

## MEROMORPHIC LÉVY PROCESSES AND THEIR FLUCTUATION IDENTITIES

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The last couple of years has seen a remarkable number of new, explicit examples of the Wiener–Hopf factorization for Lévy processes where previously there had been very few. We mention, in particular, the many cases of spectrally negative Lévy processes in [*Sixth Seminar on Stochastic Analysis, Random Fields and Applications* (2011) 119–146, *Electron. J. Probab.* **13** (2008) 1672–1701], hyper-exponential and generalized hyper-exponential Lévy processes [*Quant. Finance* **10** (2010) 629–644], Lamperti-stable processes in [*J. Appl. Probab.* **43** (2006) 967–983, *Probab. Math. Statist.* **30** (2010) 1–28, *Stochastic Process. Appl.* **119** (2009) 980–1000, *Bull. Sci. Math.* **133** (2009) 355–382], Hypergeometric processes in [*Ann. Appl. Probab.* **20** (2010) 522–564, *Ann. Appl. Probab.* **21** (2011) 2171–2190, *Bernoulli* **17** (2011) 34–59],  $\beta$ -processes in [*Ann. Appl. Probab.* **20** (2010) 1801–1830] and  $\theta$ -processes in [*J. Appl. Probab.* **47** (2010) 1023–1033].

In this paper we introduce a new family of Lévy processes, which we call Meromorphic Lévy processes, or just  $M$ -processes for short, which overlaps with many of the aforementioned classes. A key feature of the  $M$ -class is the identification of their Wiener–Hopf factors as rational functions of infinite degree written in terms of poles and roots of the Laplace exponent, all of which are real numbers. The specific structure of the  $M$ -class Wiener–Hopf factorization enables us to explicitly handle a comprehensive suite of fluctuation identities that concern first passage problems for finite and infinite intervals for both the process itself as well as the resulting process when it is reflected in its infimum. Such identities are of fundamental interest given their repeated occurrence in various fields of applied probability such as mathematical finance, insurance risk theory and queuing theory.

**1. Introduction.** The theory of Lévy processes forms the cornerstone of an enormous volume of mathematical literature which supports a wide variety of applied and theoretical stochastic models. One of the most obvious and fundamental problems that can be stated for a Lévy process, particularly in relation to its role as a modeling tool, is the distributional characterization of the time at which a Lévy process first exits either an infinite or finite interval together with its overshoot beyond the boundary of the interval. As a family of stochastic processes, Lévy processes are now well understood and the exit problem has seen many different

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approaches dating back to the 1960s. See, for example, [2, 6, 8, 13, 15, 18, 32, 39], to name but a few.

Despite the maturity of this field of study, it is surprising to note that, until very recently, there were less than a handful of examples for which explicit analytical detail concerning the first exit problem could be explored. Given the closeness in mathematical proximity of the first exit problem to the characterization of the Wiener–Hopf factorization, one might argue that the lack of concrete examples of the former was a consequence of the same being true for the latter. The landscape for both the Wiener–Hopf factorization problem and the first exit problem has changed quite rapidly in the last couple of years, however, with the discovery of a number of new, mathematically tractable families of Lévy processes. We mention, in particular, the many cases of spectrally negative Lévy processes in [20, 33], hyper-exponential and generalized hyper-exponential Lévy processes [23], Lamperti-stable processes in [9, 10, 13], Hypergeometric processes in [30, 32],  $\beta$ -processes in [28] and  $\theta$ -processes in [29].

In this paper we introduce a new family of Lévy processes, which we call Meromorphic Lévy processes, or just  $M$ -class for short, that overlaps with many of the aforementioned classes. Our definition of the  $M$ -class of processes will allow us to drive features of their Wiener–Hopf factors through to many of the fluctuation identities which are of pertinence for a wide variety of applications. In the theory of actuarial mathematics, the problem of first exit from a half-line is of fundamental interest with regard to the classical ruin problem and is typically studied within the context of an expected discounted penalty function. The latter is also known as the Gerber–Shiu function following the first article [17] of a long series of papers found within the actuarial literature. In the setting of financial mathematics, the first exit of a Lévy process, as well as a Lévy process reflected in its infimum, from an interval is of interest in the pricing of barrier options and American-type options [1] as well as certain credit risk models [19, 34]. In queueing theory exit problems for Lévy processes play a central role in understanding the trajectory of the workload during busy periods as well as in relation to buffers [16, 26]. Many optimal stopping strategies also turn out to boil down to first passage problems for both Lévy processes and Lévy processes reflected in their infimum, classic examples of which include McKean’s optimal stopping problem [37], as well as the Shepp–Shiryaev optimal stopping problem [2, 4].

It is not our purpose, however, to dwell on these applications. As alluded to above, the main objective will be to expose a comprehensive suite of fluctuation identities in explicit form for the  $M$ -class of Lévy processes. We thus conclude the introduction with an overview of the paper.

In the next section we give a formal definition of meromorphic Lévy processes, dwelling, in particular, on their relationship with discrete completely monotone functions. In Section 3 we consider several classes of existing families of Lévy processes that have appeared in recent literature. Next, in Section 4 we establish

explicit identities for the exponentially discounted first passage problem. In particular, we deal with (what is known in the actuarial literature as) the Gerber–Shiu measure describing the discounted joint triple law of the overshoot, undershoot and undershoot of the maximum at first passage over a level as well as the marginal thereof which specifies the law of the discounted overshoot. It is important to note that the discounting factor which appears in all of the identities means that one is never more than a Fourier transform away from the naturally associated space–time identity in which the additional law of the time to first passage is specified. This last Fourier inversion appears to be virtually impossible to produce analytically within the current context, but the expressions we offer are not difficult to work with in conjunction with straightforward Fourier inversion algorithms.

In Section 5 we look at the more complicated two-sided exit problem. Inspired by a technique of Rogozin [39], we solve a system of equations which characterize the discounted overshoot distribution on either side of the interval in question. The same technique also delivers explicit expressions for the discounted entrance law into an interval. In Section 6 we look in analytical detail at what can be said of the ascending and descending ladder variables. In particular, we offer expressions for their joint Laplace exponent and jump measure. Section 7 mentions some additional examples of fluctuation identities which enjoy explicit detail and, finally, in Section 8 we describe some numerical experiments in order to exhibit the genuine practical applicability of our method.

**2. Meromorphic Lévy processes.** Recall that a general one-dimensional Lévy process is a stochastic process issued from the origin with stationary and independent increments and almost sure right continuous paths. We write  $X = \{X_t : t \geq 0\}$  for its trajectory and  $\mathbb{P}$  for its law. As  $X$  is necessarily a strong Markov process, for each  $x \in \mathbb{R}$ , we will need the probability  $\mathbb{P}_x$  to denote the law of  $X$  when issued from  $x$  with the understanding that  $\mathbb{P}_0 = \mathbb{P}$ . The law  $\mathbb{P}$  of Lévy processes is characterized by its one-time transition probabilities. In particular, there always exist a triple  $(a, \sigma, \Pi)$  where  $a \in \mathbb{R}$ ,  $\sigma \in \mathbb{R}$  and  $\Pi$  is a measure on  $\mathbb{R} \setminus \{0\}$  satisfying the integrability condition  $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty$ , such that, for all  $z \in \mathbb{R}$ ,

$$(1) \quad \mathbb{E}[e^{izX_t}] = e^{t\psi(iz)},$$

where the Laplace exponent  $\psi(z)$  is given by the Lévy–Khintchine formula

$$(2) \quad \psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + \int_{\mathbb{R}} (e^{zx} - 1 - zxh(x)) \Pi(dx).$$

Here  $h(x)$  is the cutoff function which is usually taken to be  $h(x) \equiv \mathbb{I}_{\{|x| < 1\}}$ . However, everywhere in the present paper we will work with the Lévy measures  $\Pi(dx)$  which have exponentially decaying tails, thus, we will take  $h(x) \equiv 1$ . Note that the exponential decay of the tails of the Lévy measure also implies that the Laplace

exponent can be analytically continued into some vertical strip  $a < \text{Re}(z) < b$  for  $a < 0 < b$ .

Everywhere in this paper we will denote the tails of the Lévy measure as  $\bar{\Pi}^+(x) = \Pi((x, \infty))$  and  $\bar{\Pi}^-(x) = \Pi((-\infty, -x))$  for  $x > 0$ . Let us define the supremum/infimum processes  $\bar{X}_t = \sup_{s \leq t} X_s$ ,  $\underline{X}_t = \inf_{s \leq t} X_s$  and denote by  $e(q)$  an independent, exponentially distributed random variable with rate  $q > 0$ . Finally, the first passage time above  $x$  is defined as  $\tau_x^+ = \inf\{t > 0 : X_t > x\}$ , and, similarly,  $\tau_y^- = \inf\{t > 0 : X_t < y\}$ .

Let us recall some basic facts about completely monotone functions. A function  $f : (0, \infty) \mapsto \mathbb{R}$  is called *completely monotone* if  $f \in C^\infty$  and  $(-1)^n f^{(n)}(x) \geq 0$  for all  $n \in \mathbb{N} \cup \{0\}$  and  $x > 0$ . According to Bernstein’s theorem (see Theorem 1.4 in [40]), the function  $f$  is completely monotone if and only if it can be represented as the Laplace transform of a positive measure on  $[0, \infty)$ :

$$(3) \quad f(x) = \int_{[0, \infty)} e^{-zx} \mu(dz), \quad x > 0.$$

Note that if  $f(0^+) = 1$ , then  $\mu(dx)$  is a probability measure and  $1 - f(x)$  is the cumulative distribution function of a positive infinitely divisible random variable  $Z$ , whose distribution is a mixture of exponential distributions. We will denote the class of completely monotone functions as  $\mathcal{CM}$ .

Next, let us introduce a subclass of completely monotone functions, which will be important for us later. We will call  $f : (0, \infty) \mapsto \mathbb{R}$  a *discrete completely monotone* function if the measure  $\mu(dz)$  in the representation (3) is discrete, and the support of the measure  $\mu(dz)$  is infinite and does not have finite accumulation points. This implies that the measure  $\mu(dz)$  is an infinite mixture of atoms of size  $a_n$  at the points  $b_n$ ,

$$\mu(dz) = \sum_{n \geq 1} a_n \delta_{b_n}(dz),$$

where  $\delta_b(dx)$  denotes the Dirac measure at  $x = b$ , for all  $n \geq 1$  we have  $a_n > 0$ ,  $b_n \geq 0$  and  $b_n \rightarrow \infty$  as  $n \rightarrow \infty$  (without loss of generality we can also assume that the sequence  $\{b_n\}_{n \geq 1}$  is strictly increasing). From (3) it follows then that any discrete completely monotone function can be represented as an infinite series of exponential functions

$$(4) \quad f(x) = \sum_{n \geq 1} a_n e^{-b_n x}, \quad x > 0.$$

We will denote the class of discrete completely monotone functions as  $\mathcal{DCM}$ .

**DEFINITION 1 (M-class).** A Lévy process  $X$  is said to belong to the Mero-morphic class (*M-class*) if  $\bar{\Pi}^+(x), \bar{\Pi}^-(x) \in \mathcal{DCM}$ .

We see that, according to our definition of the discrete completely monotone functions, the process is Meromorphic if and only if the Lévy measure  $\Pi(dx)$  has a density with respect to the Lebesgue measure, given by

$$(5) \quad \pi(x) = \mathbb{I}_{\{x>0\}} \sum_{n \geq 1} a_n \rho_n e^{-\rho_n x} + \mathbb{I}_{\{x<0\}} \sum_{n \geq 1} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x},$$

where all the coefficients  $a_n, \hat{a}_n, \rho_n, \hat{\rho}_n$  are positive, the sequences  $\{\rho_n\}_{n \geq 1}$  and  $\{\hat{\rho}_n\}_{n \geq 1}$  are strictly increasing, and  $\rho_n \rightarrow +\infty$  and  $\hat{\rho}_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ .

**PROPOSITION 1.** *Assume that  $\pi(x)$  is given by (5). The integral  $\int_{\mathbb{R}} x^2 \pi(x) dx$  converges if and only if both series  $\sum_{n \geq 1} a_n \rho_n^{-2}$  and  $\sum_{n \geq 1} \hat{a}_n \hat{\rho}_n^{-2}$  converge.*

**PROOF.** The “if” part was established in Proposition 1 in [29]. The “only if” part follows from (5) and the Monotone Convergence Theorem.  $\square$

Recall that a function  $g : \mathbb{C} \mapsto \mathbb{C} \cup \{\infty\}$  is called *meromorphic* if it does not have any other singularities in the open complex plane except for poles. A function  $g(z)$  is called a *real meromorphic function* if it is meromorphic and  $g(z) \in \mathbb{R} \cup \{\infty\}$  for all  $z \in \mathbb{R}$ , or, equivalently, if  $\overline{g(z)} = g(\bar{z})$ .

**THEOREM 1.** *The following conditions are equivalent:*

- (i)  $X$  is Meromorphic.
- (ii)  $\bar{\Pi}^+(x), \bar{\Pi}^-(x) \in \mathcal{CM}$  and the Laplace exponent  $\psi(z)$  is meromorphic.
- (iii) For some  $q > 0$  (and then, for every  $q > 0$ ) the functions  $\mathbb{P}(\bar{X}_{e(q)} > x)$  and  $\mathbb{P}(-\underline{X}_{e(q)} > x)$  restricted to  $x > 0$  belong to the class  $\mathcal{DCM}$ .
- (iv) For some  $q > 0$  (and then, for every  $q > 0$ ) the functions  $\mathbb{P}(X_{e(q)} > x)$  and  $\mathbb{P}(-X_{e(q)} > x)$  restricted to  $x > 0$  belong to the class  $\mathcal{DCM}$ .
- (v) For some  $q > 0$  (and then, for every  $q > 0$ ) we have the factorization

$$(6) \quad q - \psi(z) = q \prod_{n \geq 1} \frac{1 - z/\zeta_n}{1 - z/\rho_n} \prod_{n \geq 1} \frac{1 + z/\hat{\zeta}_n}{1 + z/\hat{\rho}_n}, \quad z \in \mathbb{C},$$

where all roots  $\{\zeta_n, -\hat{\zeta}_n\}$  of  $\psi(z) - q$  are real and interlace with the poles  $\{\rho_n, -\hat{\rho}_n\}$

$$(7) \quad \dots - \hat{\rho}_2 < -\hat{\zeta}_2 < -\hat{\rho}_1 < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \dots$$

(vi) *The Laplace exponent  $\psi(z)$  is a real meromorphic function, which satisfies  $\text{Im}(\psi(z)/z) > 0$  for all  $z$  in the half-plane  $\text{Im}(z) > 0$ .*

**PROOF.** The main ideas and tools needed for the proof of this theorem come from the proofs of Theorem 2 in [38] and Theorem 1 in [29].

(i)  $\Rightarrow$  (ii) Using (5) and (2), we find that

$$(8) \quad \psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + z^2 \sum_{n \geq 1} \frac{a_n}{\rho_n(\rho_n - z)} + z^2 \sum_{n \geq 1} \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)}, \quad z \in \mathbb{C},$$

which shows that  $\psi(z)$  is a meromorphic function. Since  $\mathcal{DCM} \subset \mathcal{CM}$ , this proves (ii).

(ii)  $\Rightarrow$  (vi) We know that  $\psi(z)$  is a meromorphic function, and that  $\psi(0) = 0$ , therefore,  $\psi(z)$  is analytic in some neighborhood of zero. From the proof of Theorem 2 in [38] we find that (a)  $\psi(z)$  can be analytically continued in the half-planes  $\text{Im}(z) \geq 0$  and  $\text{Im}(z) < 0$ , (b)  $\text{Im}(\psi(z)/z) > 0$  for all  $z$  in the half-plane  $\text{Im}(z) > 0$ . Since  $\overline{\psi(z)} = \psi(\bar{z})$ , we conclude that  $\psi(z)$  is a real meromorphic function. Using this fact and the above statement (b), we obtain (vi).

(vi)  $\Rightarrow$  (v),  $\forall q > 0$ . For  $q > 0$  and  $\text{Im}(z) > 0$  it is true that  $\text{Im}(-q/z) > 0$ , thus, (vi) implies that  $\text{Im}((\psi(z) - q)/z) > 0$  for all  $z$  in the half-plane  $\text{Im}(z) > 0$ . Using Theorem 1 on page 220 in [35] (the statement of this result can also be found in the proof of Theorem 1 in [29]), we find that (v) is valid for all  $q > 0$ .

(v)  $\Rightarrow$  (iii) This follows from Theorem 1 in [29].

(iii)  $\Rightarrow$  (iv) One can also check that if two functions  $f(x)$  and  $g(-x)$  belong to the class  $\mathcal{DCM}$ , then the same is true for  $\mathbb{I}_{\{x>0\}}(f * g)(x)$  and  $\mathbb{I}_{\{x<0\}}(f * g)(x)$ , where  $f * g$  is the convolution. The result (iv) then follows easily from this fact and the Wiener–Hopf decomposition which says that  $X_{e(q)} \stackrel{d}{=} Y_1 + Y_2$ , where the random variables  $Y_1$  and  $Y_2$  are independent,  $Y_1 \stackrel{d}{=} \overline{X}_{e(q)}$  and  $Y_2 \stackrel{d}{=} \underline{X}_{e(q)}$ .

(iv)  $\Rightarrow$  (i) From (iv) and the definition of the  $\mathcal{DCM}$  class we know that for some  $q > 0$  there exist  $\alpha_0 \geq 0$  and positive constants  $\alpha_n, \beta_n, \hat{\alpha}_n, \hat{\beta}_n$  such that

$$(9) \quad \begin{aligned} &\mathbb{P}(X_{e(q)} \in dx) \\ &= \alpha_0 \delta_0(dx) + \left[ \mathbb{I}_{\{x>0\}} \sum_{n \geq 1} \alpha_n \beta_n^{-1} e^{-\beta_n x} + \mathbb{I}_{\{x<0\}} \sum_{n \geq 1} \hat{\alpha}_n \hat{\beta}_n^{-1} e^{\hat{\beta}_n x} \right] dx. \end{aligned}$$

Note that the condition  $\mathbb{P}(X_{e(q)} \in \mathbb{R}) = 1$  implies that both series  $\sum_{n \geq 1} \alpha_n \beta_n^{-2}$  and  $\sum_{n \geq 1} \hat{\alpha}_n \hat{\beta}_n^{-2}$  converge. Let us define  $\tilde{\psi}(z) = z^2 \mathbb{E}[\exp(zX_{e(q)})]$ . Using (9), we obtain

$$\tilde{\psi}(z) = \alpha_0 z^2 + z^2 \sum_{n \geq 1} \frac{\alpha_n}{\beta_n(\beta_n - z)} + z^2 \sum_{n \geq 1} \frac{\hat{\alpha}_n}{\hat{\beta}_n(\hat{\beta}_n + z)}, \quad z \in \mathbb{C}.$$

Comparing the above formula with (8), we conclude that  $\tilde{\psi}(z)$  is a Laplace exponent of a Meromorphic Lévy process. We have already proved that (i) implies (vi), therefore,  $\text{Im}(\tilde{\psi}(z)/z) > 0$  for all  $z$  in the half-plane  $\text{Im}(z) > 0$ .

Next, using the definition of the Laplace exponent (1), it is easy to verify that  $\mathbb{E}[\exp(zX_{e(q)})] = q/(q - \psi(z))$ . Therefore,

$$\frac{\psi(z) - q}{z} = -\frac{qz}{\tilde{\psi}(z)}.$$

As we have already established,  $\text{Im}(\tilde{\psi}(z)/z) > 0$  for all  $z$  in the half-plane  $\text{Im}(z) > 0$ . Using this fact and the above identity, we conclude that  $\text{Im}((\psi(z) - q)/z) > 0$ . Applying Theorem 1 on page 197 in [12] (the statement of this result can also be found in the proof of Theorem 1 in [29]), we find that  $\psi(z)$  admits a representation of the form (8), which in turn implies that the process  $X$  is meromorphic.

Finally, note that if (v) is true for some  $q > 0$ , then (iii) and (iv) are valid for the same value of  $q > 0$ . But as we have already demonstrated, any of the conditions (iii), (iv), (v) implies (i) which is equivalent to (v) being valid for all  $q > 0$ . Thus, if one of the conditions (iii), (iv), (v) is valid for some  $q > 0$ , then it must be valid for all  $q > 0$ .  $\square$

Statement (ii) in Theorem 1 shows that the  $M$ -class of Lévy processes might also be called “processes with completely-monotone Lévy measure and meromorphic Laplace exponent,” which explains the origin of the name “Meromorphic Lévy processes.” Note, however, that there exist Lévy processes with meromorphic Laplace exponent but not completely monotone Lévy measure.

The following technical result on partial fraction decomposition of infinite products will be very important for us later.

LEMMA 1. *Assume that we have two increasing sequences  $\rho = \{\rho_n\}_{n \geq 1}$  and  $\zeta = \{\zeta_n\}_{n \geq 1}$  of positive numbers, such that  $\rho_n \rightarrow +\infty$  as  $n \rightarrow +\infty$  and the following interlacing condition is satisfied:*

$$(10) \quad \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \dots$$

Define

$$(11) \quad \phi(z) = \prod_{n \geq 1} \frac{1 + z/\rho_n}{1 + z/\zeta_n}, \quad z > 0.$$

Then for all  $z > 0$ ,

$$(12) \quad \phi(z) = a_0(\rho, \zeta) + \sum_{n \geq 1} a_n(\rho, \zeta) \frac{\zeta_n}{\zeta_n + z},$$

$$(13) \quad \frac{1}{\phi(z)} = 1 + z b_0(\zeta, \rho) + \sum_{n \geq 1} b_n(\zeta, \rho) \left[ 1 - \frac{\rho_n}{\rho_n + z} \right],$$

where

$$(14) \quad a_0(\rho, \zeta) = \lim_{n \rightarrow +\infty} \prod_{k=1}^n \frac{\zeta_k}{\rho_k}, \quad a_n(\rho, \zeta) = \left( 1 - \frac{\zeta_n}{\rho_n} \right) \prod_{\substack{k \geq 1 \\ k \neq n}} \frac{1 - \zeta_n/\rho_k}{1 - \zeta_n/\zeta_k},$$

$$(15) \quad b_0(\zeta, \rho) = \frac{1}{\zeta_1} \lim_{n \rightarrow +\infty} \prod_{k=1}^n \frac{\rho_k}{\zeta_{k+1}}, \quad b_n(\zeta, \rho) = - \left( 1 - \frac{\rho_n}{\zeta_n} \right) \prod_{\substack{k \geq 1 \\ k \neq n}} \frac{1 - \rho_n/\zeta_k}{1 - \rho_n/\rho_k}.$$

Moreover,  $a_0(\rho, \zeta) \geq 0$ ,  $b_0(\zeta, \rho) \geq 0$  and for all  $n \geq 1$  we have  $a_n(\rho, \zeta) > 0$ ,  $b_n(\zeta, \rho) > 0$ .

PROOF. The convergence of the infinite product in (11) and the partial fraction decomposition (12) follow from Theorem 1 in [29]. To prove the second partial fraction decomposition, rewrite the infinite product as

$$\left(1 + \frac{z}{\zeta_1}\right) \prod_{n \geq 1} \frac{1 + z/\zeta_{n+1}}{1 + z/\rho_n},$$

and note that sequences  $\{\zeta_{n+1}\}_{n \geq 1}$  and  $\{\rho_n\}_{n \geq 1}$  satisfy the interlacing condition, thus, we can apply the same method. The details are left to the reader.  $\square$

Using the Monotone Convergence Theorem, one can show that formulas (12) and (13) are equivalent to

$$(16) \quad \phi(z) = a_0(\rho, \zeta) + \int_{\mathbb{R}^+} \left[ \sum_{n \geq 1} a_n(\rho, \zeta) \zeta_n e^{-\zeta_n x} \right] e^{-zx} dx,$$

$$(17) \quad \frac{1}{\phi(z)} = 1 + zb_0(\zeta, \rho) + \int_{\mathbb{R}^+} \left[ \sum_{n \geq 1} b_n(\zeta, \rho) \rho_n e^{-\rho_n x} \right] (1 - e^{-zx}) dx.$$

Before stating our next result, we recall that a Lévy process creeps upward if for some (and then all)  $x \geq 0$ ,  $\mathbb{P}(X_{\tau_x^+} = x | \tau_x^+ < \infty) > 0$ . Moreover, we say that 0 is irregular for  $(0, \infty)$  if and only if  $\mathbb{P}(\tau_0^+ > 0) = 1$ . (Note that this probability can only be 0 or 1 thanks to the Blumenthal zero–one law.) We refer to Bertoin [5], Doney [14] or Kyprianou [31] for more extensive discussion of these subtle path properties.

We also need to introduce some more notation. In the forthcoming text everything will depend on the coefficients  $\{a_n(\rho, \zeta), a_n(\hat{\rho}, \hat{\zeta})\}_{n \geq 0}$  defined using (14) and  $\{b_n(\zeta, \rho), b_n(\hat{\zeta}, \hat{\rho})\}_{n \geq 0}$  defined using (15). We define for convenience a column vector

$$\bar{a}(\rho, \zeta) = [a_0(\rho, \zeta), a_1(\rho, \zeta), a_2(\rho, \zeta), \dots]^T,$$

and similarly for  $\bar{a}(\hat{\rho}, \hat{\zeta})$ ,  $\bar{b}(\zeta, \rho)$  and  $\bar{b}(\hat{\zeta}, \hat{\rho})$ . Next, given a sequence of positive numbers  $\zeta = \{\zeta_n\}_{n \geq 1}$ , we define a column vector  $\bar{v}(\zeta, x)$  as a vector of distributions

$$\bar{v}(\zeta, x) = [\delta_0(x), \zeta_1 e^{-\zeta_1 x}, \zeta_2 e^{-\zeta_2 x}, \dots]^T,$$

where  $\delta_0(x)$  is the Dirac delta function at  $x = 0$ .

From here on, unless otherwise stated, we shall always assume that  $X$  is a Lévy process belonging to the  $M$ -class and that  $X$  is not a compound Poisson process.

THEOREM 2 (Properties of Meromorphic processes). (i) *The Wiener–Hopf factors are given by*

$$\begin{aligned}
 \phi_q^+(iz) &= \mathbb{E}[e^{-z\bar{X}_{e(q)}}] = \prod_{n \geq 1} \frac{1 + z/\rho_n}{1 + z/\zeta_n}, \\
 \phi_q^-(iz) &= \mathbb{E}[e^{z\underline{X}_{e(q)}}] = \prod_{n \geq 1} \frac{1 + z/\hat{\rho}_n}{1 + z/\hat{\zeta}_n}.
 \end{aligned}
 \tag{18}$$

(ii) *For  $x \geq 0$ ,*

$$\begin{aligned}
 \mathbb{P}(\bar{X}_{e(q)} \in dx) &= \bar{a}(\rho, \zeta)^T \times \bar{v}(\zeta, x) dx, \\
 \mathbb{P}(-\underline{X}_{e(q)} \in dx) &= \bar{a}(\hat{\rho}, \hat{\zeta})^T \times \bar{v}(\hat{\zeta}, x) dx.
 \end{aligned}
 \tag{19}$$

(iii)  $a_0(\rho, \zeta)$  ( $a_0(\hat{\rho}, \hat{\zeta})$ ) *is nonzero if and only if 0 is irregular for  $(0, \infty)$  [correspondingly,  $(-\infty, 0)$ ].*

(iv)  $b_0(\zeta, \rho)$  ( $b_0(\hat{\zeta}, \hat{\rho})$ ) *is nonzero if and only if the process  $X$  creeps upward [correspondingly, downward].*

(v) *For every  $q > 0$ ,*

$$\mathbb{P}(X_{e(q)} \in dx) = q \left[ \mathbb{I}_{\{x > 0\}} \sum_{n \geq 1} \frac{e^{-\zeta_n x}}{\psi'(\zeta_n)} - \mathbb{I}_{\{x < 0\}} \sum_{n \geq 1} \frac{e^{\hat{\zeta}_n x}}{\psi'(-\hat{\zeta}_n)} \right] dx.
 \tag{20}$$

PROOF. Parts (i) and (ii) were established in Theorem 1 in [29]; these results also follow easily from formulas (6), (16) and the structure of the Wiener–Hopf factorization. Let us prove (iii). We note that 0 is irregular for  $(0, \infty)$  if and only if, for any  $q > 0$ ,  $\bar{X}_{e(q)}$  has an atom in its distribution at 0. From part (ii) this is clearly the case if and only if  $a_0(\rho, \zeta)$  is nonzero. For part (iv) of the proof, we note that we may necessarily write

$$\phi_q^+(iz) = \frac{\kappa(q, 0)}{\kappa(q, z)},$$

where  $\kappa(q, z)$  is the Laplace exponent of the bivariate subordinator which describes the ascending ladder process of  $X$ . It is also known that a Lévy process creeps upward if and only if the subordinator describing its ladder height process has a linear drift component. The drift coefficient is then described by  $\lim_{z \uparrow \infty} \kappa(q, z)/z \in [0, \infty)$ , where the limit is independent of the value of  $q \geq 0$ . Inspecting (17), we see that the required drift coefficient is nonzero if and only if  $b_0(\zeta, \rho)$  is nonzero. The conclusion thus follows.

Finally, let us prove (v). From part (iv) of Theorem 1 we know that  $\mathbb{P}(X_{e(q)} \in dx)$  can be written in the form (9), where  $\alpha_0 = 0$  due to our assumption

that  $X$  is not a compound Poisson process. Combining this fact and the identity  $\mathbb{E}[\exp(zX_{e(q)})] = q/(q - \psi(z))$ , we conclude that

$$(21) \quad \frac{q}{q - \psi(z)} = \sum_{n \geq 1} \frac{\alpha_n}{\beta_n(\beta_n - z)} + \sum_{n \geq 1} \frac{\hat{\alpha}_n}{\hat{\beta}_n(\hat{\beta}_n + z)}, \quad z \in \mathbb{C}.$$

We see that the function on the right-hand side of the above equation has poles at the points  $\beta_n$  and  $-\hat{\beta}_n$ , while the left-hand side has poles at  $\zeta_n$  and  $-\hat{\zeta}_n$  [this follows from (6)]. Therefore, we conclude that  $\beta_n = \zeta_n$  and  $\hat{\beta}_n = \hat{\zeta}_n$ . Comparing the residues of both sides of (21) at the pole  $z = \zeta_n$ , we see that

$$\frac{\alpha_n}{\beta_n} = -\operatorname{Res}\left(\frac{q}{q - \psi(z)} : z = \zeta_n\right) = \frac{q}{\psi'(\zeta_n)}.$$

Similarly, we find

$$\frac{\hat{\alpha}_n}{\hat{\beta}_n} = \operatorname{Res}\left(\frac{q}{q - \psi(z)} : z = -\hat{\zeta}_n\right) = -\frac{q}{\psi'(-\hat{\zeta}_n)}.$$

Combining the above two identities and (9), we obtain (20).  $\square$

We would like to emphasize the importance of statement (v) of Theorem 2. It is well known that there exist only a few specific examples of Lévy processes (such as Variance Gamma or Normal Inverse Gaussian processes) for which the law of  $X_t$  is known explicitly (for every  $t > 0$ ). While we do not know the law of  $X_t$  for Meromorphic processes, the result in Theorem 2(v) shows that Meromorphic processes have an advantage that at least the distribution of  $X_{e(q)}$  can be easily computed (for every  $q > 0$ ). The formula is given in terms of the roots  $\zeta_n$  and  $\hat{\zeta}_n$ , but as we will see in Section 8, computing these numbers is a rather simple task and it can be done very efficiently.

As the next corollary shows, the Lévy measure of a Meromorphic process  $X$  can be easily reconstructed from zeros and poles of  $q - \psi(z)$ . This has the spirit of Vigon’s theory of philanthropy (cf. [41]) in that we construct the Lévy measure from the Wiener–Hopf factors.

**COROLLARY 1.** *Assume that  $q > 0$  and the Wiener–Hopf factors are given by (18). Then  $X$  is Meromorphic and its Lévy measure is given by (5), where  $\{\rho_n\}_{n \geq 1}$  and  $\{\hat{\rho}_n\}_{n \geq 1}$  are poles of the Wiener–Hopf factors  $\phi_q^+(-iz)$  and  $\phi_q^-(iz)$  and coefficients  $a_n$  and  $\hat{a}_n$  are given by*

$$(22) \quad a_n = b_n(\zeta, \rho) \frac{q}{\phi_q^-(-i\rho_n)}, \quad \hat{a}_n = b_n(\hat{\zeta}, \hat{\rho}) \frac{q}{\phi_q^+(i\hat{\rho}_n)}.$$

**PROOF.** The fact that  $X$  is Meromorphic was already established in part (iii) of Theorem 1. Assume that the density of the Lévy measure  $\pi(x)$  is given by (5), then as we have established in the proof of Theorem 1, the Laplace exponent  $\psi(z)$

can be expressed in the form (8). Comparing this equation with (6), we conclude that the exponents  $\rho_n$  and  $\hat{\rho}_n$  in (5) must coincide with the poles of  $\psi(z)$ .

From (8) it also follows that  $a_n \rho_n = -\text{Res}(\psi(z) : z = \rho_n)$ . Using the Wiener–Hopf factorization

$$(23) \quad q - \psi(z) = \frac{q}{\phi_q^+(-iz)\phi_q^-(-iz)},$$

and the fact that  $\phi_q^-(-iz)$  is nonzero at  $\rho_n$ , we see that

$$a_n \rho_n = \text{Res}(\phi_q^+(-iz)^{-1} : z = \rho_n) \times \frac{q}{\phi_q^-(-i\rho_n)}.$$

In order to finish the proof, we only need to check that  $\text{Res}(\phi_q^+(-iz)^{-1} : z = \rho_n) = \rho_n b_n(\zeta, \rho)$ , which follows from (13).  $\square$

Note that the coefficients  $a_n$  and  $\hat{a}_n$  given in (22) [which define the Lévy measure via (5)] depend on  $q$  on the right-hand side but not on the left-hand side. It is therefore tempting to take limits as  $q \downarrow 0$  on the right-hand side. This is not as straightforward as it seems, as the limits of  $a_n$  and  $\hat{a}_n$  are not easy to compute. It would be more straightforward to start with the Wiener–Hopf factorization for the case that  $q = 0$  and then perform the same analysis as in Corollary 1. The next corollary provides us with the aforementioned Wiener–Hopf factorization. We recall that  $\zeta_n = \zeta_n(q)$  [resp.  $\hat{\zeta}_n = \hat{\zeta}_n(q)$ ] denote the nonnegative solutions to equation  $\psi(z) = q$  (resp.  $\psi(-z) = q$ ).

COROLLARY 2.

(i) Assume that  $\mathbb{E}[X_1] > 0$ . As  $q \rightarrow 0^+$  we have  $q/\zeta_1(q) \rightarrow \mathbb{E}[X_1]$  and for  $n \geq 1$ ,

$$\zeta_{n+1}(q) \rightarrow \zeta_{n+1}(0) \neq 0, \quad \hat{\zeta}_n(q) \rightarrow \hat{\zeta}_n(0) \neq 0.$$

We have the Wiener–Hopf factorization  $-\psi(z) = \kappa(0, -z)\hat{\kappa}(0, z)$ , where

$$(24) \quad \begin{aligned} \kappa(0, z) &= \lim_{q \rightarrow 0^+} \frac{q}{\phi_q^+(iz)} = z\mathbb{E}[X_1] \prod_{n \geq 1} \frac{1 + z/\zeta_{n+1}(0)}{1 + z/\rho_n}, \\ \hat{\kappa}(0, z) &= \lim_{q \rightarrow 0^+} \frac{1}{\phi_q^-(-iz)} = \frac{1}{\phi_0^-(-iz)} = \prod_{n \geq 1} \frac{1 + z/\hat{\zeta}_n(0)}{1 + z/\hat{\rho}_n}. \end{aligned}$$

(ii) Assume that  $\mathbb{E}[X_1] = 0$ . As  $q \rightarrow 0^+$  we have  $\sqrt{q}/\zeta_1(q) \rightarrow \sqrt{\text{Var}(X_1)}/2$ ,  $\sqrt{q}/\hat{\zeta}_1(q) \rightarrow \sqrt{\text{Var}(X_1)}/2$  and for  $n \geq 2$ ,

$$\zeta_n(q) \rightarrow \zeta_n(0) \neq 0, \quad \hat{\zeta}_n(q) \rightarrow \hat{\zeta}_n(0) \neq 0.$$

We have the Wiener–Hopf factorization  $-\psi(z) = \kappa(0, -z)\hat{\kappa}(0, z)$ , where

$$\begin{aligned}
 \kappa(0, z) &= \lim_{q \rightarrow 0^+} \frac{\sqrt{q}}{\phi_q^+(iz)} = z \sqrt{\frac{\text{Var}(X_1)}{2}} \prod_{n \geq 1} \frac{1 + z/\zeta_{n+1}(0)}{1 + z/\rho_n}, \\
 \hat{\kappa}(0, z) &= \lim_{q \rightarrow 0^+} \frac{\sqrt{q}}{\phi_q^-(-iz)} = z \sqrt{\frac{\text{Var}(X_1)}{2}} \prod_{n \geq 1} \frac{1 + z/\hat{\zeta}_{n+1}(0)}{1 + z/\hat{\rho}_n}.
 \end{aligned}
 \tag{25}$$

PROOF. Let us prove part (i). Since  $\mathbb{E}[X_1] > 0$ , we know that  $\bar{X}_{e(q)} \rightarrow +\infty$  and  $\underline{X}_{e(q)} \rightarrow \underline{X}_\infty$  as  $q \rightarrow 0^+$ , and also that  $\underline{X}_\infty < \infty$  with probability one. Therefore, if  $z > 0$ , the Wiener–Hopf factor  $\phi_q^+(iz) = \mathbb{E}[\exp(-z\bar{X}_{e(q)})]$  must converge to 0 as  $q \rightarrow 0^+$  while  $\phi_q^-(-iz) = \mathbb{E}[\exp(z\underline{X}_{e(q)})]$  must converge to  $\mathbb{E}[\exp(z\underline{X}_\infty)]$  as  $q \rightarrow 0^+$ . Since all roots  $\zeta_n(q)$  and  $\hat{\zeta}_n(q)$  have nonzero limits as  $q \rightarrow 0^+$  if  $n \geq 2$  [this is true due to the interlacing condition (7) and the fact that  $\psi(z)$  is a meromorphic function], we conclude that  $\zeta_1(q)$  must go to zero while  $\hat{\zeta}_1(q) \rightarrow \hat{\zeta}_1(0) \neq 0$ . The function  $\psi(z)$  is analytic in the neighborhood of  $z = 0$ , and its derivative  $\psi'(0) = \mathbb{E}[X_1] \neq 0$ , thus, using the Implicit Function Theorem, we conclude that  $\zeta_1(q)$  is an analytic function of  $q$  in some neighborhood of  $q = 0$ , and taking derivative with respect to  $q$  of the equation  $\psi(z) = q$ , we find that

$$\frac{d}{dq} \zeta_1(q) = \frac{1}{\psi'(\zeta_1(q))}.$$

Since  $\psi'(0) = \mathbb{E}[X_1]$ , we conclude that  $q/\zeta_1(q) \rightarrow \mathbb{E}[X_1]$  as  $q \rightarrow 0^+$ . Using this fact and the formulae in (18) for the Wiener–Hopf factors, we obtain expressions (24).

The proof of part (ii) is very similar. We use the fact that  $\psi'(0) = \mathbb{E}[X_1] = 0$  and  $\psi''(0) = \text{Var}(X_1)$  to conclude that  $\psi(z) = \frac{1}{2} \text{Var}(X_1)z^2 + O(z^3)$  as  $z \rightarrow 0$ . This implies that when  $q$  is small and positive, the equation  $\psi(z) = q$  has two solutions in the neighborhood of zero,  $\zeta_1 = \sqrt{2q/\text{Var}(X_1)} + o(\sqrt{q})$  and  $-\hat{\zeta}_1 = -\sqrt{2q/\text{Var}(X_1)} + o(\sqrt{q})$ . Using this information, the uniqueness of the Wiener–Hopf factorization and taking the limit as  $q \rightarrow 0^+$  in (6), we obtain (25).  $\square$

**3. Examples of Meromorphic processes.** There are four particular families of Lévy processes which have featured in the recent literature, all of which have relevance to a number of applied probability models and all of which have exhibited some degree of mathematical tractability in the context that they have occurred. These processes are as follows:

*Hyper-exponential Lévy processes:* The elementary but classical Kou model [27] consists of a linear Brownian motion plus a compound Poisson process with two-sided exponentially distributed jumps. The natural generalization of this model (which therefore includes the Kou model) is the hyper-exponential Lévy process

for which the exponentially distributed jumps are replaced by hyper-exponentially distributed jumps. That is to say, the density of the Lévy measure is written

$$\pi(x) = \mathbb{I}_{\{x>0\}} \sum_{i=1}^N a_i \rho_i e^{-\rho_i x} + \mathbb{I}_{\{x<0\}} \sum_{i=1}^{\hat{N}} \hat{a}_i \hat{\rho}_i e^{\hat{\rho}_i x},$$

where  $a_i, \hat{a}_i, \rho_i$  and  $\hat{\rho}_i$  are positive numbers and  $N, \hat{N}$  are positive integers. One can verify that the Laplace exponent is a rational function of the form (8), where we have finite sums instead of infinite series, and that hyper-exponential Lévy processes have finite activity jumps and paths of bounded variation unless  $\sigma^2 > 0$ .

Strictly speaking, the Hyper-exponential Lévy processes do not belong to the class of Meromorphic processes, as in this case the Laplace exponent  $\psi(z)$  has only a finite number of poles  $\rho_n, -\hat{\rho}_n$ . However, all the results presented in this paper are still correct when we have a process with positive and/or negative hyper-exponential jumps, provided that the results are interpreted in the correct way; see Remark 1 below.

*$\beta$ -class Lévy processes:* The  $\beta$ -class of Lévy processes was introduced by Kuznetsov [28]. The Laplace exponent is given by

$$\begin{aligned} \psi(z) = & \frac{1}{2}\sigma^2 z^2 + az + \frac{c_1}{\beta_1} \left\{ B\left(\alpha_1 - \frac{z}{\beta_1}, 1 - \lambda_1\right) - B(\alpha_1, 1 - \lambda_1) \right\} \\ & + \frac{c_2}{\beta_2} \left\{ B\left(\alpha_2 + \frac{z}{\beta_2}, 1 - \lambda_2\right) - B(\alpha_2, 1 - \lambda_2) \right\}, \end{aligned}$$

where  $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$  is the Beta function, with parameter range  $a \in \mathbb{R}, \sigma \geq 0, c_i, \alpha_i, \beta_i > 0$  and  $\lambda_i \in (0, 3) \setminus \{1, 2\}$ . The corresponding Lévy measure  $\Pi(dx)$  has density

$$\pi(x) = \mathbb{I}_{\{x>0\}} c_1 \frac{e^{-\alpha_1 \beta_1 x}}{(1 - e^{-\beta_1 x})^{\lambda_1}} + \mathbb{I}_{\{x<0\}} c_2 \frac{e^{\alpha_2 \beta_2 x}}{(1 - e^{\beta_2 x})^{\lambda_2}}.$$

With the help of binomial series, one can check that  $\pi(x)$  is of the form (5), thus,  $\beta$ -processes are Meromorphic.

The large number of parameters allows one to choose Lévy processes within the  $\beta$ -class that have paths that are both of unbounded variation [when at least one of the conditions  $\sigma \neq 0, \lambda_1 \in (2, 3)$  or  $\lambda_2 \in (2, 3)$  holds] and bounded variation [when all of the conditions  $\sigma = 0, \lambda_1 \in (0, 2)$  and  $\lambda_2 \in (0, 2)$  hold] as well as having infinite and finite activity in the jump component [accordingly as both  $\lambda_1, \lambda_2 \in (1, 3)$  or  $\lambda_1, \lambda_2 \notin (1, 3)$ ]. The  $\beta$ -class of Lévy processes includes another recently introduced family of Lévy processes known as Lamperti-stable processes; cf. [9, 10, 13].

*$\theta$ -class Lévy processes:* The  $\theta$ -class of Lévy processes was also introduced by Kuznetsov [29] as a family of processes with Gaussian component and two-sided jumps, characterized by the density of the Lévy measure

$$\pi(x) = \mathbb{I}_{\{x>0\}} c_1 \beta_1 e^{-\alpha_1 x} \Theta_k(x\beta_1) + \mathbb{I}_{\{x<0\}} c_2 \beta_2 e^{\alpha_2 x} \Theta_k(-x\beta_2),$$

where  $\Theta_k(x)$  is the  $k$ th order (fractional) derivative of the theta function  $\theta_3(0, e^{-x})$ :

$$\Theta_k(x) = \frac{d^k}{dx^k} \theta_3(0, e^{-x}) = \mathbb{I}_{\{k=0\}} + 2 \sum_{n \geq 1} n^{2k} e^{-n^2 x}, \quad x > 0.$$

The parameter  $\chi = k + 1/2$  corresponds to the exponent of the singularity of  $\pi(x)$  at  $x = 0$ , and when  $\chi \in \{1/2, 3/2, 5/2\}$  ( $\chi \in \{1, 2\}$ ) the Laplace exponent of  $X$  is given in terms of trigonometric (digamma) functions. Again, due to the fact that the density of the Lévy measure is of the form (5), we conclude that  $\theta$ -processes are Meromorphic.

*Hypergeometric Lévy processes:* These processes were introduced in Kyprianou et al. [30, 32] as an example of how to use Vigon’s theory of philanthropy. Their Laplace exponent is given by

$$(26) \quad \psi(z) = \frac{1}{2} \sigma^2 z^2 + az + \Phi_1(-z) \Phi_2(z),$$

where  $a, \sigma \in \mathbb{R}$  and  $\Phi_1, \Phi_2$  are the Laplace exponents of two subordinators from the  $\beta$ -class of Lévy processes. Note that such Laplace exponents necessarily take the form

$$(27) \quad \Phi(\theta) = k + \delta\theta + \frac{c}{\beta} \{B(1 - \alpha + \gamma, -\gamma) - B(1 - \alpha + \gamma + \theta/\beta, -\gamma)\}.$$

Compared with the previous examples, it is less obvious that hypergeometric processes are Meromorphic, however, it is not hard to establish this result with the help of Theorem 1. We know that  $\beta$ -processes are meromorphic, thus, the process  $\tilde{X}$  with the Laplace exponent  $\tilde{\psi}(z) = \Phi_1(-z) \Phi_2(z)$  is Meromorphic [this is due to part (v) of Theorem 1]. Finally, the hypergeometric process defined by the Laplace exponent via (26) and the process  $\tilde{X}$  have the same Lévy measure, therefore,  $X$  is also Meromorphic.

REMARK 1. Within the scope of Definition 1, when the process  $X_t$  has hyper-exponential positive jumps (a mixture of  $N$  exponentials), we have only a finite sequence  $\{\rho_n\}_{n=1,2,\dots,N}$  and either  $N$  (or  $N + 1$ ) negative roots  $\zeta_N$ . All the formulas presented in this paper are still valid in this case if we adopt notation  $\rho_k = \infty$  for  $k > N$  and  $\zeta_k = \infty$  for  $k > N$  (or  $k > N + 1$ ). For example, expression (18) for the Wiener–Hopf factor becomes a finite product

$$\phi_q^+(iz) = \prod_{n=1}^N \frac{1 + z/\rho_n}{1 + z/\zeta_n} \quad \text{or} \quad \phi_q^+(iz) = \left[ \prod_{n=1}^N \frac{1 + z/\rho_n}{1 + z/\zeta_n} \right] \frac{1}{1 + z/\zeta_{N+1}},$$

depending on whether we have  $N$  or  $N + 1$  negative roots  $\zeta_n$ . The same remark holds true when we have hyper-exponential negative jumps.

REMARK 2. When the process  $X$  has hyper-exponential positive jumps, the calculation of coefficients  $a(\rho, \zeta)$  and  $b(\rho, \zeta)$  is much easier, since the corre-

sponding expressions in (14) and (15) are just finite products. Moreover, calculation of the coefficients  $a(\hat{\rho}, \hat{\zeta})$  and  $b(\hat{\rho}, \hat{\zeta})$  is also simplified considerably even if the sequence  $\{\hat{\rho}_n\}_{n \geq 1}$  is infinite, since we can use Wiener–Hopf factorization (23) and the fact that  $-\hat{\zeta}_n$  and  $-\hat{\rho}_n$  are simple roots (poles) of function  $\psi(z) - q$ . For example, we can compute coefficients  $a_n(\hat{\rho}, \hat{\zeta})$  as follows:

$$\begin{aligned} a_n(\hat{\rho}, \hat{\zeta}) &= \frac{1}{\hat{\zeta}_n} \operatorname{Res}(\phi_q^-( -iz) : z = -\hat{\zeta}_n) \\ &= \frac{1}{\hat{\zeta}_n} \operatorname{Res}\left(\frac{q}{(q - \psi(z))\phi_q^+(-iz)} : z = -\hat{\zeta}_n\right) = -\frac{q}{\hat{\zeta}_n \psi'(-\hat{\zeta}_n)\phi_q^+(i\hat{\zeta}_n)}, \end{aligned}$$

and this last expression is more convenient than (14), since usually we know  $\psi'(z)$  explicitly and  $\phi_q^+(z)$  is just a rational function.

**4. One-sided exit problem.** Doney and Kyprianou [15] have given a detailed characterization of the one-sided exit problem through the so-called quintuple law. The latter can be easily integrated out to give the following discounted triple law which is also known in the actuarial mathematics literature as the Gerber–Shiu measure (cf. [7]).

LEMMA 2. Fix  $c > 0$ . Define

$$\tau_c^+ = \inf\{t > 0 : X_t > c\}.$$

For all  $q, y, z > 0$  and  $u \in [0, c \vee z]$ , we have

$$\begin{aligned} &\mathbb{E}[e^{-q\tau_c^+} \mathbb{I}(X_{\tau_c^+} - c \in dy; c - X_{\tau_c^+ -} \in dz; c - \bar{X}_{\tau_c^+ -} \in du)] \\ &= \frac{1}{q} \mathbb{P}(\bar{X}_{e(q)} \in c - du) \mathbb{P}(-\underline{X}_{e(q)} \in dz - u) \Pi(dy + z). \end{aligned}$$

Taking account of formulas (19), this gives us the following immediate corollary for the  $M$ -class of Lévy processes.

COROLLARY 3. Fix  $c > 0$ . For all  $q, y, z > 0$  and  $u \in [0, c \vee z]$ , we have

$$\begin{aligned} &\mathbb{E}[e^{-q\tau_c^+} \mathbb{I}(X_{\tau_c^+} - c \in dy; c - X_{\tau_c^+ -} \in dz; c - \bar{X}_{\tau_c^+ -} \in du)] \\ &= \frac{1}{q} [\bar{a}(\rho, \zeta)^T \times \bar{v}(\zeta, c - u)] [\bar{a}(\hat{\rho}, \hat{\zeta})^T \times \bar{v}(\hat{\zeta}, z - u)] \Pi(dy + z) dz du. \end{aligned}$$

Often, one is only interested in the discounted overshoot distribution. In principle, this can be obtained from the above formula by integrating out the variables  $z$  and  $u$ . However, it turns out to be more straightforward to prove this directly, particularly as otherwise it would require us to specify in more detail the Lévy

measure in terms of poles and roots for the  $M$ -class (cf. Corollary 1). The following result does precisely this, but, in addition, it also gives us the probability of creeping.

**THEOREM 3.** *Define a matrix  $\mathbf{A} = \{a_{i,j}\}_{i,j \geq 0}$  as*

$$(28) \quad a_{i,j} = \begin{cases} 0, & \text{if } i = 0, j \geq 0, \\ a_i(\rho, \zeta)b_0(\zeta, \rho), & \text{if } i \geq 1, j = 0, \\ \frac{a_i(\rho, \zeta)b_j(\zeta, \rho)}{\rho_j - \zeta_i}, & \text{if } i \geq 1, j \geq 1. \end{cases}$$

Then for  $c > 0$  and  $y \geq 0$ , we have

$$(29) \quad \mathbb{E}[e^{-q\tau_c^+} \mathbb{I}(X_{\tau_c^+} - c \in dy)] = \bar{v}(\zeta, c)^T \times \mathbf{A} \times \bar{v}(\rho, y) dy.$$

**PROOF.** We start with the formula from Lemma 1 in [1],

$$(30) \quad \mathbb{E}[e^{-q\tau_c^+ - z(X_{\tau_c^+} - c)}] = \frac{\mathbb{E}[e^{-z(\bar{X}_{e(q)} - c)} \mathbb{I}(\bar{X}_{e(q)} > c)]}{\mathbb{E}[e^{-z\bar{X}_{e(q)}}]}.$$

Using formula (19), we find that

$$\mathbb{E}[e^{-z(\bar{X}_{e(q)} - c)} \mathbb{I}(\bar{X}_{e(q)} > c)] = \sum_{i \geq 1} a_i(\rho, \zeta) \frac{\zeta_i}{z + \zeta_i} e^{-\zeta_i c}.$$

Next, using (18) and (13), after some algebraic manipulations, we find that

$$\begin{aligned} \frac{1}{(z + \zeta_i)\mathbb{E}[e^{-z\bar{X}_{e(q)}}]} &= \frac{1}{(z + \zeta_i)} \prod_{n \geq 1} \frac{1 + z/\zeta_n}{1 + z/\rho_n} \\ &= b_0(\zeta, \rho) + \sum_{j \geq 1} \frac{b_j(\zeta, \rho)}{\rho_j - \zeta_i} \frac{\rho_j}{\rho_j + z}. \end{aligned}$$

Combining the above three equations, we find

$$\begin{aligned} \mathbb{E}[e^{-q\tau_c^+ - z(X_{\tau_c^+} - c)}] &= \sum_{i \geq 1} a_i(\rho, \zeta)b_0(\zeta, \rho)\zeta_i e^{-\zeta_i c} \\ &\quad + \sum_{i \geq 1} \sum_{j \geq 1} \frac{a_i(\rho, \zeta)b_j(\zeta, \rho)}{\rho_j - \zeta_i} \zeta_i e^{-\zeta_i c} \frac{\rho_j}{\rho_j + z}, \end{aligned}$$

which allows for a straightforward inversion in  $z$ , thereby completing the proof. □

**5. Entrance/exit problems for a finite interval.** The two-sided exit problem is a long standing problem of interest from the theory of Lévy processes and there are very few cases where explicit identities have been extracted for the (discounted) overshoot distribution on either side of the interval. Classically, the only examples with discounting which have been analytically tractable are those of jump diffusion processes with (hyper)-exponentially distributed jumps (see, e.g., [25, 42]) and, up to knowing the so-called scale function, spectrally negative processes, [31]. Otherwise, the only other examples, without discounting, are those of stable processes, [39] and Lamperti-stable processes [13].

Recall that

$$\tau_a^+ = \inf\{t > 0 : X_t > a\}, \quad \tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

For  $f \in \mathcal{L}_\infty(\mathbb{R})$ , the space of positive, measurable and uniformly bounded functions on  $\mathbb{R}$ , and  $x \in \mathbb{R}$ , define the following operators:

$$(31) \quad \begin{aligned} (\mathcal{G}f)(x) &= \mathbb{I}(0 \leq x \leq a) \mathbb{E}_x[e^{-q\tau_a^+} f(X_{\tau_a^+}) \mathbb{I}(\tau_a^+ < \tau_0^-)], \\ (\hat{\mathcal{G}}f)(x) &= \mathbb{I}(0 \leq x \leq a) \mathbb{E}_x[e^{-q\tau_0^-} f(X_{\tau_0^-}) \mathbb{I}(\tau_0^- < \tau_a^+)], \end{aligned}$$

and for any subset  $I \in \mathbb{R}$  define

$$(32) \quad \begin{aligned} (\mathcal{P}_I f)(x) &= \mathbb{I}(x \in I) \mathbb{E}_x[e^{-q\tau_a^+} f(X_{\tau_a^+})], \\ (\hat{\mathcal{P}}_I f)(x) &= \mathbb{I}(x \in I) \mathbb{E}_x[e^{-q\tau_0^-} f(X_{\tau_0^-})]. \end{aligned}$$

These are bounded operators from  $\mathcal{L}_\infty(\mathbb{R})$  into  $\mathcal{L}_\infty(\mathbb{R})$ . Note also that if  $X_0 < 0$ , then  $\tau_0^- = 0$  and, similarly, if  $X_0 > a$ , then  $\tau_a^+ = 0$ . For example, when  $x > a$ ,  $\mathbb{E}_x[e^{-q\tau_a^+} f(X_{\tau_a^+})] = f(x)$  and when  $x < 0$ ,  $\mathbb{E}_x[e^{-q\tau_0^-} f(X_{\tau_0^-})] = f(x)$ .

Following similar reasoning to Rogozin [39] (see also [24]), an application of the Markov property tells us that for all  $x \in [0, a]$  we have

$$\begin{aligned} \mathbb{E}_x[e^{-q\tau_a^+} f(X_{\tau_a^+})] &= \mathbb{E}_x[e^{-q\tau_a^+} f(X_{\tau_a^+}) \mathbb{I}(\tau_a^+ < \tau_0^-)] \\ &\quad + \mathbb{E}_x[e^{-q\tau_0^-} \mathbb{I}(\tau_0^- < \tau_a^+) \mathbb{E}_{X_{\tau_0^-}}[e^{-q\tau_a^+} f(X_{\tau_a^+})]]. \end{aligned}$$

Thus, we have the following operator identities:

$$(33) \quad \begin{cases} \mathcal{P}_{[0,a]} = \mathcal{G} + \hat{\mathcal{G}}\mathcal{P}_{(-\infty,0]}, \\ \hat{\mathcal{P}}_{[0,a]} = \hat{\mathcal{G}} + \mathcal{G}\hat{\mathcal{P}}_{[a,\infty)}. \end{cases}$$

Consider a space  $\mathcal{H}$  of bounded linear operators on  $\mathcal{L}_\infty(\mathbb{R})$  with the norm

$$\|\mathcal{G}\| = \sup_{\|f\|_\infty \leq 1} \|\mathcal{G}f\|_\infty.$$

In the next theorem, which holds for general Lévy processes, we give a series representation for  $\mathcal{G}$  in terms of  $\hat{\mathcal{P}}_{[0,a]}$ ,  $\mathcal{P}_{(-\infty,0]}$ ,  $\hat{\mathcal{P}}_{[a,\infty)}$ ,  $\mathcal{P}_{(-\infty,0]}$  and a similar one

for  $\hat{\mathcal{G}}$ . In the case of stable processes and general Lévy processes, respectively, Rogozin [39] and Kadankov and Kadankova [24] have shown similar series representations. The presentation we give above only differs in that it takes the form of linear operators. This turns out to be more convenient for the particular application to the  $M$ -class of processes.

**THEOREM 4.** *Assume  $q > 0$ . A pair of operators  $(\mathcal{G}, \hat{\mathcal{G}})$  given by (31) is the unique solution in  $\mathcal{H}$  to the system of equations (33). Moreover, this solution can also be represented in series form*

$$(34) \quad \begin{aligned} \mathcal{G} = & \mathcal{P}_{[0,a]} - \hat{\mathcal{P}}_{[0,a]}\mathcal{P}_{(-\infty,0]} + \mathcal{P}_{[0,a]}\hat{\mathcal{P}}_{[a,\infty)}\mathcal{P}_{(-\infty,0]} \\ & - \hat{\mathcal{P}}_{[0,a]}\mathcal{P}_{(-\infty,0]}\hat{\mathcal{P}}_{[a,\infty)}\mathcal{P}_{(-\infty,0]} + \dots \end{aligned}$$

and

$$\begin{aligned} \hat{\mathcal{G}} = & \hat{\mathcal{P}}_{[0,a]} - \mathcal{P}_{[0,a]}\hat{\mathcal{P}}_{[a,\infty)} + \hat{\mathcal{P}}_{[0,a]}\mathcal{P}_{(-\infty,0]}\hat{\mathcal{P}}_{[a,\infty)} \\ & - \mathcal{P}_{[0,a]}\hat{\mathcal{P}}_{[a,\infty)}\mathcal{P}_{(-\infty,0]}\hat{\mathcal{P}}_{[a,\infty)} + \dots, \end{aligned}$$

where, in both cases, convergence is exponential in the following sense. There exist  $\eta \in (0, 1)$  and a constant  $C > 0$  such that if  $S_n$  is the sum of the first  $n$  terms in the series describing  $\mathcal{G}$ , then  $\|\mathcal{G} - S_n\| \leq C\eta^{n+1}$ , with a similar statement holding for  $\hat{\mathcal{G}}$ .

**PROOF.** It suffices to prove the result for  $\mathcal{G}$ , the proof for  $\hat{\mathcal{G}}$  follows by duality. For uniqueness, we note first that the norm of operators  $\mathcal{P}_{(-\infty,0]}$  is strictly less than one:

$$\begin{aligned} \|\mathcal{P}_{(-\infty,0]}\| &= \sup_{\|f\|_\infty \leq 1} \left[ \sup_{x \in \mathbb{R}} \mathbb{I}(x \leq 0) \mathbb{E}_x [e^{-q\tau_a^+} f(X_{\tau_a^+})] \right] \\ &\leq \sup_{x \in \mathbb{R}} \mathbb{I}(x \leq 0) \mathbb{E}_x [e^{-q\tau_a^+}] = \mathbb{E}_0 [e^{-q\tau_a^+}] < 1 \end{aligned}$$

and, similarly, we find  $\|\hat{\mathcal{P}}_{[a,\infty)}\| < 1$ . Uniqueness also follows from the latter fact. Indeed, if we assume that we have another pair of solutions  $(G, \hat{G})$ , then using the system of equations (33), we find that the difference  $\mathcal{G} - G$  must satisfy the equation

$$\mathcal{G} - G = (\mathcal{G} - G)\hat{\mathcal{P}}_{[a,\infty)}\mathcal{P}_{(-\infty,0]}.$$

Thus, we find

$$\|\mathcal{G} - G\| \leq \|\mathcal{G} - G\| \times \|\mathcal{P}_{[a,\infty)}\| \times \|\mathcal{P}_{(-\infty,0]}\|$$

and since  $\|\mathcal{P}_{[a,\infty)}\| \vee \|\mathcal{P}_{(-\infty,0]}\| < 1$ , we conclude that  $\|\mathcal{G} - G\| = 0$  and  $G = \mathcal{G}$ . To establish the series representation (34), we use (33) to find

$$\mathcal{G} = \mathcal{P}_{[0,a]} - \hat{\mathcal{P}}_{[0,a]}\mathcal{P}_{(-\infty,0]} + \mathcal{G}\hat{\mathcal{P}}_{[a,\infty)}\mathcal{P}_{(-\infty,0]}.$$

By iterating this equation, we obtain series representation (34), which converges at an exponential rate as described in the statement of the theorem with  $\eta = \|\hat{\mathcal{P}}_{[a,\infty)}\| \vee \|\mathcal{P}_{(-\infty,0]}\| \in (0, 1)$ .  $\square$

The next theorem converts the above general setting to the specific setting of  $M$ -processes.

**THEOREM 5** (First exit from a finite interval). *Let  $a > 0$  and define a matrix  $\mathbf{B} = \mathbf{B}(\hat{\rho}, \zeta, a) = \{b_{i,j}\}_{i,j \geq 0}$  with*

$$(35) \quad b_{i,j} = \begin{cases} \zeta_j e^{-a\zeta_j}, & \text{if } i = 0, j \geq 1, \\ 0, & \text{if } i \geq 0, j = 0, \\ \frac{\hat{\rho}_i \zeta_j}{\hat{\rho}_i + \zeta_j} e^{-a\zeta_j}, & \text{if } i \geq 1, j \geq 1, \end{cases}$$

and, similarly,  $\hat{\mathbf{B}} = \mathbf{B}(\rho, \hat{\zeta}, a)$ . There exist matrices  $\mathbf{C}_1, \mathbf{C}_2$  and  $\hat{\mathbf{C}}_1, \hat{\mathbf{C}}_2$  such that for  $x \in (0, a)$  we have

$$(36) \quad \begin{aligned} &\mathbb{E}_x[e^{-q\tau_a^+} \mathbb{I}(X_{\tau_a^+} \in dy; \tau_a^+ < \tau_0^-)] \\ &= [\bar{v}(\zeta, a-x)^T \times \mathbf{C}_1 + \bar{v}(\hat{\zeta}, x)^T \times \mathbf{C}_2] \times \bar{v}(\rho, y-a) dy, \\ &\mathbb{E}_x[e^{-q\tau_0^-} \mathbb{I}(X_{\tau_0^-} \in dy; \tau_0^- < \tau_a^+)] \\ &= [\bar{v}(\hat{\zeta}, x)^T \times \hat{\mathbf{C}}_1 + \bar{v}(\zeta, a-x)^T \times \hat{\mathbf{C}}_2] \times \bar{v}(\hat{\rho}, -y) dy. \end{aligned}$$

These matrices satisfy the following system of linear equations:

$$(37) \quad \begin{cases} \mathbf{C}_1 = \mathbf{A} - \hat{\mathbf{C}}_2 \mathbf{B} \mathbf{A}, & \begin{cases} \hat{\mathbf{C}}_1 = \hat{\mathbf{A}} - \mathbf{C}_2 \hat{\mathbf{B}} \hat{\mathbf{A}}, \\ \mathbf{C}_2 = -\hat{\mathbf{C}}_1 \mathbf{B} \mathbf{A}, \end{cases} \\ \hat{\mathbf{C}}_2 = -\mathbf{C}_1 \hat{\mathbf{B}} \hat{\mathbf{A}}, & \end{cases}$$

where  $\mathbf{A}$  and  $\hat{\mathbf{A}}$  are defined by (28). The system of linear equations (37) can be solved iteratively with exponential convergence with respect to the matrix norm  $\|\cdot\|_\infty$ , where for any square matrix  $\mathbf{M}$ ,

$$(38) \quad \|\mathbf{M}\|_\infty = \max_{i \geq 0} \sum_{j \geq 0} |M_{i,j}|.$$

Before proceeding to the proof of the above theorem, there is one technical lemma we must first address.

**LEMMA 3.** *Let  $\mathbf{H} = \mathbf{B} \times \mathbf{A}$ . Then  $\|\mathbf{H}\|_\infty < 1$ .*

**PROOF.** Using definitions (28) and (35) of  $\mathbf{A}$  and  $\mathbf{B}$ , we find that for  $i, j \geq 1$ ,

$$h_{0,0} = b_0(\zeta, \rho) \sum_{k \geq 1} a_k(\rho, \zeta) \zeta_k e^{-a\zeta_k},$$

$$\begin{aligned}
 h_{0,j} &= b_j(\zeta, \rho) \sum_{k \geq 1} a_k(\rho, \zeta) \frac{\zeta_k e^{-a\zeta_k}}{\rho_j - \zeta_k}, \\
 h_{i,0} &= \hat{\rho}_i b_0(\zeta, \rho) \sum_{k \geq 1} a_k(\rho, \zeta) \frac{\zeta_k e^{-a\zeta_k}}{\zeta_k + \hat{\rho}_i}, \\
 h_{i,j} &= \hat{\rho}_i b_j(\zeta, \rho) \sum_{k \geq 1} a_k(\rho, \zeta) \frac{\zeta_k e^{-a\zeta_k}}{(\rho_j - \zeta_k)(\zeta_k + \hat{\rho}_i)}.
 \end{aligned}$$

Our first goal is to prove that  $h_{i,j}$  are positive for all  $i, j \geq 0$  (note that  $h_{i,0} > 0$  for all  $i \geq 0$ ). Let us consider a random variable  $\xi = e(\rho_j) + \bar{X}_{e(q)}$ , where as usual  $e(\rho_j)$  is an exponentially distributed random variable with rate  $\rho_j$ , independent of  $X$  and  $e(q)$ . The Laplace transform of  $\xi$  is given by

$$\mathbb{E}[e^{-z\xi}] = \mathbb{E}[e^{-ze(\rho_j)}] \times \mathbb{E}[e^{-z\bar{X}_{e(q)}}] = \frac{\rho_j}{\rho_j + z} \prod_{n \geq 1} \frac{1 + z/\rho_n}{1 + z/\zeta_n},$$

and using the partial fraction decomposition (12), after some algebraic manipulations we find that  $\xi$  has a probability density function

$$\frac{d}{dx} \mathbb{P}(\xi \leq x) = \rho_j \sum_{k \geq 1} a_k(\rho, \zeta) \frac{\zeta_k e^{-x\zeta_k}}{\rho_j - \zeta_k}.$$

This proves that  $h_{0,j}$  are all positive. To prove that  $h_{i,j}$  are positive, one can use identity

$$\mathbb{E}[e^{-\hat{\rho}_i(\xi-a)} \mathbb{I}(\xi > a)] = \rho_j \sum_{k \geq 1} a_k(\rho, \zeta) \frac{\zeta_k e^{-a\zeta_k}}{(\rho_j - \zeta_k)(\zeta_k + \hat{\rho}_i)}.$$

Next we need to prove that  $\|\mathbf{H}\|_\infty < 1$ . Define a row vector  $\bar{h}_i$  as the  $i$ th row of the matrix  $\mathbf{H}$ . Using Theorem 3, we check that

$$\mathbb{E}[e^{-q\tau_a^+} \mathbb{I}(X_{\tau_a^+} - a \in dy) | X_0 = -\eta_i] = \bar{h}_i \times \bar{v}(\rho, y) dy,$$

where  $\eta_0 \equiv 0$  and  $\eta_i$  is independent and exponentially distributed with parameter  $\hat{\rho}_i$ . Thus, we have

$$\sum_{j \geq 0} h_{i,j} = \mathbb{E}[e^{-q\tau_a^+} | X_0 = -\eta_i] < \mathbb{E}[e^{-q\tau_a^+} | X_0 = 0] < 1,$$

and using the already established fact that  $h_{i,j}$  are positive, we conclude that  $\|\mathbf{H}\|_\infty < 1$ .  $\square$

**PROOF OF THEOREM 5.** The operator  $\mathcal{P}_I$  is an integral operator, with kernel

$$\begin{aligned}
 &\mathbb{I}(x \in I) \mathbb{E}_x[e^{-q\tau_a^+} \mathbb{I}(X_{\tau_a^+} \in dy)] \\
 &= \mathbb{I}(x \in I, y \geq a) \bar{v}(\zeta, a-x)^T \times \mathbf{A} \times \bar{v}(\rho, y-a) dy,
 \end{aligned}$$

(see Theorem 3). We also have a similar formula for  $\hat{\mathcal{P}}_I$ . Using the infinite series representation (34), we find that there exist some matrices of coefficients  $\mathbf{C}_1, \mathbf{C}_2$  and  $\hat{\mathbf{C}}_1, \hat{\mathbf{C}}_2$  so that integral kernels of operators  $\mathcal{G}$  and  $\hat{\mathcal{G}}$  can be represented in the form (36). Matrix equations (37) follow from operator identities (33) and the fact that

$$\int_0^\infty \bar{v}(\hat{\rho}, z) \times \bar{v}(\zeta, a + z)^T dz = \mathbf{B}.$$

Equations (37) can be solved iteratively as follows:

$$\begin{aligned} \mathbf{C}_1^{(n+1)} &= \mathbf{A} + \mathbf{C}_1^{(n)} \hat{\mathbf{B}} \hat{\mathbf{A}} \mathbf{B} \mathbf{A}, & \mathbf{C}_1^{(0)} &= \mathbf{0}, \\ \hat{\mathbf{C}}_1^{(n+1)} &= \hat{\mathbf{A}} + \hat{\mathbf{C}}_1^{(n)} \mathbf{B} \mathbf{A} \hat{\mathbf{B}} \hat{\mathbf{A}}, & \hat{\mathbf{C}}_1^{(0)} &= \mathbf{0}, \end{aligned}$$

and the convergence  $\mathbf{C}_1^{(n)} \rightarrow \mathbf{C}_1$  and  $\hat{\mathbf{C}}_1^{(n)} \rightarrow \hat{\mathbf{C}}_1$  is exponential because of Lemma 3. Once we find  $\mathbf{C}_1$  and  $\hat{\mathbf{C}}_1$ , we find  $\mathbf{C}_2 = -\hat{\mathbf{C}}_1 \mathbf{B} \mathbf{A}$  and  $\hat{\mathbf{C}}_2 = -\mathbf{C}_1 \hat{\mathbf{B}} \hat{\mathbf{A}}$ . □

REMARK 3. Assume that within the  $M$ -class the positive (negative) jumps come from a mixture of  $N < \infty$  ( $\hat{N} < \infty$ ) exponential distributions. Then matrices  $\mathbf{B}$  and  $\hat{\mathbf{B}}$  have size  $(\hat{N} + 1) \times (N + 1)$  and  $(N + 1) \times (\hat{N} + 1)$ , while matrices  $\mathbf{C}_1$  and  $\hat{\mathbf{C}}_1$  have sizes  $(N + 1) \times (N + 1)$  and  $(\hat{N} + 1) \times (\hat{N} + 1)$  and can be found by the relations

$$\mathbf{C}_1 = \mathbf{A}(\mathbf{I} - \hat{\mathbf{B}} \hat{\mathbf{A}} \mathbf{B} \mathbf{A})^{-1}, \quad \hat{\mathbf{C}}_1 = \hat{\mathbf{A}}(\mathbf{I} - \mathbf{B} \mathbf{A} \hat{\mathbf{B}} \hat{\mathbf{A}})^{-1}.$$

The idea of writing down a pair of simultaneous equations (31) also works when considering the problem of first entrance *into* an interval. This problem has been considered earlier by [39] for the setting of stable processes and [25] for two-sided Lévy processes whose upward jumps are exponentially distributed.

THEOREM 6 (First entrance into a finite interval). *Let  $a > 0$  and define  $\tau = \inf\{t \geq 0 : X_t \in (0, a)\}$ . Define a matrix  $\mathbf{M} = \mathbf{B}(\hat{\zeta}, \rho, a)^T$  and, similarly,  $\hat{\mathbf{M}} = \mathbf{B}(\zeta, \hat{\rho}, a)^T$  [see (35)]. There exist matrices  $\mathbf{N}_1, \mathbf{N}_2$  and  $\hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2$  such that for  $y \in (0, a)$  we have*

$$(39) \quad \mathbb{E}_x[e^{-q\tau} \mathbb{I}(X_\tau \in dy)] = \begin{cases} \bar{v}(\zeta, -x)^T \times [\mathbf{N}_1 \times \bar{v}(\rho, y) \\ \quad + \mathbf{N}_2 \times \bar{v}(\hat{\rho}, a - y)] \times dy, & x \leq 0, \\ \bar{v}(\hat{\zeta}, x - a)^T \times [\hat{\mathbf{N}}_1 \times \bar{v}(\hat{\rho}, a - y) \\ \quad + \hat{\mathbf{N}}_2 \times \bar{v}(\rho, y)] \times dy, & x \geq a. \end{cases}$$

These matrices satisfy the following system of linear equations:

$$(40) \quad \begin{cases} \mathbf{N}_1 = \mathbf{A} + \mathbf{A} \mathbf{M} \hat{\mathbf{N}}_2, \\ \hat{\mathbf{N}}_2 = \hat{\mathbf{A}} \hat{\mathbf{M}} \mathbf{N}_1, \end{cases} \quad \begin{cases} \hat{\mathbf{N}}_1 = \hat{\mathbf{A}} + \hat{\mathbf{A}} \hat{\mathbf{M}} \mathbf{N}_2, \\ \mathbf{N}_2 = \mathbf{A} \mathbf{M} \hat{\mathbf{N}}_1, \end{cases}$$

where  $\mathbf{A}$  and  $\hat{\mathbf{A}}$  are defined by (28). This system of linear equations (40) can be solved iteratively with exponential convergence with respect to the matrix norm  $\|\cdot\|_\infty$ .

The proof of this theorem is very similar to the proof of Theorem 5, and we leave the details to the reader.

**6. Ladder processes.** In this section we derive several results related to the Laplace exponent  $\kappa(q, z)$  of the bivariate ladder process  $(L, H)$ . We are interested in numerical evaluation of this object since it is the key to many important fluctuation identities, such as the quintuple law at the first passage which was introduced in [15].

Everywhere in this section we will denote the characteristic exponent of the Lévy process  $X$  by  $\Psi(z) = -\ln \mathbb{E}[\exp(izX_1)]$ . Note that the Laplace exponent  $\psi(z)$  can be expressed in terms of the characteristic exponent as  $\psi(z) = -\Psi(-iz)$ , this fact easily follows from (1).

The following theorem, which holds for general Lévy processes, is a generalization of the expression for the Wiener–Hopf factors which can be found in Lemma 4.2 in [36].

**THEOREM 7.** *Assume that for some  $\varepsilon_1 > 0$  and for all  $|z| \in \mathbb{R}$  large enough we have*

$$(41) \quad |\Psi(z)| > c|z|^{\varepsilon_1},$$

for some  $c > 0$ , and

$$(42) \quad \int_{-\varepsilon_2}^{\varepsilon_2} \left| \frac{\Psi(z)}{z} \right| dz,$$

exists for some (and then for all)  $\varepsilon_2 > 0$ . Then for  $\operatorname{Re}(q) > 0$  and  $\operatorname{Re}(z) > 0$  we have

$$(43) \quad \kappa(q, z) = \exp \left[ \frac{1}{2\pi i} \int_{\mathbb{R}} \left( \frac{\ln(q + \Psi(u))}{u - iz} - \frac{\ln(1 + \Psi(u))}{u} \right) du \right].$$

**PROOF.** We start with the following integral representation for  $\kappa(q, z)$  (see Theorem 6.16 in [31]):

$$\kappa(q, z) = \exp \left[ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} (e^{-t} - e^{-qt-zx}) \frac{1}{t} \mathbb{P}(X_t \in dx) dt \right].$$

Assume that the process  $X_t$  has a nonzero Gaussian component ( $\sigma \neq 0$ ), then  $\mathbb{P}(X_t \in dx)$  has a density which can be obtained as the inverse Fourier transform of the right-hand side of (1). Using this fact and assuming that  $\gamma > 0$ , we obtain

$$(44) \quad \begin{aligned} & \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} (e^{-t-\gamma x} - e^{-qt-zx}) \frac{1}{t} \mathbb{P}(X_t \in dx) dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} (e^{-t-\gamma x} - e^{-qt-zx}) \frac{1}{t} \int_{\mathbb{R}} e^{-t\Psi(u)-iux} du dx dt. \end{aligned}$$

Applying the Fubini theorem and performing integration in  $x$ , we find that the above integral is equal to

$$\begin{aligned}
 & \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{dt}{t} \int_{\mathbb{R}} du \left( \frac{e^{-t(1+\Psi(u))}}{\gamma + iu} - \frac{e^{-t(q+\Psi(u))}}{z + iu} \right) \\
 (45) \quad &= \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{dt}{t} \int_{\mathbb{R}} du \left( \frac{1}{\gamma + iu} (e^{-t(1+\Psi(u))} - e^{-t}) \right. \\
 & \quad \left. - \frac{1}{z + iu} (e^{-t(q+\Psi(u))} - e^{-t}) \right),
 \end{aligned}$$

where in the last step we have used the fact that for  $\gamma > 0$  and  $\text{Re}(z) > 0$ ,

$$\int_{\mathbb{R}} \left[ \frac{1}{\gamma + iu} - \frac{1}{z + iu} \right] du = 0$$

(which can be easily verified by residue calculus). Next, we apply the Fubini theorem and Frullani integral to the last integral in (45) and combining this result with (44), we obtain

$$\begin{aligned}
 & \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} (e^{-t-\gamma x} - e^{-qt-zx}) \frac{1}{t} \mathbb{P}(X_t \in dx) dt \\
 (46) \quad &= \frac{1}{2\pi i} \int_{\mathbb{R}} \left( \frac{\ln(q + \Psi(u))}{u - iz} - \frac{\ln(1 + \Psi(u))}{u - i\gamma} \right) du.
 \end{aligned}$$

Next we need to take the limit of the above identity as  $\gamma \rightarrow 0^+$ . For  $u$  small the integrand in the right-hand side of (46) is bounded by

$$C \frac{\ln(1 + |\Psi(u)|)}{|u|},$$

uniformly in  $\gamma \in [0, \tilde{\gamma}]$  for each  $\tilde{\gamma} > 0$ , and for  $u$  large it can be bounded by

$$\begin{aligned}
 & \left| \frac{\ln(q + \Psi(u))}{u - iz} - \frac{\ln(1 + \Psi(u))}{u - i\gamma} \right| \\
 &= \left| i(z - \gamma) \frac{\ln(q + \Psi(u))}{(u - i\gamma)(u - iz)} - \frac{\ln(1 + (1 - q)/(q + \Psi(u)))}{u - i\gamma} \right|, \\
 &= O(\ln(|\Psi(u)|)u^{-2}) + O((u\Psi(u))^{-1}),
 \end{aligned}$$

again uniformly in  $\gamma \in [0, \tilde{\gamma}]$ . Using the above two estimates and assumption (41), we conclude that the integrand in the right-hand side of (46) is bounded by

$$\tilde{C} \frac{\ln(1 + |\Psi(u)|)}{|u|} (1 + |u|)^{-\varepsilon_1},$$

for some  $\tilde{C} > 0$  uniformly in  $\gamma \in [0, \tilde{\gamma}]$ , and the above function is integrable on  $\mathbb{R}$  due to assumption (42). Thus, we can apply the Dominated Convergence Theorem

on the right-hand side of (46) together with the Monotone Convergence Theorem on the left-hand side while taking limits as  $\gamma \rightarrow 0^+$ . To finish the proof, we only need to take the limit  $\sigma \rightarrow 0^+$ , which can be justified in exactly the same way using the Dominated Convergence Theorem on the right-hand side of (46) and weak convergence on the right-hand side of (46). Note, in particular, that the characteristic exponent of  $X$  is continuous in the Gaussian coefficient and, hence, by the Continuity Theorem for Fourier transforms, the law of  $X$  is weakly continuous in the Gaussian coefficient.  $\square$

Assumption (42) is a very mild one. It is satisfied by all processes except those which have unusually heavy tails of the Lévy measure. It is satisfied by all processes in the  $M$ -class, and more generally, by all processes for which there exists an  $\varepsilon > 0$  such that  $\Pi(\mathbb{R} \setminus (-x, x)) = O(x^{-\varepsilon})$  as  $x \rightarrow +\infty$  (e.g., all stable processes have this property). While condition (41) is more restrictive, one can see that it only excludes compound Poisson processes and some processes of bounded variation which are equal to the sum of its jumps. One example of such a process is a pure jump Variance Gamma process with no linear drift  $X_t = \sigma W_{\Gamma_t} + \mu \Gamma_t$ , which satisfies  $|\Psi(z)| \sim c \ln(|z|)$  as  $z \rightarrow \infty$ . One can see that condition (41) is satisfied for the processes of bounded variation with nonzero drift, processes with nonzero Gaussian component and all processes with the Lévy measure satisfying  $\Pi(\mathbb{R} \setminus (-x, x)) > x^{-\varepsilon}$  as  $x \rightarrow 0^+$  for some  $\varepsilon > 0$ . In particular, this condition is satisfied for all processes in the  $\beta$ -,  $\theta$ -, hypergeometric or hyperexponential family excluding compound Poisson processes.

Theorem 7 allows us to derive an expression for the Laplace exponent of the bivariate ladder process  $(L, H)$ .

**THEOREM 8.** *Assume that  $X$  belongs to  $\beta$ -,  $\theta$ -, hyperexponential or hypergeometric family of Levy processes and that  $X$  is not a compound Poisson process. Then*

$$(47) \quad \kappa(q, z) = [\phi_q^+(iz)]^{-1} \prod_{n \geq 1} \frac{\zeta_n(q)}{\zeta_n(1)}.$$

**PROOF.** If  $X$  belongs to  $\beta$ -,  $\theta$ - or hypergeometric family of Levy processes, we can use asymptotics for  $\zeta_n$  as  $n \rightarrow +\infty$  (see [28], [29] and [30]) to find that  $\zeta_n(q)/\zeta_n(1) = 1 + O(n^{-1-\varepsilon})$  and  $\hat{\zeta}_n(q)/\hat{\zeta}_n(1) = 1 + O(n^{-1-\varepsilon})$ , which implies that both infinite products

$$\prod_{n \geq 1} \frac{\zeta_n(q)}{\zeta_n(1)}, \quad \prod_{n \geq 1} \frac{\hat{\zeta}_n(q)}{\hat{\zeta}_n(1)},$$

converge. Next, using the same technique as in the proof of Lemma 6 in [28], one can prove that

$$\prod_{n \geq 1} \frac{1 - iz/\zeta_n(q)}{1 - iz/\zeta_n(1)} \prod_{n \geq 1} \frac{1 + iz/\hat{\zeta}_n(q)}{1 + iz/\hat{\zeta}_n(1)} \rightarrow \prod_{n \geq 1} \frac{\zeta_n(1)}{\zeta_n(q)} \prod_{n \geq 1} \frac{\hat{\zeta}_n(1)}{\hat{\zeta}_n(q)},$$

as  $z \rightarrow \infty, z \in \mathbb{R}$ . Using the Wiener–Hopf factorization  $q/(q + \Psi(z)) = \phi_q^+(z) \times \phi_q^-(z)$  and (18), we obtain

$$\frac{q + \Psi(z)}{1 + \Psi(z)} = q \prod_{n \geq 1} \frac{1 - iz/\zeta_n(q)}{1 - iz/\zeta_n(1)} \prod_{n \geq 1} \frac{1 + iz/\hat{\zeta}_n(q)}{1 + iz/\hat{\zeta}_n(1)}.$$

Since  $X$  is not a compound Poisson process, we have  $|\Psi(z)| \rightarrow \infty$  as  $z \rightarrow \infty, z \in \mathbb{R}$ , thus, we finally conclude that

$$\prod_{n \geq 1} \frac{\zeta_n(q)}{\zeta_n(1)} \prod_{n \geq 1} \frac{\hat{\zeta}_n(q)}{\hat{\zeta}_n(1)} = q.$$

The next step is to use the Wiener–Hopf factorization  $q/(q + \Psi(z)) = \phi_q^+(z)\phi_q^-(z)$ , (18) and the above identity to rewrite  $q + \Psi(z)$  as

$$q + \Psi(z) = \prod_{n \geq 1} \frac{(\zeta_n(q) - iz)/\zeta_n(1)}{1 - iz/\rho_n} \prod_{n \geq 1} \frac{(\hat{\zeta}_n(q) + iz)/\hat{\zeta}_n(1)}{1 + iz/\hat{\rho}_n}.$$

Now we can use the above factorization and the following integral identity (which can be proved by shifting the contour of integration in the complex plane):

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\mathbb{R}} \left( \frac{\ln((1/b)(a - u))}{u - iz} - \frac{\ln((1/b)(b - u))}{u} \right) du \\ &= \begin{cases} \ln\left(\frac{1}{b}(a - iz)\right), & \text{if } \text{Im}(a) < 0, \text{Im}(b) < 0, \\ 0, & \text{if } \text{Im}(a) > 0, \text{Im}(b) > 0, \end{cases} \end{aligned}$$

to deduce that for  $\text{Re}(z) > 0$ ,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\mathbb{R}} \left( \frac{\ln(q + \Psi(u))}{u - iz} - \frac{\ln(1 + \Psi(u))}{u} \right) du \\ &= \sum_{n \geq 1} \left[ \ln\left(\frac{\zeta_n(q) + z}{\zeta_n(1)}\right) - \ln\left(1 + \frac{z}{\rho_n}\right) \right], \end{aligned}$$

which is equivalent to (47).  $\square$

**COROLLARY 4.** *Let  $\nu(ds, dx)$  be the Lévy measure of the ascending ladder process  $(L_t^{-1}, H_t)$ . Then for  $x > 0$  we have*

$$(48) \quad \int_{\mathbb{R}^+} e^{-qs} \nu(ds, dx) = \left[ \prod_{n \geq 1} \frac{\zeta_n(q)}{\zeta_n(1)} \right] \bar{b}(\zeta, \rho)^T \times \bar{\nu}(\rho, x) dx.$$

**PROOF.** Formula (48) is a corollary of (47), (17) and the fact that  $\nu(ds, dx)$  is related to  $\kappa(q, z)$  through the formula

$$\kappa(q, z) = \kappa(q, 0) + az + \int_0^\infty (1 - e^{-zx}) \int_0^\infty e^{-qs} \nu(ds, dx). \quad \square$$

**7. More fluctuation identities.** We offer some more fluctuation identities. Although they are slightly more complex, they are still equally straightforward for the purpose of numerical work.

We assume throughout this section that  $X$  is regular for both  $(0, \infty)$  and  $(-\infty, 0)$ . Equivalently, we assume that  $a_0(\rho, \zeta) = a_0(\hat{\rho}, \hat{\zeta}) = 0$ . This is the case if, for example,  $X$  has paths of unbounded variation. It will be clear from the proofs of the results given below how this assumption may be removed.

For  $a > 0$  and for  $y \leq a$  we define resolvent for  $X$  killed on leaving  $(-\infty, a]$  as

$$R^{(q)}(a, dy) := \int_0^\infty e^{-qt} \mathbb{P}(X_t \in dy; t < \tau_a^+) dt.$$

**THEOREM 9.** Define a matrix  $\mathbf{D} = \{d_{i,j}\}_{i,j \geq 0}$  as follows:

$$d_{i,j} = \begin{cases} 0, & \text{if } i = 0 \text{ or } j = 0, \\ a_i(\rho, \zeta) \frac{1}{\zeta_i + \hat{\zeta}_j} a_j(\hat{\rho}, \hat{\zeta}), & \text{if } i \geq 1, j \geq 1. \end{cases}$$

Then if  $y \leq a$  we have

$$qR^{(q)}(a, dy) = [\bar{v}(\zeta, 0 \vee y)^T \times \mathbf{D} \times \bar{v}(\hat{\zeta}, 0 \vee (-y)) - \bar{v}(\zeta, a)^T \times \mathbf{D} \times \bar{v}(\hat{\zeta}, a - y)] dy.$$

**PROOF.** From the proof of Theorem 20 on page 176 of [5], it can be seen that

$$\begin{aligned} q \int_{(-\infty, a]} f(y) R^{(q)}(a, dy) &= \int_{[0, a]} \mathbb{P}(\bar{X}_{e(q)} \in dz) \int_{[0, \infty)} \mathbb{P}(-\underline{X}_{e(q)} \in du) f(-u + z) \\ &= \int_{[0, a]} \mathbb{P}(\bar{X}_{e(q)} \in dz) \int_{(-\infty, z]} f(y) \mathbb{P}(\underline{X}_{e(q)} \in -z + dy) \\ &= \int_{(-\infty, a]} f(y) \int_{[0 \vee y, a]} \mathbb{P}(\bar{X}_{e(q)} \in dz) \mathbb{P}(\underline{X}_{e(q)} \in -z + dy). \end{aligned}$$

Thus, we obtain an alternative representation of the Spitzer–Bertoin identity

$$qR^{(q)}(a, dy) = \int_{[0 \vee y, a]} \mathbb{P}(\bar{X}_{e(q)} \in dz) \mathbb{P}(\underline{X}_{e(q)} \in -z + dy).$$

To finish the proof, we have to use the formulas in (19) and perform the integration in the above expression [noting that some terms are lost on account of the fact that we have assumed  $a_0(\rho, \zeta) = a_0(\hat{\rho}, \hat{\zeta}) = 0$ ].  $\square$

Next define

$$\Theta^{(q)}(a, x, dy) = \int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in dy, t < \tau_a^+ \wedge \tau_0^-) dt.$$

THEOREM 10. Assume  $q > 0$  and  $y \in [0, a]$ , then

$$\begin{aligned}
 q\Theta^{(q)}(a, x, dy) &= [\bar{v}(\zeta, 0 \vee y - x)^T \times \mathbf{D} \times \bar{v}(\hat{\zeta}, 0 \vee (x - y)) \\
 &\quad - \bar{v}(\zeta, a - x)^T \times \mathbf{D} \times \bar{v}(\hat{\zeta}, a - y)] dy \\
 &\quad - [\bar{v}(\hat{\zeta}, x)^T \times \hat{\mathbf{C}}_1 + \bar{v}(\zeta, a - x)^T \times \hat{\mathbf{C}}_2] \\
 &\quad \times [\mathbf{B}(y) \times \mathbf{D} \times \bar{v}(\hat{\zeta}, 0) - \mathbf{B}(a) \times \mathbf{D} \times \bar{v}(\hat{\zeta}, a - y)] dy,
 \end{aligned}$$

where matrix  $\mathbf{B}(y) = \mathbf{B}(\hat{\rho}, \zeta, y)$  is defined in (35),  $\mathbf{D}$  is defined in Theorem 9, while matrices  $\hat{\mathbf{C}}_1$  and  $\hat{\mathbf{C}}_2$  come from Theorem 5. Also,  $\bar{v}(\hat{\zeta}, 0)$  is interpreted as  $[0, \hat{\zeta}_1, \hat{\zeta}_2, \dots]^T$ .

PROOF. Define

$$(49) \quad \hat{g}^{(q)}(a, x, dy) = \mathbb{E}_x[e^{-q\tau_0^-} \mathbb{I}(-X_{\tau_0^-} \in dy; \tau_0^- < \tau_a^+)].$$

Note, moreover, that for  $f$  supported in  $[0, a]$ ,

$$\begin{aligned}
 &\int_{[0,a]} f(y)\Theta^{(q)}(a, x, dy) \\
 &= \frac{1}{q}\mathbb{E}_x[f(X_{e(q)})\mathbb{I}(e(q) < \tau_0^- \wedge \tau_a^+)] \\
 &= \frac{1}{q}\mathbb{E}_x[f(X_{e(q)})\mathbb{I}(e(q) < \tau_a^+)] - \frac{1}{q}\mathbb{E}_x[f(X_{e(q)})\mathbb{I}(\tau_0^- < e(q) < \tau_a^+)] \\
 &= \int_0^a f(y)R^{(q)}(a - x, dy - x) - \frac{1}{q}\mathbb{E}_x[f(X_{e(q)})\mathbb{I}(\tau_0^- < e(q) < \tau_a^+)].
 \end{aligned}$$

The second expectation in the above expression can be rewritten as

$$\begin{aligned}
 &\int_0^\infty e^{-qt} \mathbb{E}_x[f(X_t)\mathbb{I}(\tau_0^- < t < \tau_a^+)] dt \\
 &= \mathbb{E}_x\left[\int_{\tau_0^-}^\infty e^{-qt} f(X_t)\mathbb{I}(t < \tau_a^+) dt\right] \\
 &= \mathbb{E}_x\left[e^{-q\tau_0^-} \mathbb{I}(\tau_0^- < \tau_a^+) \mathbb{E}_{X_{\tau_0^-}}\left[\int_0^\infty e^{-qt} f(X_t)\mathbb{I}(t < \tau_a^+) dt\right]\right] \\
 &= \int_{\mathbb{R}^+} \hat{g}^{(q)}(a, x, dz) \int_{\mathbb{R}^+} R^{(q)}(a + z, z + dy) f(y).
 \end{aligned}$$

So, in conclusion, we have for  $y \in [0, a]$ ,

$$\Theta^{(q)}(a, x, dy) = R^{(q)}(a - x, dy - x) - \int_{\mathbb{R}^+} \hat{g}^{(q)}(a, x, dz) R^{(q)}(a + z, z + dy),$$

and to end the proof, one should use results of Theorems 5 and 9 and compute the above integral. The details are left to the reader.  $\square$

REMARK 4. The previous result also allows us to write down the discounted joint overshoot, undershoot distribution for the two-sided exit problem:

$$g^{(q)}(a, x, dy, dz) = \mathbb{E}_x[e^{-q\tau_a^+} \mathbb{I}(\tau_a^+ < \tau_0^-; X_{\tau_a^+} \in dy; X_{\tau_a^+} \in dz)].$$

(Again this relates to the so-called Gerber–Shiu measure for classical risk theory). Indeed, for  $y > a$  and  $z \in [0, a]$ , using the compensation formula, we have

$$\begin{aligned} g^{(q)}(a, x, dy, dz) &= \mathbb{E}_x \left[ \sum_{t \geq 0} e^{-qt} \mathbb{I}(\bar{X}_{t-} \leq a, \underline{X}_{t-} \geq 0, X_{t-} \in dz) \mathbb{I}(X_t \in dy) \right] \\ &= \mathbb{E}_x \left[ \int_{\mathbb{R}^+} e^{-qt} \mathbb{I}(\bar{X}_{t-} \leq a, \underline{X}_{t-} \geq 0, X_{t-} \in dz) dt \right] \Pi(dy - z) \\ &= \Theta^{(q)}(a, x, dz) \Pi(dy - z). \end{aligned}$$

REMARK 5. Define the reflected process  $Y := X - \underline{X}$  and its resolvent when killed on exiting  $[0, a]$ ,

$$\Lambda^{(q)}(a, x, dy) = \int_0^\infty e^{-qt} \mathbb{P}_x(Y_t \in dy, t < \sigma_a) dt,$$

where  $\sigma_a = \inf\{s > 0 : Y_s > a\}$ . According to [3], one may write for all  $q \geq 0$ ,

$$\Lambda^{(q)}(a, x, dy) = \Theta^{(q)}(a, x, dy) + \hat{g}^{(q)}(a, x) \cdot \lim_{z \downarrow 0} \frac{\Theta^{(q)}(a, z, dy)}{1 - \hat{g}^{(q)}(a, z)},$$

where  $\hat{g}^{(q)}(a, x) = \mathbb{E}_x[e^{-q\tau_0^-} \mathbb{I}(\tau_a^+ > \tau_0^-)]$ . Moreover, note that we may continue the computations in a similar way to before with the help of the compensation formula,

$$\mathbb{E}_0[e^{-q\sigma_a} \mathbb{I}(Y_{\sigma_a} \in dy; Y_{\sigma_a} \in dz)] = \Lambda^{(q)}(a, 0, dz) \Pi(dy - z),$$

and, hence,

$$\begin{aligned} &\mathbb{E}_x[e^{-q\sigma_a} \mathbb{I}(Y_{\sigma_a} \in dy; Y_{\sigma_a} \in dz)] \\ &= g^{(q)}(a, x, dy, dz) + \hat{g}^{(q)}(a, x) \Lambda^{(q)}(a, 0, dz) \Pi(dy - z). \end{aligned}$$

**8. Numerical results.** For all our numerical examples we will use a process  $X$  from the  $\beta$ -family (see the [Introduction](#) or [28]) having parameters

$$(\sigma, \mu, \alpha_1, \beta_1, \lambda_1, c_1, \alpha_2, \beta_2, \lambda_2, c_2) = (\sigma, \mu, 1, 1.5, 1.5, 1, 1, 1.5, 1.5, 1).$$

Here  $\mu = \mathbb{E}[X_1]$  and  $\sigma$  is the Gaussian coefficient; the other parameters define the density of a Lévy measure, which has exponentially decaying tails and  $O(|x|^{-3/2})$  singularity at  $x = 0$ , thus, this process has jumps of infinite activity but finite variation. We define the following four parameter sets:

$$\begin{aligned} \text{Set 1: } \sigma &= 0.5, \mu = 1, & \text{Set 2: } \sigma &= 0.5, \mu = -1, \\ \text{Set 3: } \sigma &= 0, \mu = 1, & \text{Set 4: } \sigma &= 0, \mu = -1. \end{aligned}$$

As a first illustration of the efficiency of our algorithm we will compute the following three quantities related to the two sided exit problem:

(i) density of the overshoot on the event that the process exists at the upper boundary

$$f_1(x, y) = \frac{d}{dy} \mathbb{E}_x[e^{-q\tau_1^+} \mathbb{I}(X_{\tau_1^+} \leq y; \tau_1^+ < \tau_0^-)],$$

(ii) probability of exiting from the interval  $[0, 1]$  at the upper boundary

$$f_2(x) = \mathbb{E}_x[e^{-q\tau_1^+} \mathbb{I}(\tau_1^+ < \tau_0^-)],$$

(iii) probability of exiting the interval  $[0, 1]$  by creeping across the upper boundary

$$f_3(x) = \mathbb{E}_x[e^{-q\tau_1^+} \mathbb{I}(X_{\tau_1^+} = 1; \tau_1^+ < \tau_0^-)].$$

In order to compute these expressions, we use methods described in Theorem 5. We truncate all the matrices  $C_i, \hat{C}_i$  so that they have size  $200 \times 100$  [this corresponds to truncating coefficients  $a_i(\rho, \zeta)$  and  $a_i(\hat{\rho}, \hat{\zeta})$  at  $i = 200$  and coefficients  $b_j(\zeta, \rho)$  and  $b_j(\hat{\zeta}, \hat{\rho})$  at  $j = 100$ ]. In order to compute coefficients  $a_i(\rho, \zeta)$ ,  $a_i(\hat{\rho}, \hat{\zeta})$ ,  $b_j(\zeta, \rho)$  and  $b_j(\hat{\zeta}, \hat{\rho})$ , we truncate infinite products in (14) and (15) at  $k = 400$ , thus, all the computations depend on precomputing  $\{\zeta_n, \hat{\zeta}_n\}$  for  $n = 1, 2, \dots, 400$ . All the code was written in Fortran and the computations were performed on a standard laptop (Intel Core 2 Duo 2.5 GHz processor and 3 GB of RAM).

We present the results for  $q = 1$  in Figures 1 and 2. Computations required to produce graphs for each parameter set took around 0.15 seconds. The numerical results clearly show the effects that we would expect to see. In Figures 1(b), 1(d) and 2(b) we see a positive probability of creeping, which is expected since the process  $X$  has a Gaussian component in the first two cases and a bounded variation and positive drift in the third case. Parameter Set 4 corresponds to a process with bounded variation and negative drift, thus, we do not have any upward creeping, and this is exactly what we obtain in Figure 2(d). Also, Figures 1(a), 1(c) and 2(a) show that  $f_1(x, y) \rightarrow 0$  as  $x \rightarrow 1^-$ , which confirms our expectation, as in this case the upper half line is regular and as  $x \rightarrow 1^-$  the process will cross the barrier at 1 by creeping, not by jumping over it. This is different from figure 2(c), where, because of bounded variation and negative drift, the upper half line is irregular and the only way to cross the barrier at 1 is by jumping over it. Next, Figure 1(b) and 1(d) shows that  $f_i(x) \rightarrow 0$  as  $x \rightarrow 0^+$  and  $f_i(x) \rightarrow 1$  as  $x \rightarrow 1^-$ , which again agrees with the theory, as for parameter Sets 1 and 2 the process  $X$  has a Gaussian component, therefore, 0 is regular for  $(-\infty, 0)$  and  $(0, \infty)$ . This is not so for parameter Sets 3 and 4, since now the process  $X$  has bounded variation and drift, and depending on the sign of the drift, 0 is regular for either  $(0, \infty)$  or  $(-\infty, 0)$ , and this is what we observe on Figure 2(b) and 2(d).

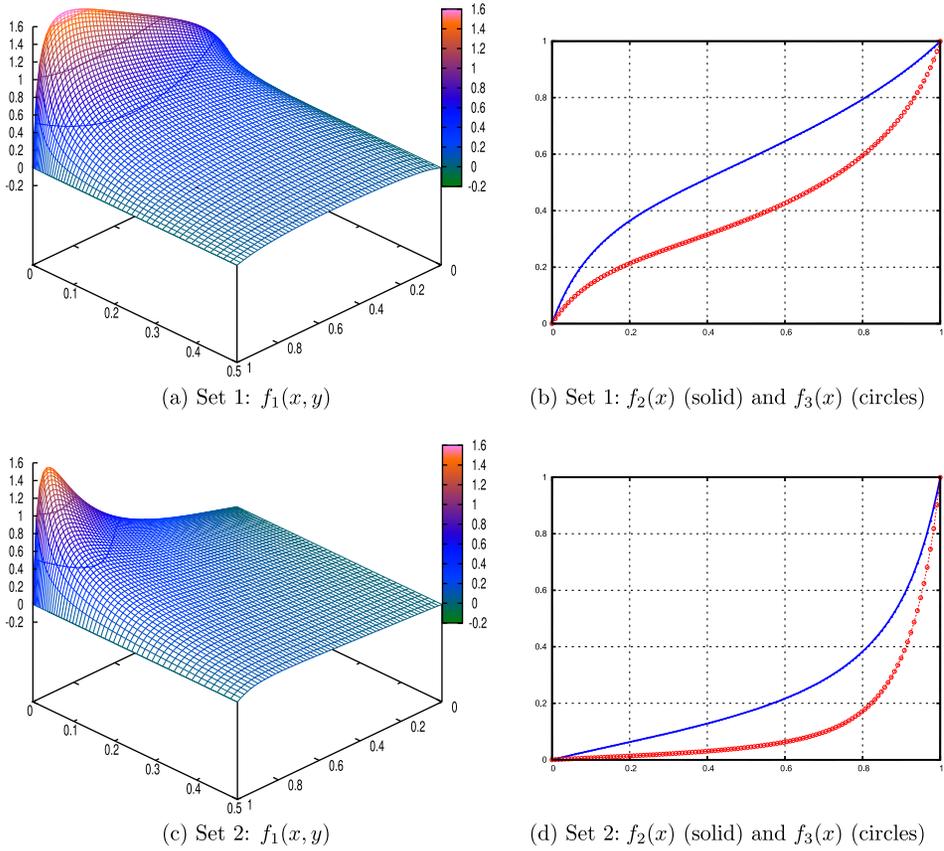


FIG. 1. Unbounded variation case ( $\sigma = 0.5$ ): computing the density of the overshoot  $f_1(x, y)$  ( $x \in (0, 1)$ ,  $y \in (0, 0.5)$ ), probability of first exit  $f_2(x)$  and probability of creeping  $f_3(x)$  for parameter Set 1 (positive drift  $\mu = 1$ ) and Set 2 (negative drift  $\mu = -1$ ).

For the next example, we compute the density  $u(s, x)$  of the bivariate renewal measure  $\mathcal{U}(ds, dx)$  defined by

$$\mathcal{U}(ds, dx) = \int_{\mathbb{R}^+} \mathbb{P}(L_t^{-1} \in ds, H_t \in dx) dt,$$

where  $(L, H)$  is the ascending ladder process; see [15, 31] and [32]. This measure is a very important object, as it gives us full knowledge of the quintuple law at the first passage; see [15]. Using formulas 6.18 and 7.10 in [31], we see that  $\mathcal{U}(ds, dx)$  satisfies

$$\int_{\mathbb{R}^+} e^{-qs} \mathcal{U}(ds, dx) = \frac{\mathbb{P}(\bar{X}_{e(q)} \in dx)}{\kappa(q, 0)},$$

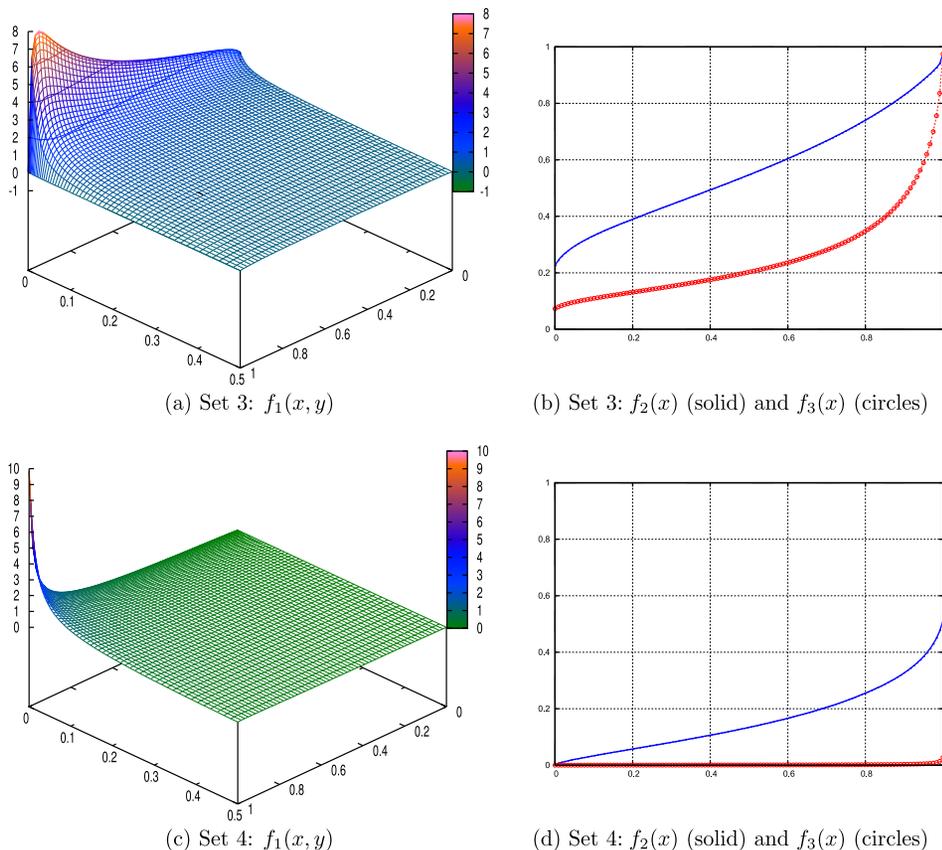


FIG. 2. Bounded variation case ( $\sigma = 0$ ): computing the density of the overshoot  $f_1(x, y)$  ( $x \in (0, 1)$ ,  $y \in (0, 0.5)$ ), probability of first exit  $f_2(x)$  and probability of creeping  $f_3(x)$  for parameter Set 3 (positive drift  $\mu = 1$ ) and Set 4 (negative drift  $\mu = -1$ ).

therefore, using Theorem 8 and (19), we find that the density of  $\mathcal{U}(ds, dx)$  can be computed as

$$u(s, x) = \frac{1}{2\pi i} \int_{q_0+i\mathbb{R}} \prod_{n \geq 1} \frac{\zeta_n(1)}{\zeta_n(q)} [\bar{a}(\rho, \zeta(q))^T \times \bar{v}(\zeta(q), x)] e^{qs} dq,$$

for any  $q_0 > 0$ . It turns out that it is quite easy to compute the above integral numerically using the technique discussed in [28] coupled with a Filon-type method (see [22]). Producing each graph on Figure 3 takes around 1.2 seconds. In order to compute  $u(s, x)$ , we truncated  $a_i(\rho, \zeta)$  at  $i = 100$  and used 200 roots  $\zeta_k$  to compute these coefficients using formulas (14). We chose  $q_0 = 0.25$ , truncated integral in  $q$  at  $|\text{Im}(q)| < 10^4$ . The numerical results are presented in Figure 3. Note that both the ascending ladder height and time processes behind Figure 3(c) have a linear drift, where, as in Figure 3(d), they are both driftless compound Poisson

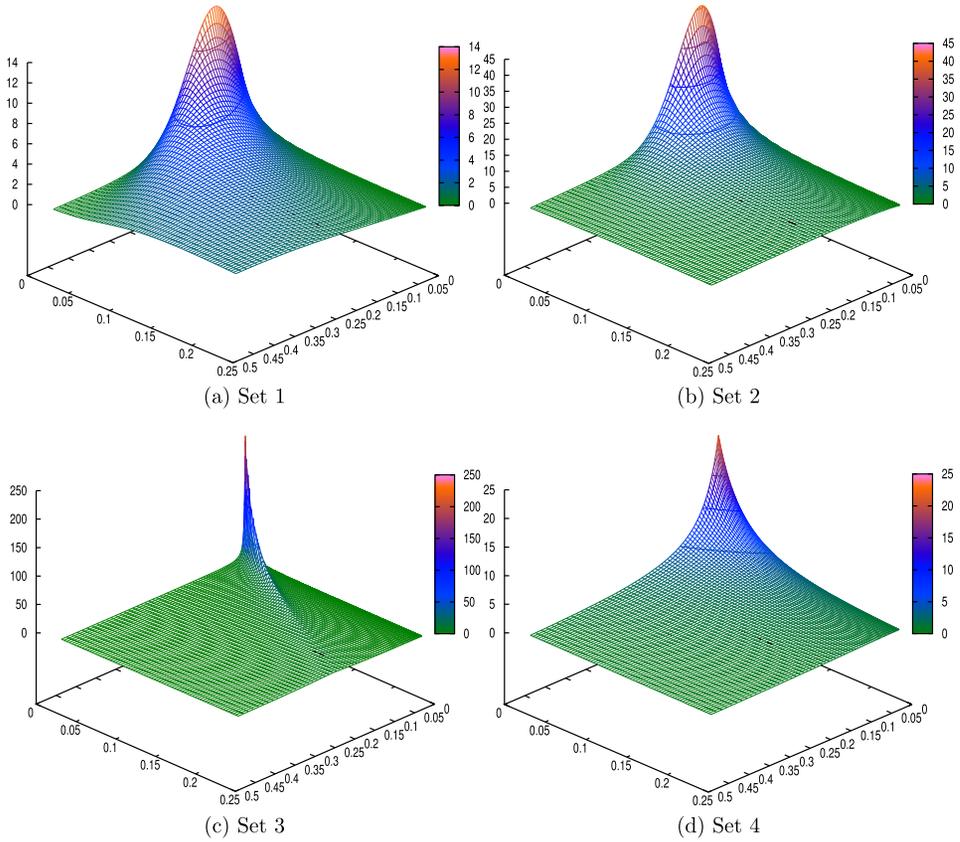


FIG. 3. Computing the density of the renewal measure  $U(ds, dx)$ ,  $s \in (0, 0.25)$  and  $x \in (0, 0.5)$ .

processes. In the former case this explains the strong concentration of mass around a linear trend, and in the latter case there exists an atom at  $x = s = 0$ , which is not visible on the graph since we are only plotting the absolutely continuous part.

As we have mentioned in the introduction, all expressions related to fluctuation identities presented in this paper have the following properties: (i) they are computed explicitly in terms of roots/poles of  $q + \Psi(iz)$  and possibly some linear algebra operations, (ii) all of them have the law of the space variables (e.g., overshoot or location of the last maximum) in closed form and (iii) they involve Laplace transform of the first passage time  $\tau_a^+$  or  $\tau_0^-$ . The third condition implies that if we want to compute joint distribution of both space and time functionals of the process (e.g., joint density of the first passage time and the overshoot), we would have to perform a Fourier transform in the  $q$ -variable, and it has to be done numerically. See [28] and [30] for examples of application of this technique. It turns out that, using exactly the same method, we can also obtain similar results for Lévy processes with stochastic volatility, which are very popular models in Mathematical Finance; see, for example, [11]. We will briefly present this technique for

the case of the two-sided exit problem considered above. Let  $T_s$  be an increasing continuous process satisfying  $T_0 = 0$ . We require that  $\mathbb{E}[\exp(qT_s)]$  is known in closed form. A classical example is the integral of the Cox–Ingersoll–Ross diffusion process, however, one can also choose several other processes; see [21]. Define a time-changed process  $Z_s = X_{T_s}$ ,  $s \geq 0$ , where we assume that  $T_s$  is independent of  $X_t$ . As before, define  $s_0^- \{s_a^+\}$  to be the first passage time of process  $Z_s$  below 0 {above  $a$ }. Since  $T_s$  is continuous, we have  $T_{s_a^+} = \tau_a^+$ , thus, we obtain for any positive  $q_0$

$$\begin{aligned} & \mathbb{E}_z[\mathbb{I}(Z_{s_a^+} \in dy; s_a^+ \leq u; s_a^+ < s_0^-)] \\ &= \mathbb{E}_z[\mathbb{I}(X_{\tau_a^+} \in dy; \tau_a^+ \leq T_u; \tau_a^+ < \tau_0^-)] \\ &= \int_{\mathbb{R}^+} \mathbb{E}_z[\mathbb{I}(X_{\tau_a^+} \in dy; \tau_a^+ \leq t; \tau_a^+ < \tau_0^-)] \mathbb{P}(T_u \in dt) \\ &= \frac{1}{2\pi i} \int_{q_0+i\mathbb{R}} g^{(q)}(a, z, dy) \mathbb{E}[e^{qT_u}] q^{-2} dq, \end{aligned}$$

where we have used the fact that

$$\mathbb{E}_z[\mathbb{I}(X_{\tau_a^+} \in dy; \tau_a^+ \leq t; \tau_a^+ < \tau_0^-)] = \frac{1}{2\pi i} \int_{q_0+i\mathbb{R}} g^{(q)}(a, z, dy) e^{qt} q^{-2} dq,$$

which follows from (49) by inverting the Laplace transform in  $q$ . Thus, we see that if we choose the time change process  $\{T_s : t \geq 0\}$  for which the Laplace transform  $\mathbb{E}[\exp(qT_s)]$  is known in closed form, then computing quantities for the time-changed process  $Z$  is essentially identical to computing the same quantities for the process  $X$  itself.

## REFERENCES

- [1] ALILI, L. and KYPRIANOU, A. E. (2005). Some remarks on first passage of Lévy processes, the American put and pasting principles. *Ann. Appl. Probab.* **15** 2062–2080. [MR2152253](#)
- [2] AVRAM, F., KYPRIANOU, A. E. and PISTORIUS, M. R. (2004). Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Probab.* **14** 215–238. [MR2023021](#)
- [3] BAURDOUX, E. J. (2009). Some excursion calculations for reflected Lévy processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **6** 149–162. [MR2506862](#)
- [4] BAURDOUX, E. J. and KYPRIANOU, A. E. (2009). The Shepp–Shiryaev stochastic game driven by a spectrally negative Lévy process. *Theory Probab. Appl.* **53** 481–499.
- [5] BERTOIN, J. (1996). *Lévy Processes. Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. [MR1406564](#)
- [6] BERTOIN, J. (1997). Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval. *Ann. Appl. Probab.* **7** 156–169. [MR1428754](#)
- [7] BIFFIS, E. and KYPRIANOU, A. E. (2010). A note on scale functions and the time value of ruin for Lévy insurance risk processes. *Insurance Math. Econom.* **46** 85–91. [MR2586158](#)
- [8] BOROVKOV, A. A. (1976). *Stochastic Processes in Queueing Theory. Applications of Mathematics* **4**. Springer, New York. [MR0391297](#)

- [9] CABALLERO, M. E. and CHAUMONT, L. (2006). Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.* **43** 967–983. [MR2274630](#)
- [10] CABALLERO, M. E., PARDO, J. C. and PEREZ, J. L. (2010). On the Lamperti stable processes. *Probab. Math. Statist.* **30** 1–28.
- [11] CARR, P., GEMAN, H., MADAN, D. B. and YOR, M. (2003). Stochastic volatility for Lévy processes. *Math. Finance* **13** 345–382. [MR1995283](#)
- [12] ČEBOTAREV, N. G. and MEĬIMAN, N. N. (1949). The Routh–Hurwitz problem for polynomials and entire functions. Real quasipolynomials with  $r = 3, s = 1$ . *Tr. Mat. Inst. Steklova* **26** 331. [MR0033890](#)
- [13] CHAUMONT, L., KYPRIANOU, A. E. and PARDO, J. C. (2009). Some explicit identities associated with positive self-similar Markov processes. *Stochastic Process. Appl.* **119** 980–1000. [MR2499867](#)
- [14] DONEY, R. A. (2007). *Fluctuation Theory for Lévy Processes. Lecture Notes in Math.* **1897**. Springer, Berlin. [MR2320889](#)
- [15] DONEY, R. A. and KYPRIANOU, A. E. (2006). Overshoots and undershoots of Lévy processes. *Ann. Appl. Probab.* **16** 91–106. [MR2209337](#)
- [16] ES-SAGHOUBANI, A. and MANDJES, M. (2008). On the correlation structure of a Lévy-driven queue. *J. Appl. Probab.* **45** 940–952. [MR2484153](#)
- [17] GERBER, H. U. and SHIU, E. S. W. (1997). The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. *Insurance Math. Econom.* **21** 129–137. [MR1604928](#)
- [18] GETTOOR, R. K. (1961). First passage times for symmetric stable processes in space. *Trans. Amer. Math. Soc.* **101** 75–90. [MR0137148](#)
- [19] HILBERINK, B. and ROGERS, L. C. G. (2002). Optimal capital structure and endogenous default. *Finance Stoch.* **6** 237–263. [MR1897961](#)
- [20] HUBALEK, F. and KYPRIANOU, A. E. (2011). Old and new examples of scale functions for spectrally negative Lévy processes. In *Sixth Seminar on Stochastic Analysis, Random Fields and Applications* (R. Dalang, M. Dozzi and F. Russo, eds.) 119–146. Birkhäuser, Boston.
- [21] HURD, T. R. and KUZNETSOV, A. (2008). Explicit formulas for Laplace transforms of stochastic integrals. *Markov Process. Related Fields* **14** 277–290. [MR2437532](#)
- [22] ISERLES, A. (2004). On the numerical quadrature of highly-oscillating integrals. I. Fourier transforms. *IMA J. Numer. Anal.* **24** 365–391. [MR2068828](#)
- [23] JEANNIN, M. and PISTORIUS, M. (2010). A transform approach to compute prices and Greeks of barrier options driven by a class of Lévy processes. *Quant. Finance* **10** 629–644. [MR2676789](#)
- [24] KADANKOV, V. F. and KADANKOVA, T. V. (2005). On the distribution of the first exit time from an interval and the value of the overjump across a boundary for processes with independent increments and random walks. *Ukrain. Mat. Zh.* **57** 1359–1384. [MR2219768](#)
- [25] KADANKOVA, T. and VERAVERBEKE, N. (2007). On several two-boundary problems for a particular class of Lévy processes. *J. Theoret. Probab.* **20** 1073–1085. [MR2359069](#)
- [26] KONSTANTOPOULOS, T., KYPRIANOU, A. E., SALMINEN, P. and SIRVIÖ, M. (2008). Analysis of stochastic fluid queues driven by local-time processes. *Adv. in Appl. Probab.* **40** 1072–1103. [MR2488533](#)
- [27] KOU, S. (2002). A jump diffusion model for option pricing. *Management Science* **48** 1086–1101.
- [28] KUZNETSOV, A. (2010). Wiener–Hopf factorization and distribution of extrema for a family of Lévy processes. *Ann. Appl. Probab.* **20** 1801–1830. [MR2724421](#)
- [29] KUZNETSOV, A. (2010). Wiener–Hopf factorization for a family of Lévy processes related to theta functions. *J. Appl. Probab.* **47** 1023–1033. [MR2752893](#)

- [30] KUZNETSOV, A., KYPRIANOU, A. E., PARDO, J. C. and VAN SCHAİK, K. (2011). A Wiener–Hopf Monte Carlo simulation technique for Lévy processes. *Ann. Appl. Probab.* **21** 2171–2190. [MR2895413](#)
- [31] KYPRIANOU, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin. [MR2250061](#)
- [32] KYPRIANOU, A. E., PARDO, J. C. and RIVERO, V. (2010). Exact and asymptotic  $n$ -tuple laws at first and last passage. *Ann. Appl. Probab.* **20** 522–564. [MR2650041](#)
- [33] KYPRIANOU, A. E. and RIVERO, V. (2008). Special, conjugate and complete scale functions for spectrally negative Lévy processes. *Electron. J. Probab.* **13** 1672–1701. [MR2448127](#)
- [34] KYPRIANOU, A. E. and SURYA, B. A. (2007). Principles of smooth and continuous fit in the determination of endogenous bankruptcy levels. *Finance Stoch.* **11** 131–152. [MR2284015](#)
- [35] LEVIN, B. Y. (1996). *Lectures on Entire Functions. Translations of Mathematical Monographs* **150**. Amer. Math. Soc., Providence, RI. [MR1400006](#)
- [36] LEWIS, A. L. and MORDECKI, E. (2008). Wiener–Hopf factorization for Lévy processes having positive jumps with rational transforms. *J. Appl. Probab.* **45** 118–134. [MR2409315](#)
- [37] MCKEAN, H. (1965). Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics. *Ind. Manag. Rev.* **6** 32–39.
- [38] ROGERS, L. C. G. (1983). Wiener–Hopf factorization of diffusions and Lévy processes. *Proc. Lond. Math. Soc.* (3) **47** 177–191. [MR0698932](#)
- [39] ROGOZIN, B. A. (1972). The distribution of the first hit for stable and asymptotically stable walks on an interval. *Theor. Probab. Appl.* **17** 332–338.
- [40] SCHILLING, R. L., SONG, R. and VONDRAČEK, Z. (2010). *Bernstein Functions: Theory and Applications. de Gruyter Studies in Mathematics* **37**. de Gruyter, Berlin. [MR2598208](#)
- [41] VIGON, V. (2002). Simplifiez vos Lévy en titillant la factorisation de Wiener–Hopf. Thèse de doctorat de l’INSA de Rouen.
- [42] WANG, S. and ZHANG, C. (2007). First-exit times and barrier strategy of a diffusion process with two-sided jumps. Preprint.

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