Self-similar Markov Processes

BUC14
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Related material:
https://arxiv.org/abs/1707.04343
https://arxiv.org/abs/1511.06356
https://arxiv.org/abs/1706.09924

§1. Quick review of Lévy processes

(Killed) Lévy process

- $(\xi_t, t \geq 0)$ is a (killed) Lévy process if it has stationary and independents with RCLL paths (and is sent to a cemetery state after and independent and exponentially distributed time).
- Process is entirely characterised by its one-dimensional transitions, which are coded by the Lévy–Khinchine formula [Exercise! Show that the exponent must factorise]:

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta\cdot\xi_t}] = \mathrm{e}^{-\Psi(\theta)t}, \qquad \theta \in \mathbb{R}^d,$$

where,

$$\Psi(\theta) = q + \mathrm{i}\mathrm{a}\cdot\theta + \frac{1}{2}\theta\cdot\mathbf{A}\theta + \int_{\mathbb{R}^d}(1 - \mathrm{e}^{\mathrm{i}\theta\cdot x} + \mathrm{i}(\theta\cdot x)\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x),$$

where $\mathrm{a} \in \mathbb{R}$, $\mathbf{A}$ is a $d \times d$ Gaussian covariance matrix and Π is a measure satisfying $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\Pi(\mathrm{d}x) < \infty$. Think of Π as the intensity of jumps in the sense of

$$\mathbb{P}(X \text{ has jump at time } t \text{ of size } \mathrm{d}x) = \Pi(\mathrm{d}x)\mathrm{d}t + o(\mathrm{d}t).$$

- In one dimension the path of a Lévy process can be monotone, in which case it is called a *subordinator* and we work with the Laplace exponent

$$\mathbb{E}[\mathrm{e}^{-\lambda\xi_t}] = \mathrm{e}^{-\Phi(\lambda)t}, \qquad t \geq 0$$

where

$$\Phi(\lambda) = q + \delta\lambda + \int_{(0,\infty)}(1 - \mathrm{e}^{-\lambda x})\Upsilon(\mathrm{d}x), \qquad \lambda \geq 0.$$

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Lévy process: One dimension

Two examples in one dimension:

- **Stable subordinator** $(\xi_t, t \geq 0)$ is a subordinator which satisfies the additional scaling property: For $c > 0$

$$\text{under } \mathbb{P}, \text{ the law of } (c\xi_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P},$$

where $\alpha \in (0,1)$. We have

$$\Phi(\lambda) = \lambda^{\alpha}, \qquad \lambda \geq 0, \qquad \text{and} \qquad \Pi(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{1}{x^{1+\alpha}} \mathrm{d}x, \qquad x > 0.$$

- **Hypgergeometric Lévy process:** For $\beta \leq 1, \gamma \in (0,1), \hat{\beta} \geq 0, \hat{\gamma} \in (0,1)$

$$\Psi(\theta) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}\theta)}{\Gamma(1-\beta-\mathrm{i}\theta)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}\theta)}{\Gamma(\hat{\beta}+\mathrm{i}\theta)} \qquad \theta \in \mathbb{R}.$$

The Lévy measure has a density with respect to Lebesgue measure which is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2F_1\left(1+\gamma, \eta; \eta-\hat{\gamma}; \mathrm{e}^{-x}\right), & \text{if } x > 0, \\ -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1\left(1+\hat{\gamma}, \eta; \eta-\gamma; \mathrm{e}^{x}\right), & \text{if } x < 0, \end{cases}$$

where $\eta := 1-\beta+\gamma+\hat{\beta}+\hat{\gamma}$.

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Lévy process: One dimension

- If ξ has a characteristic exponent Ψ then necessarily

$$\Psi(\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R}.$$

where κ and $\hat{\kappa}$ are Bernstein functions, e.g.

$$\kappa(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\Upsilon(\mathrm{d}x), \qquad \lambda \geq 0.$$

- The factorisation has a physical interpretation:
  - range of the κ-subordinator agrees with the range of $\sup_{s\leq t} \xi_s$, $t \geq 0$
  - range $\hat{\kappa}$-subordinator agrees with the range of $-\inf_{s\leq t} \xi_s$, $t \geq 0$.
- Note if $\delta > 0$, then $\mathbb{P}(\xi_{\tau_x^+} = x) > 0$, where $\tau_x^+ = \inf\{t > 0 : \xi_t = x\}$, $x > 0$.
- We have already seen the hypergeometric example

$$\Psi(\theta) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}\theta)}{\Gamma(1-\beta-\mathrm{i}\theta)} \qquad \times \qquad \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}\theta)}{\Gamma(\hat{\beta}+\mathrm{i}\theta)} \qquad \theta \in \mathbb{R}.$$

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α-STABLE PROCESS

Definition

A Lévy process X is called (strictly) α-stable if it is also a self-similar Markov process.

- Necessarily $\alpha \in (0,2]$. [$\alpha = 2 \rightarrow$ BM, exclude this.]
- The characteristic exponent $\Psi(\theta) := -t^{-1} \log \mathbb{E}(e^{i\theta X_t})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha (e^{\pi i\alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta>0)} + e^{-\pi i\alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

  where $\rho = P_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$
- Assume jumps in both directions ($0 < \alpha\rho, \alpha\hat{\rho} < 1$), so that the Lévy **density** takes the form

$$\frac{\Gamma(1+\alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} \left(\sin(\pi\alpha\rho)\mathbf{1}_{\{x>0\}} + \sin(\pi\alpha\hat{\rho})\mathbf{1}_{\{x<0\}}\right)$$

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- Note that, for $c > 0$, $c^{-\alpha}\Psi(c\theta) = \Psi(\theta)$,
- which is equivalent to saying that $cX_{c^{-\alpha}t} =^d X_t$,
- which by stationary and independent increments is equivalent to saying $(cX_{c^{-\alpha}t}, t \geq 0) =^d (X_t, t \geq 0)$ when $X_0 = 0$,
- or equivalently is equivalent to saying $(cX^{(x)}_{c^{-\alpha}t}, t \geq 0) =^d (X^{(cx)}_t, t \geq 0)$, where we have indicated the point of issue as an additional index.

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HITTING POINTS

- We say that ξ *can hit a point* $x \in \mathbb{R}$ if

$$\mathbb{P}(\xi_t = x \text{ for at least one } t > 0) > 0.$$

- Creeping is one way to hit a point, but not the only way

Theorem (Kesten (1969)/Bretagnolle (1971))

Suppose that ξ is not a compound Poisson process. Then ξ can hit points if and only if

$$\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{1+\Psi(z)}\right) \mathrm{d}z < \infty.$$

If the Kesten-Bretagnolle integral test is satisfied, then

$$\mathbb{P}(\tau^{\{x\}} < \infty) = \frac{u(x)}{u(0)},$$

where $\tau^{\{x\}} = \inf\{t > 0 : \xi_t = x\}$, providing we can compute the inversion

$$u(x) = \int_{c+\mathrm{i}\mathbb{R}} \frac{\mathrm{e}^{-zx}}{\Psi(-\mathrm{i}z)} \mathrm{d}z$$

for some $c \in \mathbb{R}$.

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Mini-exercise

Prove that:

- For $\alpha \in (0,1]$ stable processes cannot hit points
- For $\alpha \in (1,2)$ stable processes hit points with probability 1

§2. Self-similar Markov processes

Self-similar Markov processes (ssMp)

Definition

A regular strong Markov process $(Z_t : t \geq 0)$ on $\mathbb{R}^d$, with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}^d$, is a rssMp if there exists an index $\alpha \in (0, \infty)$ such that for all $c > 0$ and $x \in \mathbb{R}^d$,

$$(cZ_{tc^{-\alpha}} : t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal in law to } (Z_t : t \geq 0) \text{ under } \mathbb{P}_{cx}.$$

Some of your best friends are ssMp

Suppose that $(X_t : t \geq 0)$ is a one-dimensional stable process with two-sided jumps:

- [Exercise!] Write $\underline{X}_t := \inf_{s \leq t} X_s$. Then $(X_t, \underline{X}_t)$, $t \geq 0$ is a Markov process (in fact true for all Lévy proceses).
- For $c > 0$ and $\alpha = 2$,

$$\begin{pmatrix} c\underline{X}_{c^{-\alpha}t} \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} c\inf_{s \leq c^{-\alpha}t} X_s \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} \inf_{u \leq t} cX_{c^{-\alpha}u} \\ cX_{c^{-\alpha}t} \end{pmatrix}, \qquad t \geq 0,$$

  and the latter is equal in law to $(X, \underline{X})$, because of the scaling property of X.
- [Exercise!] $\Rightarrow$ Markov process $Z_t := X_t - (-x \wedge \underline{X}_t)$, $t \geq 0$ is also a ssMp on $[0, \infty)$ issued from $x > 0$ with index 2.
- [Exercise!] $\Rightarrow Z_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$ is also a ssMp, again on $[0, \infty)$.
- [Exercise!] Suppose that X is symmetric, then $Z_t := |X_t|$, $t \geq 0$. Because of rotational invariance, it is a Markov process.

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- [Exercise!] Suppose that X is symmetric, then $Z_t := |X_t|$, $t \geq 0$. Because of rotational invariance, it is a Markov process.

Some of your best friends are ssMp

Suppose that $(X_t : t \geq 0)$ is a one-dimensional stable process with two-sided jumps:

- [Exercise!] Write $\underline{X}_t := \inf_{s \leq t} X_s$. Then $(X_t, \underline{X}_t)$, $t \geq 0$ is a Markov process (in fact true for all Lévy proceses).
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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Note when $d = 3$, $|X_t|, t \geq 0$ is also equal in law to a Brownian motion conditioned to stay positive: i.e if we define, for a 1-d Brownian motion $(B_t : t \geq 0)$,

$$\mathbb{P}_x^{\uparrow}(A) = \lim_{s \to \infty} \mathbb{P}_x(A | \underline{B}_{t+s} > 0) = \mathbb{E}_x \left[\frac{B_t}{x} \mathbf{1}_{(\underline{B}_t > 0)} \mathbf{1}_{(A)} \right]$$

  where $A \in \sigma\{B_t : u \leq t\}$, then

$$(|X_t|, t \geq 0) \text{ with } |X_0| = x \text{ is equal in law to } (B, \mathbb{P}_x^{\uparrow}).$$

- [Exercise!] Prove that

$$\frac{B_t}{x} \mathbf{1}_{(\underline{B}_t > 0)}, \qquad t \geq 0,$$

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CONDITIONED α-STABLE PROCESSES

- Recall that each Lévy processes, $\xi = \{\xi_t : t \geq 0\}$, enjoys the Wiener-Hopf factorisation i.e. up to a multiplicative constant, $\Psi_\xi(\theta) := t^{-1}\log \mathbb{E}[e^{i\theta\xi_t}]$ respects the factorisation

$$\Psi_\xi(\theta) = \kappa(-i\theta)\hat{\kappa}(i\theta), \qquad \theta \in \mathbb{R},$$

where κ and $\hat{\kappa}$ are Bernstein functions. That is e.g. κ takes the form

$$\kappa(\lambda) = q + \mathrm{a}\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\nu(dx), \qquad \lambda \geq 0$$

where ν is a measure satisfying $\int_{(0,\infty)}(1 \wedge x)\nu(dx) < \infty$.

- The probabilistic significance of these subordinators, is that their range corresponds precisely to the range of the running maximum of ξ and of $-\xi$ respectively.
- In the case of α-stable processes, up to a multiplicative constant,

$$\kappa(\lambda) = \lambda^{\alpha\rho} \text{ and } \hat{\kappa}(\lambda) = \lambda^{\alpha\hat{\rho}}, \qquad \lambda \geq 0.$$

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- Associated to the descending ladder subordinator $\hat{\kappa}$ is its potential measure $\hat{U}$, which satisfies

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  [Note in the excluded case that $\alpha = 2$ and $\rho = 1/2$, i.e. Brownian motion, $\hat{U}(x) = x$.]

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- For $c, x > 0$, $t \geq 0$ and appropriately bounded, measurable and non-negative f, we can write,

$$\begin{aligned} &\mathbb{E}^{\uparrow}_x[f(\{cX_{c^{-\alpha}s} : s \leq t\})] \\ &\quad = \mathbb{E}\left[f(\{cX^{(x)}_{c^{-\alpha}s} : s \leq t\})\frac{(X^{(x)}_{c^{-\alpha}t})^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}^{(x)}_{c^{-\alpha}t} \geq 0)}\right] \\ &\quad = \mathbb{E}\left[f(\{X^{(cx)}_s : s \leq t\}\frac{(X^{(cx)}_t)^{\alpha\hat{\rho}}}{(cx)^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}^{(cx)}_t \geq 0)}\right] \\ &\quad = \mathbb{E}^{\uparrow}_{cx}[f(\{X_s : s \leq t\})]. \end{aligned}$$

- This also makes the process $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, a self-similar Markov process on $[0, \infty)$.
- Unlike the case of Brownian motion, the conditioned stable process does not have the law of the radial part of a 3-dimensional stable process (the analogue to the Brownian case).

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§3. Lamperti Transform

NOTATION

- Use $\xi := \{\xi_t : t \geq 0\}$ to denote a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time, $\mathbf{e}_q$, with rate in $q \in [0, \infty)$. The characteristic exponent of ξ is thus written

$$-\log E(\mathrm{e}^{\mathrm{i}\theta\xi_1}) = \Psi(\theta) = q + \text{Lévy–Khintchine}$$

- Define the associated integrated exponential Lévy process

$$I_t = \int_0^t \mathrm{e}^{\alpha\xi_s} \mathrm{d}s, \qquad t \geq 0. \tag{1}$$

and its limit, $I_\infty := \lim_{t\uparrow\infty} I_t$.

- Also interested in the inverse process of I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0. \tag{2}$$

As usual, we work with the convention $\inf \emptyset = \infty$.

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LAMPERTI TRANSFORM FOR POSITIVE SSMP

Theorem (Part (i))

Fix $\alpha > 0$. If $Z^{(x)}$, $x > 0$, is a positive self-similar Markov process with index of self-similarity α, then up to absorption at the origin, it can be represented as follows. For $x > 0$,

$$Z_t^{(x)} \mathbf{1}_{(t<\zeta^{(x)})} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}, \qquad t \geq 0,$$

where $\zeta^{(x)} = \inf\{t > 0 : Z_t^{(x)} = 0\}$ and either

(1) *$\zeta^{(x)} = \infty$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\limsup_{t\uparrow\infty} \xi_t = \infty$,*

(2) *$\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} = 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\lim_{t\uparrow\infty} \xi_t = -\infty$, or*

(3) *$\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} > 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process killed at an independent and exponentially distributed random time.*

In all cases, we may identify $\zeta^{(x)} = x^\alpha I_\infty$.

LAMPERTI TRANSFORM FOR POSITIVE SSMP

Theorem (Part (ii))

Conversely, suppose that ξ is a given (killed) Lévy process. For each $x > 0$, define

$$Z_t^{(x)} = x \exp\{\xi_{\varphi(x^{-\alpha}t)}\}\mathbf{1}_{(t<x^{\alpha}I_\infty)}, \qquad t \geq 0.$$

Then $Z^{(x)}$ defines a positive self-similar Markov process, up to its absorption time $\zeta^{(x)} = x^{\alpha}I_\infty$, with index α.

Lamperti transform for positive ssMp

$(Z, \mathrm{P}_x)_{x>0}$ pssMp $Z_t = \exp(\xi_{S(t)})$, S a random time-change	$\leftrightarrow$	$(\xi, \mathbb{P}_y)_{y\in\mathbb{R}}$ killed Lévy $\xi_s = \log(Z_{T(s)})$, T a random time-change

$$\left.\begin{array}{c} Z \text{ never hits zero} \\ Z \text{ hits zero continuously} \\ Z \text{ hits zero by a jump} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

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§4. Positive self-similar Markov processes

STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

Consider a stable process X with two-sided jumps:

- The stable process cannot 'creep' downwards across the threshold 0 and so must do so with a jump.
- This puts $Z_t^* := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$, in the class of pssMp for which the underlying Lévy process experiences exponential killing.
- Write $\xi^* = \{\xi_t^* : t \geq 0\}$ for the underlying Lévy process and denote its killing rate by q^*.
- Let's try and decode the characteristics of ξ^*.

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- We know that the α-stable process experiences downward jumps at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\frac{1}{|x|^{1+\alpha}}\mathrm{d}x, \qquad x<0.$$

- Given that we know the value of Z^*_{t-}, on $\{\underline{X}_t > 0\}$, the stable process will pass over the origin at rate

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- On the other hand, the Lamperti transform says that on $\{t < \zeta\}$, as a pssMp, Z is sent to the origin at rate

$$q^*\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = q^*\mathrm{e}^{-\alpha\xi^*_{\varphi(t)}} = q^*(Z^*)^{-\alpha}_t.$$

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$$q^*\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = q^*\mathrm{e}^{-\alpha\xi^*_{\varphi(t)}} = q^*(Z^*)^{-\alpha}_t.$$

- Comparing gives us

$$q^* = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}.$$

Stable process killed on entry to $(-\infty, 0)$

- We know that the α-stable process experiences downward jumps at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\frac{1}{|x|^{1+\alpha}}\mathrm{d}x, \qquad x<0.$$

- Given that we know the value of Z^*_{t-}, on $\{\underline{X}_t > 0\}$, the stable process will pass over the origin at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\left(\int_{Z^*_{t-}}^{\infty}\frac{1}{|x|^{1+\alpha}}\mathrm{d}x\right) = \frac{\Gamma(1+\alpha)}{\alpha\pi}\sin(\pi\alpha\hat{\rho})(Z^*)^{-\alpha}_{t-}.$$

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STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

- Referring again to the Lamperti transform, we know that, under $\mathbb{P}_1$ (so that $\mathbb{P}_1(\xi_0^* = 0) = 1$),

$$Z^*_{\zeta-} = X_{\tau_0^- -} = e^{\xi^*_{\mathbf{e}_{q^*}}},$$

where $\mathbf{e}_{q^*}$ is an exponentially distributed random variable with rate q^*.

- This motivates the computation

$$\mathbb{E}_1[(Z^*_{\zeta-})^{i\theta}] = \mathbb{E}_1[e^{i\theta\xi^*_{\mathbf{e}_{q^*}-}}] = \frac{q^*}{(\Psi^*(z) - q^*) + q^*}, \qquad \theta \in \mathbb{R},$$

where Ψ^* is the characteristic exponent of ξ^*.

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Stable process killed on entry to $(-\infty, 0)$

Setting

$$K = \frac{\sin \alpha\hat{\rho}\pi}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})},$$

Remembering the "overshoot-undershoot" distributional law at first passage (well known in the literature - see e.g. Chapter 8 of my book) and deduce that, for all $v \in [0,1]$,

$$\begin{aligned}
&\mathbb{P}_1(X_{\tau_0^- -} \in \mathrm{d}v) \\
&= \hat{\mathbb{P}}_0(1 - X_{\tau_1^+ -} \in \mathrm{d}v) \\
&= K\left(\int_0^\infty \int_0^\infty \mathbf{1}_{(y \le 1 \wedge v)} \frac{(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}}{(v+u)^{1+\alpha}} \mathrm{d}u\mathrm{d}y\right)\mathrm{d}v \\
&= \frac{K}{\alpha}\left(\int_0^1 \mathbf{1}_{(y\le v)} v^{-\alpha}(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}\mathrm{d}y\right)\mathrm{d}v,
\end{aligned}$$

where $\hat{\mathbb{P}}_0$ is the law of $-X$ issued from 0.

STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

We are led to the conclusion that

$$\begin{aligned}
\frac{q*}{\Psi^*(\theta)} &= \frac{K}{\alpha}\int_0^1 (1-y)^{\alpha\hat{\rho}-1}\int_0^\infty \mathbf{1}_{(y\le v)} v^{\mathrm{i}\theta-\alpha\hat{\rho}-1}\left(1-\frac{y}{v}\right)^{\alpha\rho-1} \mathrm{d}v\mathrm{d}y \\
&= \frac{K}{\alpha}\int_0^1 (1-y)^{\alpha\hat{\rho}-1} y^{\mathrm{i}\theta-\alpha\hat{\rho}}\mathrm{d}y \frac{\Gamma(\alpha\hat{\rho}-\mathrm{i}\theta)\Gamma(\alpha\rho)}{\Gamma(\alpha-\mathrm{i}\theta)} \\
&= \frac{\Gamma(\alpha\hat{\rho}-\mathrm{i}\theta)\Gamma(\alpha\rho)\Gamma(1-\alpha\hat{\rho}+\mathrm{i}\theta)\Gamma(\alpha\hat{\rho})\Gamma(\alpha+1)}{\alpha\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})\Gamma(\alpha\hat{\rho})\Gamma(1+\mathrm{i}\theta)\Gamma(\alpha-\mathrm{i}\theta)},
\end{aligned}$$

where in the first equality Fubini's Theorem has been used, in the second equality a straightforward substitution $w = y/v$ has been used for the inner integral on the preceding line together with the classical beta integral and, finally, in the third equality, the Beta integral has been used for a second time. Inserting the respective values for the constants q^* and K, we come to rest at the following result:

STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

Theorem

For the pssMp constructed by killing a stable process on first entry to $(-\infty, 0)$, the underlying ***killed*** *Lévy process, ξ^*, that appears through the Lamperti transform has characteristic exponent given by*

$$\Psi^*(z) = \frac{\Gamma(\alpha - iz)}{\Gamma(\alpha\hat{\rho} - iz)} \frac{\Gamma(1 + iz)}{\Gamma(1 - \alpha\hat{\rho} + iz)}, \qquad z \in \mathbb{R}.$$

MASSIVE EXERCISE NR. 1: SHUNTED STABLE PROCESSES

The following concept comes from Tim Budd, Nijmegen University. Consider a two-sided jumping stable process, X, and define the shunted stable process from the following path transformation.

1. When it first enters into $(-\infty, 0)$, suppose its overshoot is $-\Delta$ where $\Delta > 0$.
2. At the first passage time shunt the process upwards by adding 2Δ to its path (more generally $n\Delta$).
3. From its shunted position, allow it to continue with the dynamics of a stable process until it next goes negative.
4. Goto step 1.

Problems

- How we can write the shunted stable process pathwise in terms of X?
- Show that the shunted stable process is a self-similar Markov process.
- What is its underlying Lévy process?

Stable processes conditioned to stay positive

- Use the Lamperti representation of the α-stable process X to write, for $A \in \sigma(X_u : u \leq t)$,

$$\mathbb{P}^{\uparrow}_x(A) = \mathbb{E}_x\left[\frac{X_t^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}}\mathbf{1}_{(\underline{X}_t>0)}\mathbf{1}_{(A)}\right] = E\left[e^{\alpha\hat{\rho}\xi^*_\tau}\mathbf{1}_{(\tau<\mathbf{e}_{q^*})}\mathbf{1}_{(A)}\right],$$

where $\tau = \varphi(x^{-\alpha}t)$ is a stopping time in the natural filtration of ξ^*.

- Noting that $\Psi^*(-\mathrm{i}\alpha\hat{\rho}) = 0$, the change of measure constitutes an Esscher transform at the level of ξ^*.

Theorem

The underlying Lévy process, $\xi^{\uparrow}$, that appears through the Lamperti transform applied to $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, has characteristic exponent given by

$$\Psi^{\uparrow}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)}\frac{\Gamma(1+\alpha\hat{\rho}+\mathrm{i}z)}{\Gamma(1+\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

- In particular $\Psi^{\uparrow}(z) = \Psi^*(z - \mathrm{i}\alpha\hat{\rho})$, $z \in \mathbb{R}$ so that $\Psi^{\uparrow}(0) = 0$ (i.e. no killing!)
- One can also check by hand that $\Psi^{\uparrow\prime}(0+) = E[\xi^{\uparrow}_1] > 0$ so that $\lim_{t\to\infty}\xi^{\uparrow}_t = \infty$.

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Did you spot the other root?

- In essence, the case of the stable process conditioned to stay positive boils down to an Esscher transform in the underlying (Lamperti-transformed) Lévy process.
- It was important that we identified a root of $\Psi^*(z) = 0$ in order to avoid involving a 'time component' of the Esscher transform.
- However, there is another root of the equation

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)} = 0,$$

namely $z = -\mathrm{i}(1 - \alpha\hat{\rho})$.

- And this means that

$$\mathrm{e}^{(1-\alpha\hat{\rho})\xi^*}, \qquad t \geq 0,$$

is a unit-mean Martingale, which can also be used to construct an Esscher transform:

$$\Psi^{\downarrow}(z) = \Psi^*(z - \mathrm{i}(1 - \alpha\hat{\rho})) = \Psi^{\downarrow}(z) = \frac{\Gamma(1 + \alpha\rho - \mathrm{i}z)}{\Gamma(1 - \mathrm{i}z)} \frac{\Gamma(\mathrm{i}z + \alpha\hat{\rho})}{\Gamma(\mathrm{i}z)}.$$

- The choice of notation is pre-emptive since we can also check that $\Psi^{\downarrow}(0) = 0$ and $\Psi^{\downarrow\prime}(0) < 0$ so that if $\xi^{\downarrow}$ is a Lévy process with characteristic exponent $\Psi^{\downarrow}$, then $\lim_{t\to\infty} \xi^{\downarrow}_t = -\infty$.

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REVERSE ENGINEERING

- What now happens if we define for $A \in \sigma(X_u : u \le t)$,

$$\mathbb{P}^{\downarrow}_x(A) = E\left[e^{(1-\alpha\hat{\rho})\xi^*_\tau}\mathbf{1}_{(\tau < \mathbf{e}_{q^*})}\mathbf{1}_{(A)}\right] = \mathbb{E}_x\left[\frac{X_t^{(1-\alpha\hat{\rho})}}{x^{(1-\alpha\hat{\rho})}}\mathbf{1}_{(\underline{X}_t > 0)}\mathbf{1}_{(A)}\right],$$

where $\tau = \varphi(x^{-\alpha}t)$ is a stopping time in the natural filtration of ξ^*.

- In the same way we checked that $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, is a pssMp, we can also check that $(X, \mathbb{P}^{\downarrow}_x)$, $x > 0$ is a pssMp.
[Exercise!] Do it!
- In an appropriate sense, it turns out that $(X, \mathbb{P}^{\downarrow}_x)$, $x > 0$ is the law of a stable process conditioned to continuously approach the origin from above.

Reverse engineering

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Censored stable processes

- Start with X, the stable process.
- Let $A_t = \int_0^t \mathbf{1}_{(X_t > 0)} \, \mathrm{d}t$.
- Let γ be the right-inverse of A, and put $\check{Z}_t := X_{\gamma(t)}$.
- Finally, make zero an absorbing state: $Z_t = \check{Z}_t \mathbf{1}_{(t < T_0)}$ where

$$T_0 = \inf\{t > 0 : X_t = 0\}.$$

  Note $T_0 < \infty$ a.s. if and only if $\alpha \in (1, 2)$ and otherwise $T_0 = \infty$ a.s.
- This is the censored stable process.

Censored stable processes

Theorem

Suppose that the underlying Lévy process for the censored stable process is denoted by $\overset{\rightsquigarrow}{\xi}$. *Then* $\overset{\rightsquigarrow}{\xi}$ *is equal in law to* $\xi^{**} \oplus \xi^{C}$, *with*

- ξ^{**} *equal in law to* ξ^{*} *with the killing removed,*
- ξ^{C} *a compound Poisson process with jump rate* $q^{*} = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi$.

Moreover, the characteristic exponent of $\overset{\rightsquigarrow}{\xi}$ *is given by*

$$\overset{\rightsquigarrow}{\Psi}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)} \frac{\Gamma(1-\alpha\rho+\mathrm{i}z)}{\Gamma(1-\alpha+\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

The radial part of a stable process

- Suppose that X is a symmetric stable process, i.e $\rho = 1/2$.
- We know that $|X|$ is a pssMp.

Theorem

Suppose that the underlying Lévy process for $|X|$ is written ξ, then it characteristic exponent is given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + \alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + 1))}{\Gamma(\frac{1}{2}(\mathrm{i}z + 1 - \alpha))}, \qquad z \in \mathbb{R}.$$

[Exercise!] This is quite hard to prove for $\alpha \in (1, 2)$, but could can be proved in a straightforward way for $\alpha \in (0, 1]$. Try it!
[Hint!] Think about what happens after X first crosses the origin and apply the Markov property as well as symmetry. You will need to use the law of the overshoot of X below the origin given a few slides back.

§5. Real valued self-similar Markov processes

- So far we only spoke about $[0, \infty)$.
- What can we say about $\mathbb{R}$-valued self-similar Markov processes.
- This requires us to first investigate Markov Additive (Lévy) Processes

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Markov additive processes (MAPs)

- E is a finite state space
- $(J(t))_{t\geq 0}$ is a continuous-time, irreducible Markov chain on E
- process (ξ, J) in $\mathbb{R} \times E$ is called a *Markov additive process (MAP)* with probabilities $\mathbf{P}_{x,i}$, $x \in \mathbb{R}, i \in E$, if, for any $i \in E$, $s, t \geq 0$: Given $\{J(t) = i\}$,
  $(\xi(t+s) - \xi(t), J(t+s)) \stackrel{d}{=} (\xi(s), J(s))$ with law $\mathbf{P}_{0,i}$.

PATHWISE DESCRIPTION OF A MAP

The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$,

- there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$
- and a sequence of iid random variables $(U_{ij}^n)_{n\geq 0}$, independent of the chain J,
- such that if $T_0 = 0$ and $(T_n)_{n\geq 1}$ are the jump times of J,
  the process ξ has the representation

$$\xi(t) = \mathbf{1}_{(n>0)}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n),$$

  for $t \in [T_n, T_{n+1})$, $n \geq 0$.
- [Exercise!] Show that the property above implies the definition on the previous slide.

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $\mathbf{Q} = (q_{ij})_{i,j\in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$ with $i \neq j$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Otherwise define $U_{i,i} \equiv 0$, for each $i \in E$.
- Write $G(z)$ for the $N \times N$ matrix whose (i,j)th element is $G_{ij}(z)$.
- Let

$$\boldsymbol{\Psi}(z) = \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) + \mathbf{Q} \circ G(z),$$

  (when it exists), where $\circ$ indicates elementwise multiplication.
- The matrix exponent of the MAP (ξ, J) is given by

$$\mathbf{E}_{0,i}(e^{z\xi(t)}; J(t) = j) = \left(e^{\boldsymbol{\Psi}(z)t}\right)_{i,j}, \qquad i, j \in E,$$

  (when it exists).

LAMPERTI-KIU TRANSFORM

- Take J to be irreducible on $E = \{1, -1\}$.
- Let
$$Z_t = |x|e^{\xi(\tau(|x|^{-\alpha}t))}J(\tau(|x|^{-\alpha}t)) \qquad 0 \le t < T_0,$$
where
$$\tau(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))\mathrm{d}u > t\right\}$$
and
$$T_0 = |x|^{-\alpha}\int_0^\infty e^{\alpha\xi(u)}\mathrm{d}u.$$
- Then Z_t is a real-valued self-similar Markov process in the sense that the law of $(cZ_{tc^{-\alpha}} : t \ge 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{cx}$.
- The converse (within a special class of rssMps) is also true.
- [Exercise!] Explain what happens if e.g. J is an absorbing Markov Chain on $\{1, -1\}$ with $\{1\}$ as an absorbing state?

Lamperti-Kiu transform

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- [Exercise!] Explain what happens if e.g. J is an absorbing Markov Chain on $\{1, -1\}$ with $\{1\}$ as an absorbing state?

LAMPERTI-KIU TRANSFORM

- Take J to be irreducible on $E = \{1, -1\}$.
- Let

$$Z_t = |x|e^{\xi(\tau(|x|^{-\alpha}t))}J(\tau(|x|^{-\alpha}t)) \qquad 0 \leq t < T_0,$$

where

$$\tau(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))\mathrm{d}u > t\right\}$$

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ENTRANCE AT ZERO

- Given the Lamperti-Kiu representation

$$Z_t = e^{\xi(\tau(|x|^{-\alpha}t)) + \log|x|} J(\tau(|x|^{-\alpha}t)) \qquad 0 \le t < T_0,$$

it is clear that we can think of a similar construction from zero to the case of pssMp.

- We need to construct a stationary version of the pair (ξ, J) which is indexed by $\mathbb{R}$ and pinned at space-time point $(-\infty, \infty)$.
- Just like the theory of Lévy processes, by observing the range of the process (ξ_t, J_t) $t \ge 0$, **only** at the points of its new suprema, we see a process (H_t^+, J_t^+), $t \ge 0$, which is also a MAP, where H^+ is has increasing paths.

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AN α-STABLE PROCESS IS A RSSMP

- An α-stable process up to absorption in the origin is a rssMp.
- When $\alpha \in (0,1]$, the process never hits the origin a.s.
- When $\alpha \in (1,2)$, the process is absorbs at the origin a.s.
- The matrix exponent of the underlying MAP is given by:

$$\begin{bmatrix} \frac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1 - \alpha\hat{\rho} + z)} & -\frac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})} \\ -\frac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} & \frac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho - z)\Gamma(1 - \alpha\rho + z)} \end{bmatrix},$$

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ESSCHER TRANSFORM FOR MAPS

- If $\boldsymbol{\Psi}(z)$ is well defined then it has a real simple eigenvalue $\chi(z)$, which is larger than the real part of all its other eigenvalues.
- Furthermore, the corresponding right-eigenvector $\mathbf{v}(z) = (v_1(z), \cdots, v_N(z))$ has strictly positive entries and may be normalised such that $\pi \cdot \mathbf{v}(z) = 1$.

Theorem

Let $\mathcal{G}_t = \sigma\{(\xi(s), J(s)) : s \leq t\}$, $t \geq 0$, *and*

$$M_t := \mathrm{e}^{\gamma\xi(t) - \chi(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_i(\gamma)}, \qquad t \geq 0,$$

for some $\gamma \in \mathbb{R}$ *such that* $\chi(\gamma)$ *is defined. Then,* M_t, $t \geq 0$, *is a unit-mean martingale. Moreover, under the change of measure*

$$\mathrm{d}\mathbf{P}^{\gamma}_{0,i}\Big|_{\mathcal{G}_t} = M_t \, \mathrm{d}\mathbf{P}_{0,i}\big|_{\mathcal{G}_t}, \qquad t \geq 0,$$

the process (ξ, J) *remains in the class of MAPs with new exponent given by*

$$\boldsymbol{\Psi}_\gamma(z) = \boldsymbol{\Delta}_v(\gamma)^{-1}\boldsymbol{\Psi}(z+\gamma)\boldsymbol{\Delta}_v(\gamma) - \chi(\gamma)\mathbf{I}.$$

Here, $\mathbf{I}$ *is the identity matrix and* $\boldsymbol{\Delta}_v(\gamma) = \mathrm{diag}(\boldsymbol{v}(\gamma))$.

Esscher and drift

- Suppose that χ is defined in some open interval D of $\mathbb{R}$, then, it is smooth and convex on D.
- Since $\Psi(0) = -\mathbf{Q}$, if, moreover, J is irreducible, we always have $\chi(0) = 0$ and $\mathbf{v}(0) = (1, \cdots, 1)$. So $0 \in D$ and $\chi'(0)$ is well defined and finite.
- With all of the above

$$\lim_{t\to\infty} \frac{\xi_t}{t} = \chi'(0) \qquad \text{a.s.}$$

- [Exercise!] Show that in the above circumstances, if $\chi'(0) < 0$, then the associated ssMp hits the origin in an almost surely finite time, independently of its point of issue $x \in \mathbb{R}$.

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ESSCHER AND THE STABLE-MAP

- For the MAP that underlies the stable process $D = (-1, \alpha)$, it can be checked that $\det\Psi(\alpha - 1) = 0$ i.e. $\chi(\alpha - 1) = 0$, which makes

$$\Psi^\circ(z) = \Delta^{-1}\Psi(z + \alpha - 1)\Delta$$
$$= \begin{bmatrix} \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\rho-z)\Gamma(\alpha\rho+z)} & -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \\ -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} & \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\hat{\rho}-z)\Gamma(\alpha\hat{\rho}+z)} \end{bmatrix},$$

where $\Delta = \mathrm{diag}(\sin(\pi\alpha\hat{\rho}), \sin(\pi\alpha\rho))$.

- When $\alpha \in (0,1)$, $\chi'(0) > 0$ (because the stable process never touches the origin a.s.) and $\Psi^\circ(z)$-MAP drifts to $-\infty$
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RIESZ-BOGDAN-ZAK TRANSFORM

Theorem (Riesz–Bogdan–Zak transform)

Suppose that X is an α-stable process as outlined in the introduction. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}\backslash\{0\}$, $(-1/X_{\eta(t)})_{t\geq 0}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{-1/x})$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \left(\frac{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(X_t)}{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(x)}\right) \left|\frac{X_t}{x}\right|^{\alpha-1} \mathbf{1}_{(t<\tau^{\{0\}})}$$

and $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$. Moreover, the process $(X, \mathbb{P}^\circ_x)$, $x \in \mathbb{R}\backslash\{0\}$ is a self-similar Markov process with underlying MAP via the Lamperti-Kiu transform given by $\mathbf{\Psi}^\circ(z)$.

WHAT IS THE Ψ°-MAP?

Thinking of the affect on the long term behaviour of the underlying MAP of the Esscher transform

- When $\alpha \in (0,1)$, $(X, \mathbb{P}_x^\circ)$, $x \neq 0$ has the law of the the stable process conditioned to absorb continuously at the origin in the sense,

$$\mathbb{P}_y^\circ(A) = \lim_{a\to 0} \mathbb{P}_y(A, t < T_0 \,|\, \tau_{(-a,a)} < \infty),$$

  for $A \in \mathcal{F}_t = \sigma(X_s, s \leq t)$,
  $\tau_{(-a,a)} = \inf\{t > 0 : |X_t| < a\}$ and $T_0 = \inf\{t > 0 : X_t = 0\}$.
- When $\alpha \in (1,2)$, $(X, \mathbb{P}_x^\circ)$, $x \neq 0$ has the law of the stable process conditioned to avoid the origin in the sense

$$\mathbb{P}_y^\circ(A) = \lim_{s\to\infty} \mathbb{P}_y(A \,|\, T_0 > t + s),$$

  for $A \in \mathcal{F}_t = \sigma(X_s, s \leq t)$ and $T_0 = \inf\{t > 0 : X_t = 0\}$.

[Exercise!] Explain this change in behaviour heuristically.

MASSIVE PROBLEM NR. 2: BOGDAN-BURDZY-CHEN CENSORED STABLE PROCESS

Consider a two-sided jumping stable process, X started anywhere in $(-1, 1)$.

Every time the process exits the interval $(-1, 1)$ the jump that brings it out of the interval is deleted and the process continues from its 'pre-jump' position.

Problems

- Does the resulting process have a stationary limit?
- Or does is the resulting process non-conservative and end up at the boundary of $(-1, 1)$?
- If it is conservative, what is its stationary distribution on $(-1, 1)$?

Later you can consider this question in higher dimensions for an isotropic d-dimensional stable process, replacing $(-1, 1)$ by $\mathbb{S}_{d-1}$.

§6. Isotropic stable processes in dimension $d \geq 2$ seen as Lévy processes

Isotropic α-stable process in dimension $d \geq 2$

For $d \geq 2$, let $X := (X_t : t \geq 0)$ be a d-dimensional isotropic stable process.

- X has stationary and independent increments (it is a Lévy process)
- Characteristic exponent $\Psi(\theta) = -\log \mathbb{E}_0(e^{i\theta \cdot X_1})$ satisfies

$$\Psi(\theta) = |\theta|^\alpha, \qquad \theta \in \mathbb{R}.$$

- Necessarily, $\alpha \in (0, 2]$, we exclude 2 as it pertains to the setting of a Brownian motion.
- Associated Lévy measure satisfies, for $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}\Pi(B) &= \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} \mathrm{d}y \\ &= \frac{2^{\alpha-1}\Gamma((d+\alpha)/2)\Gamma(d/2)}{\pi^d \left|\Gamma(-\alpha/2)\right|} \int_{\mathbb{S}_{d-1}} r^{d-1}\sigma_1(\mathrm{d}\theta) \int_0^\infty \mathbf{1}_B(r\theta) \frac{1}{r^{\alpha+d}} \mathrm{d}r,\end{aligned}$$

  where $\sigma_1(\mathrm{d}\theta)$ is the surface measure on $\mathbb{S}_{d-1}$ normalised to have unit mass.
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Isotropic α-stable process in dimension $d \geq 2$

- Stable processes are also self-similar. For $c > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$\text{under } \mathbb{P}_x, \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx}.$$

- Isotropy means, for all orthogonal transformations (e.g. rotations) $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

$$\text{under } \mathbb{P}_x, \text{ the law of } (UX_t, t \geq 0) \text{ is equal to } \mathbb{P}_{Ux}.$$

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ISOTROPIC α-STABLE PROCESS IN DIMENSION $d \geq 2$

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Some classical properties: Transience

We are interested in the potential measure

$$U(x, \mathrm{d}y) = \int_0^\infty \mathbb{P}_x(X_t \in \mathrm{d}y)\mathrm{d}t = \left(\int_0^\infty \mathrm{p}_t(y-x)\mathrm{d}t\right)\mathrm{d}y, \qquad x, y \in \mathbb{R}.$$

Note: stationary and independent increments means that it suffices to consider $U(0, \mathrm{d}y)$.

Theorem

The potential of X is absolutely continuous with respect to Lebesgue measure, in which case, its density in collaboration with spatial homogeneity satisfies $U(x, \mathrm{d}y) = u(y-x)\mathrm{d}y$, $x, y \in \mathbb{R}^d$, where

$$u(z) = 2^{-\alpha}\pi^{-d/2}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}|z|^{\alpha-d}, \qquad z \in \mathbb{R}^d.$$

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Proof of Theorem

Now note that, for bounded and measurable $f : \mathbb{R}^d \mapsto \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty f(X_t)\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty f(\sqrt{2}B_{S_t})\mathrm{d}t\right] \\
&= \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}t\, \mathbb{P}(S_t \in \mathrm{d}s) \int_{\mathbb{R}} \mathbb{P}(B_s \in \mathrm{d}x) f(\sqrt{2}x) \\
&= \frac{1}{\Gamma(\alpha/2)\pi^{d/2}2^d} \int_{\mathbb{R}} \mathrm{d}y \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-|y|^2/4s} s^{-1+(\alpha-d)/2} f(y) \\
&= \frac{1}{2^\alpha \Gamma(\alpha/2)\pi^{d/2}} \int_{\mathbb{R}} \mathrm{d}y\, |y|^{(\alpha-d)} \int_0^\infty \mathrm{d}u\, e^{-u} u^{-1+(d-\alpha/2)} f(y) \\
&= \frac{\Gamma((d-\alpha)/2)}{2^\alpha \Gamma(\alpha/2)\pi^{d/2}} \int_{\mathbb{R}} \mathrm{d}y\, |y|^{(\alpha-d)} f(y).
\end{aligned}$$

SOME CLASSICAL PROPERTIES: POLARITY

- Kesten-Bretagnolle integral test, in dimension $d \geq 2$,

$$\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{1+\Psi(z)}\right) \mathrm{d}z = \int_{\mathbb{R}} \frac{1}{1+|z|^{\alpha}} \mathrm{d}z \propto \int_{\mathbb{R}} \frac{1}{1+r^{\alpha}} r^{d-1} \mathrm{d}r\, \sigma_1(\mathrm{d}\theta) = \infty.$$

- $\mathbb{P}_x(\tau^{\{y\}} < \infty) = 0$, for $x, y \in \mathbb{R}^d$.
- i.e. the stable process cannot hit individual points almost surely.

§8. Isotropic stable processes in dimension $d \geq 2$ seen as a self-similar Markov process

The radial part of a stable process

Lemma
The process $(|X_t|, t \geq 0)$ is strong Markov and self-similar.

- Temporarily write $(X_t^{(x)}, t \geq 0)$ in place of $(X, \mathbb{P}_x)$
- Markov property of X tells us that, for $s, t \geq 0$,

$$X_{t+s}^{(x)} = \tilde{X}_s^{(X_t^{(x)})},$$

where $\tilde{X}^{(x)}$ is an independent copy of $X^{(x)}$.
- Isotropy implies that

$$|X_{t+s}^{(x)}| = |\tilde{X}_s^{(y)}|_{y=X_t^{(x)}} =^d |\tilde{X}_s^{(z)}|_{z=(|X_t^{(x)}|,0,0\cdots,0)}$$

- Hence Markov property holds, strong Markov property (and Feller property) can be developed from this argument
- Self-similarity of $|X|$ follows directly from the self-similarity of X.

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LAMPERTI-TRANSFORM OF $|X|$

Theorem (Caballero-Pardo-Perez (2011))

For the pssMp constructed using the radial part of an isotropic d-dimensional stable process, the underlying Lévy process, ξ that appears through the Lamperti has characteristic exponent given by

$$\Psi(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + \alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + d))}{\Gamma(\frac{1}{2}(\mathrm{i}z + d - \alpha))}, \qquad z \in \mathbb{R}.$$

- The fact that $\lim_{t\to\infty} |X_t| = \infty$ implies that $\lim_{t\to\infty} \xi_t = \infty$
- If we write $\psi(\lambda) = -\Psi(-\mathrm{i}\lambda) = \log \mathbb{E}[\mathrm{e}^{\lambda X_1}]$ for the Laplace exponent of ξ, then it is well defined for $\lambda \in (-d, \alpha)$ with roots at $\lambda = 0$ and $\lambda = \alpha - d$.
- Note that

$$\exp((\alpha - d)\xi_t), \qquad t \geq 0,$$

is a martingale

- Recalling that $|X_t| = \exp(\xi_{\varphi_t})$ and that φ_t is an almost surely finite stopping time (because $\lim_{t\to\infty} \xi_t = \infty$) we can deduce that

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Conditioned stable process

- We can define the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^{\circ}_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0, x \neq 0$$

- Suppose that f is a bounded measurable function then, for all $c > 0$,

$$\mathbb{E}^{\circ}_x[f(cX_{c^{-\alpha}s}, s \leq t)] = \mathbb{E}_x\left[\frac{|cX_{c^{-\alpha}t}|^{\alpha-d}}{|cx|^{d-\alpha}} f(cX_{c^{-\alpha}s}, s \leq t)\right]$$
$$= \mathbb{E}_{cx}\left[\frac{|X_t|^{\alpha-d}}{|cx|^{d-\alpha}} f(X_s, s \leq t)\right] = \mathbb{E}^{\circ}_{cx}[f(X_s, , s \leq t)]$$

- Markovian, isotropy and self-similarity properties pass through to $(X, \mathbb{P}^{\circ}_x)$, $x \neq 0$.
- Similarly $(|X|, \mathbb{P}^{\circ}_x)$, $x \neq 0$ is a positive self-similar Markov process.

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CONDITIONED STABLE PROCESS

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Conditioned stable process

- It turns out that $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, corresponds to the stable process conditioned to be continuously absorbed at the origin.
- More precisely, for $A \in \sigma(X_s, s \leq t)$, if we set $\{0\}$ to be 'cemetery' state and $\mathtt{k} = \inf\{t > 0 : X_t = 0\}$, then

$$\mathbb{P}^\circ_x(A, t < \mathtt{k}) = \lim_{a \downarrow 0} \mathbb{P}_x(A,\ t < \mathtt{k} | \tau^\oplus_a < \infty),$$

where $\tau^\oplus_a = \inf\{t > 0 : |X_t| < a\}$.
- In light of the associated Esscher transform on ξ, we note that the Lamperti transform of $(|X|, \mathbb{P}^\circ_x)$, $x \neq 0$, corresponds to the Lévy process with characteristic exponent

$$\Psi^\circ(z) = 2^\alpha \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + d))}{\Gamma(-\frac{1}{2}(\mathrm{i}z + \alpha - d))} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + \alpha))}{\Gamma(\frac{1}{2}\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

- Given the pathwise interpretation of $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, it follows immediately that $\lim_{t \to \infty} \xi_t = -\infty$, $\mathbb{P}^\circ_x$ almost surely, for any $x \neq 0$.

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CONDITIONED STABLE PROCESS

- It turns out that $(X, \mathbb{P}^\circ_x)$, $x \neq 0$, corresponds to the stable process conditioned to be continuously absorbed at the origin.
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$\mathbb{R}^d$-SELF-SIMILAR MARKOV PROCESSES

Definition

A $\mathbb{R}^d$-valued regular Feller process $Z = (Z_t, t \geq 0)$ is called a $\mathbb{R}^d$*-valued self-similar Markov process* if there exists a constant $\alpha > 0$ such that, for any $x > 0$ and $c > 0$,

$$\text{the law of } (cZ_{c^{-\alpha}t}, t \geq 0) \text{ under } P_x \text{ is } P_{cx},$$

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LAMPERTI–KIU TRANSFORM

In order to introduce the analogue of the Lamperti transform in d-dimensions, we need to remind ourselves of what we mean by a Markov additive process in this context.

Definition

An $\mathbb{R} \times E$ valued regular Feller process $(\xi, \Theta) = ((\xi_t, \Theta_t) : t \geq 0)$ with probabilities $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in E$, and cemetery state $(-\infty, \dagger)$ is called a *Markov additive process* (MAP) if Θ is a regular Feller process on E with cemetery state $\dagger$ such that, for every bounded measurable function $f : (\mathbb{R} \cup \{-\infty\}) \times (E \cup \{\dagger\}) \to \mathbb{R}$, $t, s \geq 0$ and $(x, \theta) \in \mathbb{R} \times E$, on $\{t < \varsigma\}$,

$$\mathbf{E}_{x,\theta}[f(\xi_{t+s} - \xi_t, \Theta_{t+s}) | \sigma((\xi_u, \Theta_u), u \leq t)] = \mathbf{E}_{0,\Theta_t}[f(\xi_s, \Theta_s)],$$

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- Roughly speaking, one thinks of a MAP as a 'Markov modulated' Lévy process
- It has 'conditional stationary and independent increments'
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LAMPERTI–KIU TRANSFORM

Theorem

Fix $\alpha > 0$. The process Z is a ssMp with index α if and only if there exists a (killed) MAP, (ξ, Θ) on $\mathbb{R} \times \mathbb{S}_{d-1}$ such that

$$Z_t := \mathrm{e}^{\xi_{\varphi(t)}} \Theta_{\varphi(t)} \qquad , \qquad t \le I_\varsigma,$$

where

$$\varphi(t) = \inf\left\{ s > 0 : \int_0^s \mathrm{e}^{\alpha \xi_u}\, \mathrm{d}u > t \right\}, \qquad t \le I_\varsigma,$$

and $I_\varsigma = \int_0^\varsigma \mathrm{e}^{\alpha\xi_s} \mathrm{d}s$ is the lifetime of Z until absorption at the origin. Here, we interpret $\exp\{-\infty\} \times \dagger := 0$ and $\inf \emptyset := \infty$.

- In the representation, the time to absorption in the origin,

$$\zeta = \inf\{t > 0 : Z_t = 0\},$$

  satisfies $\zeta = I_\varsigma$.
- Note $x \in \mathbb{R}^d$ if and only if

$$x = (|x|, \mathrm{Arg}(x)),$$

  where $\mathrm{Arg}(x) = x/|x| \in \mathbb{S}_{d-1}$. The Lamperti–Kiu decomposition therefore gives us a d-dimensional skew product decomposition of self-similar Markov processes.

LAMPERTI-STABLE MAP

- The stable process X is an $\mathbb{R}^d$-valued self-similar Markov process and therefore fits the description above
- **How do we characterise its underlying MAP** (ξ, Θ)?
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MAP ISOTROPY

Theorem

Suppose (ξ, Θ) is the MAP underlying the stable process. Then $((\xi, U^{-1}\Theta), \mathbf{P}_{x,\theta})$ is equal in law to $((\xi, \Theta), \mathbf{P}_{x,U^{-1}\theta})$, for every orthogonal d-dimensional matrix U and $x \in \mathbb{R}^d$, $\theta \in \mathbb{S}_{d-1}$.

Proof.

First note that $\varphi(t) = \int_0^t |X_u|^{-\alpha}\mathrm{d}u$. It follows that

$$(\xi_t, \Theta_t) = (\log |X_{A(t)}|, \operatorname{Arg}(X_{A(t)})), \qquad t \geq 0,$$

where the random times $A(t) = \inf\left\{s > 0 : \int_0^s |X_u|^{-\alpha}\mathrm{d}u > t\right\}$ are stopping times in the natural filtration of X.

Now suppose that U is any orthogonal d-dimensional matrix and let $X' = U^{-1}X$. Since X is isotropic and since $|X'| = |X|$, and $\operatorname{Arg}(X') = U^{-1}\operatorname{Arg}(X)$, we see that, for $x \in \mathbb{R}$ and $\theta \in \mathbb{S}_{d-1}$

$$\begin{aligned}
((\xi, U^{-1}\Theta), \mathbf{P}_{\log|x|,\theta}) &= ((\log |X_{A(\cdot)}|,\, U^{-1}\operatorname{Arg}(X_{A(\cdot)})), \mathbb{P}_x) \\
&\overset{d}{=} ((\log |X_{A(\cdot)}|,\, \operatorname{Arg}(X_{A(\cdot)})), \mathbb{P}_{U^{-1}x}) \\
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MAP CORROLATION

- We will work with the increments $\Delta\xi_t = \xi_t - \xi_{t-} \in \mathbb{R}$, $t \geq 0$,

Theorem (Bo Li, Victor Rivero, Bertoin-Werner (1996))

Suppose that f is a bounded measurable function on $[0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}_{d-1} \times \mathbb{S}_{d-1}$ such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, then, for all $\theta \in \mathbb{S}_{d-1}$,

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MAP OF $(X, \mathbb{P}^\circ_\cdot)$

- Recall that $(|X_t|^{\alpha-d}, t \geq 0)$, is a martingale.
- Informally, we should expect $\mathcal{L}h = 0$, where $h(x) = |x|^{\alpha-d}$ and $\mathcal{L}$ is the infinitesimal generator of the stable process, which has action

$$\mathcal{L}f(x) = \mathrm{a} \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x+y) - f(x) - \mathbf{1}_{(|y|\leq 1)} y \cdot \nabla f(x)]\Pi(\mathrm{d}y), \qquad |x| > 0,$$

for appropriately smooth functions.

- Associated to $(X, \mathbb{P}_x)$, $x \neq 0$ is the generator

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for $|x| > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

MAP OF $(X, \mathbb{P}^\circ_\cdot)$

- Recall that $(|X_t|^{\alpha-d}, t \geq 0)$, is a martingale.
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for appropriately smooth functions.
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MAP OF $(X, \mathbb{P}^\circ_\cdot)$

Theorem

Suppose that f is a bounded measurable function on $[0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}_{d-1} \times \mathbb{S}_{d-1}$ such that $f(\cdot,\cdot,0,\cdot,\cdot) = 0$, then, for all $\theta \in \mathbb{S}_{d-1}$,

$$\mathbf{E}^\circ_{0,\theta}\left(\sum_{s>0} f(s, \xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\right)$$
$$= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{S}_{d-1}} \int_{\mathbb{R}} V^\circ_\theta(\mathrm{d}s, \mathrm{d}x, \mathrm{d}\vartheta)\sigma_1(\mathrm{d}\phi)\mathrm{d}y \frac{c(\alpha)\mathrm{e}^{yd}}{|\mathrm{e}^y\phi - \vartheta|^{\alpha+d}} f(s, x, -y, \vartheta, \phi),$$

where

$$V^\circ_\theta(\mathrm{d}s, dx, \mathrm{d}\vartheta) = \mathbf{P}^\circ_{0,\theta}(\xi_s \in \mathrm{d}x, \Theta_s \in \mathrm{d}\vartheta)\mathrm{d}s, \qquad x \in \mathbb{R}, \vartheta \in \mathbb{S}_{d-1}, s \geq 0,$$

is the space-time potential of (ξ, Θ) under $\mathbf{P}^\circ_{0,\theta}$.

Comparing the right-hand side above with that of the previous Theorem, it now becomes immediately clear that the the jump structure of (ξ, Θ) under $\mathbf{P}^\circ_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}_{d-1}$, is precisely that of $(-\xi, \Theta)$ under $\mathbf{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathbb{S}_{d-1}$.

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§9. Riesz–Bogdan–Żak transform

Riesz–Bogdan–Żak transform

- Define the transformation $K : \mathbb{R}^d \mapsto \mathbb{R}^d$, by

$$Kx = \frac{x}{|x|^2}, \qquad x \in \mathbb{R}^d \backslash \{0\}.$$

- This transformation inverts space through the unit sphere $\{x \in \mathbb{R}^d : |x| = 1\}$.
- Write $x \in \mathbb{R}^d$ in skew product form $x = (|x|, \mathrm{Arg}(x))$, and note that

$$Kx = (|x|^{-1}, \mathrm{Arg}(x)), \qquad x \in \mathbb{R}^d \backslash \{0\},$$

showing that the K-transform 'radially inverts' elements of $\mathbb{R}^d$ through $\mathbb{S}_{d-1}$.
- In particular $K(Kx) = x$

Theorem (d-dimensional Riesz–Bogdan–Żak Transform, $d \geq 2$)

Suppose that X is a d-dimensional isotropic stable process with $d \geq 2$. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0. \qquad (3)$$

Then, for all $x \in \mathbb{R}^d \backslash \{0\}$, $(KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{Kx})$.

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Proof of Riesz–Bogdan–Żak transform

We give a proof, different to the original proof of Bogdan and Żak (2010).

- Recall that $X_t = e^{\xi_{\varphi(t)}}\Theta_{\varphi(t)}$, where

$$\int_0^{\varphi(t)} e^{\alpha\xi_u}\,du = t, \qquad t \geq 0.$$

- Note also that, as an inverse,

$$\int_0^{\eta(t)} |X_u|^{-2\alpha} du = t, \qquad t \geq 0.$$

- Differentiating,

$$\frac{d\varphi(t)}{dt} = e^{-\alpha\xi_{\varphi(t)}} \text{ and } \frac{d\eta(t)}{dt} = e^{2\alpha\xi_{\varphi\circ\eta(t)}}, \qquad \eta(t) < \tau^{\{0\}}.$$

and chain rule now tells us that

$$\frac{d(\varphi\circ\eta)(t)}{dt} = \left.\frac{d\varphi(s)}{ds}\right|_{s=\eta(t)} \frac{d\eta(t)}{dt} = e^{\alpha\xi_{\varphi\circ\eta(t)}}.$$

- Said another way,

$$\int_0^{\varphi\circ\eta(t)} e^{-\alpha\xi_u} du = t, \qquad t \geq 0,$$

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PROOF OF RIESZ–BOGDAN–ŻAK TRANSFORM

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