution to the mild equation

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The mild solution $(\phi_t, t \geq 0)$, is dual to $(\psi(t,\cdot,\cdot), t \geq 0)$ on $L_2(D \times V)$, i.e.

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Mild equation

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Eigenfunctions of the expectation semi-group?

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- We want to play with the eigenfunction $\tilde{h} \in L_2(D \times V)$, e.g.
$$\langle f, \phi_t[\tilde{h}]\rangle = \langle \psi_f(t,\cdot,\cdot), \tilde{h}\rangle = \mathrm{e}^{\lambda t}\langle f, \tilde{h}\rangle$$
suggesting (at least in the $L_2(D \times V)$ sense)
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$\Rightarrow$ points us towards Monte-Carlo methods - especially when $\lambda = 0$

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Perron-Frobenius again

Recent work of Champagnat and Villemonais on quasi-stationary distributions for Markov semigroups (in the spirit of Tweedie's R-theory) allows us to conclude the following

Theorem

Suppose that D is non-empty and convex,

$$\underline{\beta} := \inf_{r \in D, \upsilon \in V} \sigma_{\mathcal{f}}(r, \upsilon) \left(\int_V \pi_{\mathcal{f}}(r, \upsilon, \upsilon') \mathrm{d}\upsilon' - 1 \right) > 0.$$

Then there exists a $\lambda_* \in \mathbb{R}$, a positive right eigenfunction $\varphi \in L^+_\infty(D \times V)$ and a left eigenmeasure which is absolutely continuous with respect to Lebesgue measure on $D \times V$ with density $\tilde{\varphi} \in L^+_\infty(D \times V)$, both having associated eigenvalue $\mathrm{e}^{\lambda_c t}$, and such that φ (resp. $\tilde{\varphi}$) is uniformly (resp. a.e. uniformly) bounded away from zero on each compactly embedded subset of $D \times V$. In particular for all $g \in L^+_\infty(D \times V)$

$$\langle \tilde{\varphi}, \phi_t[g] \rangle = \mathrm{e}^{\lambda_* t} \langle \tilde{\varphi}, g \rangle \quad (\textit{resp. } \phi_t[\varphi] = \mathrm{e}^{\lambda_* t} \varphi) \quad t \geq 0.$$

Moreover, there exists $\varepsilon > 0$ such that, for all $g \in L^+_\infty(D \times V)$,

$$\left\| \mathrm{e}^{-\lambda_* t} \varphi^{-1} \phi_t[g] - \langle \tilde{\varphi}, g \rangle \right\|_\infty = O(\mathrm{e}^{-\varepsilon t}) \textit{ as } t \to +\infty.$$

Theorem

For all $g \in L^+_\infty(D \times V)$ such that, up to a multiplicative constant, $g \leq \varphi$, under the assumptions as the previous Theorem,

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda_* t}\langle g, X_t\rangle = \langle g, \tilde{\varphi}\rangle W_\infty.$$

almost surely, where W_∞ is a special random variable (in fact a martingale limit). Moreover, W_∞ is positive with positive probability if and only if $\lambda_* > 0$, otherwise $W_\infty = 0$.

WE ARE NOW MONTE-CARLO-READY

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- and e.g.

$$\frac{\varphi(r,\upsilon)}{\varphi(r_0,\upsilon_0)} = \lim_{t\to\infty} \frac{\phi_t[g](r,\upsilon)}{\phi_t[g](r_0,\upsilon_0)} = \lim_{t\to\infty} \frac{\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle g, X_t\rangle]}{\mathbb{E}_{\delta_{(r_0,\upsilon_0)}}[\langle g, X_t\rangle]}$$

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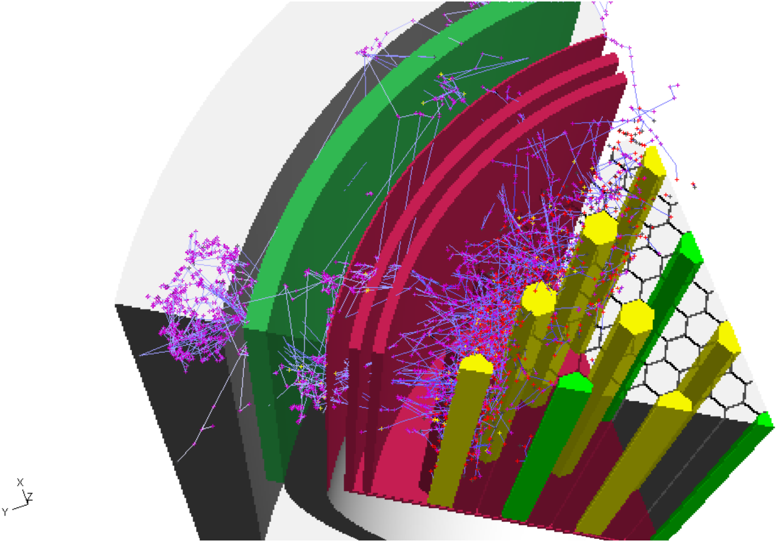
Monte-Carlo, importance map and supercomputers



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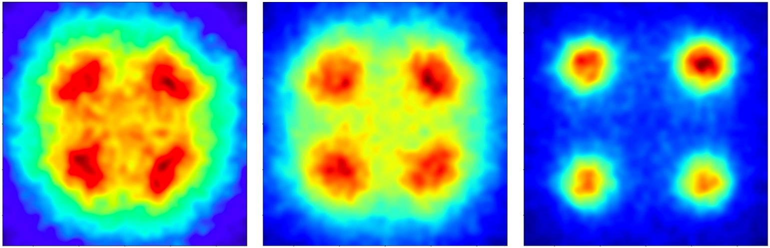

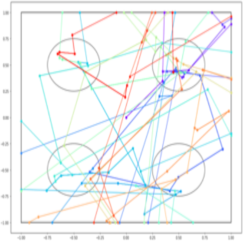

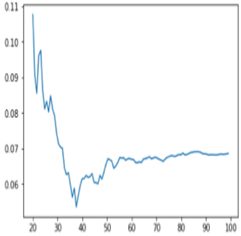

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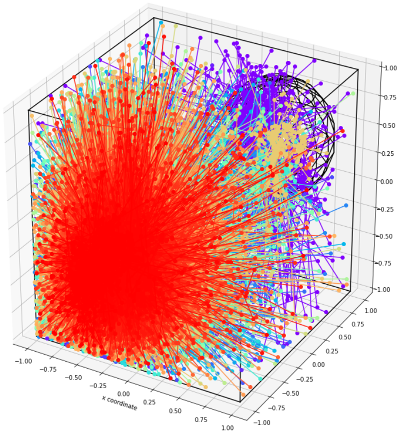

PROBLEM!

- Needs a massive supercomputer to deal with an industrial scale simulation

- Simulating (inhomogeneous) branching trees is no joke: cannot be parallelised.

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Problem!

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Many-to-one and Monte-Carlo parallelisation

- Recall semigroup operators

$$\mathcal{T}f(r,\upsilon) := \upsilon\cdot\nabla f(r,\upsilon) \quad \text{(backwards transport)}$$

$$\mathcal{S}f(r,\upsilon) := \sigma_{\mathrm{s}}(r,\upsilon)\int_V (f(r,\upsilon')-f(r,\upsilon))\pi_{\mathrm{s}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon' \quad \text{(backwards scattering)}$$

$$\mathcal{F}f(r,\upsilon) := \sigma_{\mathrm{f}}(r,\upsilon)\int_V f(r,\upsilon')\pi_{\mathrm{f}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon' - \sigma_{\mathrm{f}}(r,\upsilon)f(r,\upsilon) \quad \text{(backwards fission)}$$

- Basic algebra gives

$$\mathcal{T}+\mathcal{S}+\mathcal{F} = \upsilon\cdot\nabla f(r,\upsilon,t)+\alpha(r,\upsilon)\int_V \left(f(r,\upsilon',t)-f(r,\upsilon,t)\right)\pi(r,\upsilon,\upsilon')\mathrm{d}\upsilon'+\beta(r,\upsilon)f(r,\upsilon)$$

where

$$\alpha(r,\upsilon) := \sigma_{\mathrm{s}}(r,\upsilon)+\sigma_{\mathrm{f}}(r,\upsilon)\int_V \pi_{\mathrm{f}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon',$$

$$\pi(r,\upsilon,\upsilon')\mathrm{d}\upsilon' := \alpha^{-1}(r,\upsilon)\left[\sigma_{\mathrm{s}}(r,\upsilon)\pi_{\mathrm{s}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon'+\sigma_{\mathrm{f}}(r,\upsilon)\pi_{\mathrm{f}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon'\right],$$

$$\beta(r,\upsilon) := \alpha(r,\upsilon)-\sigma_{\mathrm{s}}(r,\upsilon)-\sigma_{\mathrm{f}}(r,\upsilon).$$

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where

$$\begin{aligned}\alpha(r,\upsilon) &:= \sigma_{\mathtt{s}}(r,\upsilon)+\sigma_{\mathtt{f}}(r,\upsilon)\int_V \pi_{\mathtt{f}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon',\\ \pi(r,\upsilon,\upsilon')\mathrm{d}\upsilon' &:= \alpha^{-1}(r,\upsilon)\left[\sigma_{\mathtt{s}}(r,\upsilon)\pi_{\mathtt{s}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon' + \sigma_{\mathtt{f}}(r,\upsilon)\pi_{\mathtt{f}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon'\right],\\ \beta(r,\upsilon) &:= \alpha(r,\upsilon)-\sigma_{\mathtt{s}}(r,\upsilon)-\sigma_{\mathtt{f}}(r,\upsilon).\end{aligned}$$

Many-to-one and Monte-Carlo parallelisation

- The representation $\mathcal{T} + \mathcal{S} + \mathcal{F} = \mathcal{L} + \beta$, where

$$\mathcal{L}f(r,\upsilon) = \upsilon \cdot \nabla f(r,\upsilon,t) + \alpha(r,\upsilon)\int_V \big(f(r,\upsilon',t) - f(r,\upsilon,t)\big)\pi(r,\upsilon,\upsilon')\mathrm{d}\upsilon'$$

implies

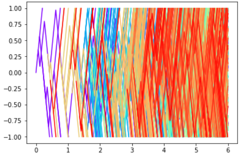
$$\phi_t[g](r,\upsilon) = \mathbb{E}_{\delta_{(r,\upsilon)}}[\langle g, X_t\rangle] = \mathbf{E}_{(r,\upsilon)}\left[\mathrm{e}^{\int_0^t \beta(R_u,\Upsilon_u)\mathrm{d}u} g(R_t,\Upsilon_t)\mathbf{1}_{(t<\tau^D)}\right],$$

for $t \geq 0, r \in D, \upsilon \in V$, where

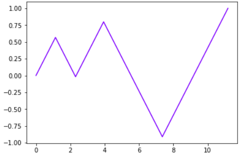
$$\tau^D = \inf\{t > 0 : R_t \notin D\}.$$

and $((R_t,\Upsilon_t), t \geq 0)$ with probabilities $\mathbf{P}_{(r,\upsilon)}, r \in V, \upsilon \in D$, is the $\mathcal{L}$-neutron random walk.

- This affords new parallelisable opportunities to Monte-Carlo solve numerically for h:



can be replaced by



Importance sampling

- Pick a 'first guess' of φ, denoted here by η, that satisfies $\eta(r,\upsilon) = 0$ for $r \in \partial D$ if $\upsilon \cdot \mathbf{n}_r > 0$.
- Perform the Doob η-transform

$$\left.\frac{\mathrm{d}\mathbf{P}^{\eta}_{(r,\upsilon)}}{\mathrm{d}\mathbf{P}_{(r,\upsilon)}}\right|_{\sigma((R_s,\Upsilon_s),s\le t)} := \exp\left(-\int_0^t \frac{\mathcal{L}\eta(R_s,\Upsilon_s)}{\eta(R_s,\Upsilon_s)}\mathrm{d}s\right)\frac{\eta(R_t,\Upsilon_t)}{\eta(r,\upsilon)}\mathbf{1}_{(t<\tau^D)}$$

- Gives new neutron random walk characterised by

$$\mathcal{L}_\eta f(r,\upsilon) = \upsilon\cdot\nabla f(r,\upsilon) + \alpha(r,\upsilon)\int_V (g(r,\upsilon') - g(r,\upsilon))\,\frac{\eta(r,\upsilon')}{\eta(r,\upsilon)}\pi(r,\upsilon,\upsilon')\mathrm{d}\upsilon'.$$

Lemma

Moreover,

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$$\left.\times\prod_{i=1}^{N_t}\frac{\eta(R_{T_i},\Upsilon_{T_{i-1}})}{\eta(R_{T_i},\Upsilon_{T_i})}g(R_t,\Upsilon_t)\mathbf{1}_{(t<\tau^D)}\right]$$

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Want to choose η so that the Neutron Random Walk $\mathcal{L}_\eta$ remains trapped in D

Theorem

A sufficient condition on η for (R, Υ) under $\mathbf{P}^\eta$ to be conservative is that

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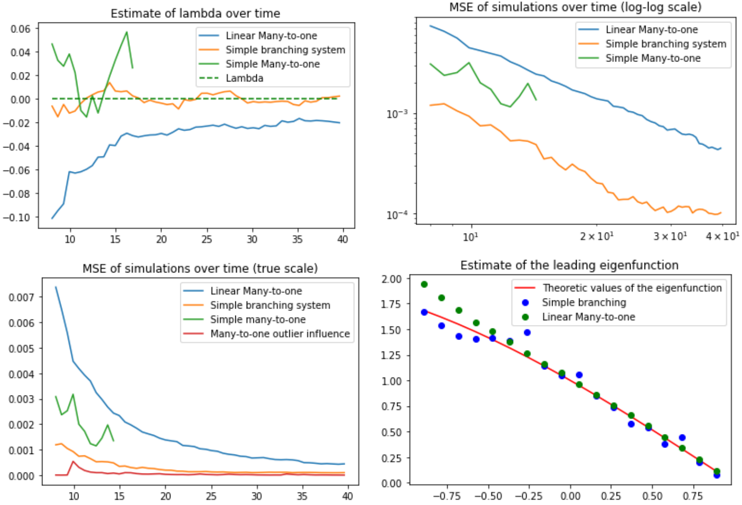
Importance sampling: Interval reactor



Ongoing work

- Complexity analysis of rates of convergence of Monte-Carlo schemes
- Hybrid constrained neutron branching / random walk methods
- Stochastic growth methods at criticality e.g. conditionally on survival,

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  where $\mathbf{e}$ is an exponential distribution.
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Thank you!