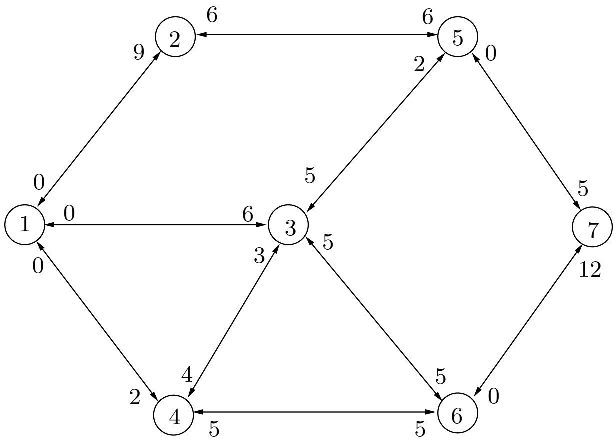
Solutions: Week 9

Solution 28 The given matrix can be represented with the following network. Next we introduce the flows

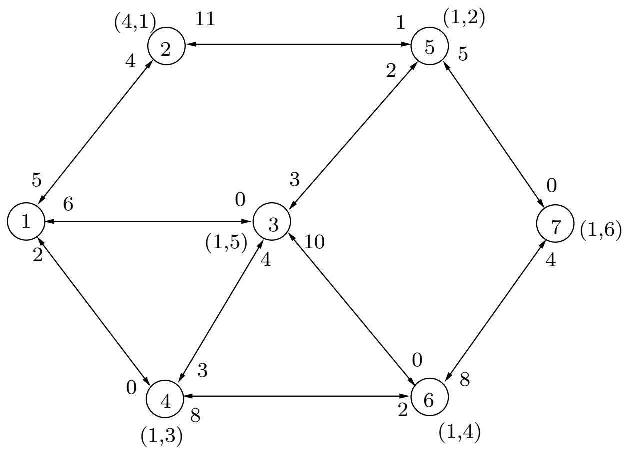


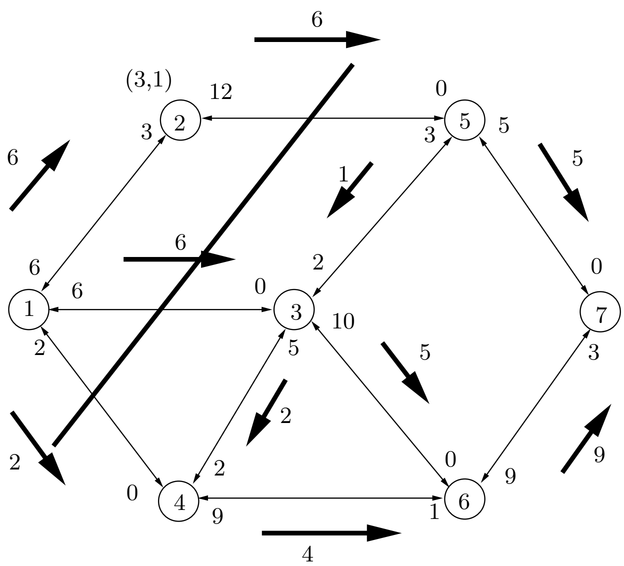
$$
\begin{aligned}
1 \to 2 \to 5 \to 7 &\quad : \quad \text{flow } 5 \\
1 \to 3 \to 6 \to 7 &\quad : \quad \text{flow } 5 \\
1 \to 4 \to 6 \to 7 &\quad : \quad \text{flow } 2 \\
1 \to 3 \to 4 \to 6 \to 7 &\quad : \quad \text{flow } 1
\end{aligned}
$$

which results in the following labelled diagram of residual capacities. From this diagram we identify an outstanding arc from source to sink,

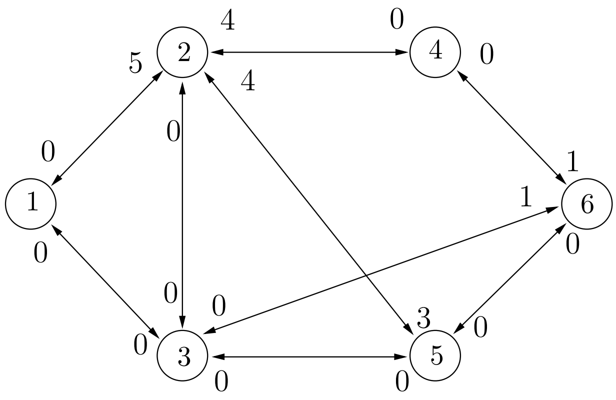
$$1 \to 2 \to 5 \to 3 \to 4 \to 6 \to 7 : \text{ flow } 1$$

and we come to a final labelled diagram The minimal cut has been indicated along with final flows. Note that the flow from the source agrees with the original capacities across the minimal cut and that is 14.

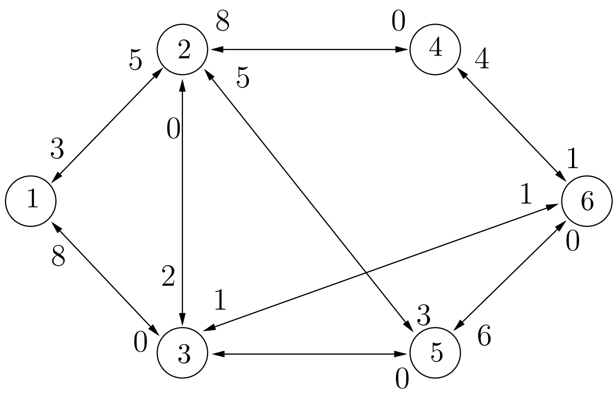

(4,1)
11
1
(1,2)
2
5
4
2
5
5
3
0
6
0
1
3
7
(1,6)
(1,5)
10
2
4
4
3
0
0
8
4
6
8
2
(1,3)
(1,4)


6
(3,1)
12
0
2
5
5
3
3
6
5
1
6
2
0
6
6
0
1
3
7
2
10
5
3
5
2
2
2
0
9
9
0
2
4
6
9
1
4

Solution 29 Constructing the diagram of the residual capacities **only in the direction of the specified flows** one obtains the following picture.



If one augments the capacities correctly then in fact one should obtain the following diagram



and the path

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 6: \text{ flow } 1$$

becomes clear.

Solution 30 As usual suppose that for a network (γ, N), if $N = \{1, ..., n\}$ we can let the source node be 1 and the sink node be n. The problem becomes to

$$
\begin{aligned}
&\text{maximise } z = \textstyle\sum_{j=2}^{n} x_{1j} \\
&\text{subject to:} \\
&\textstyle\sum_{i=1}^{n} x_{ik} - \sum_{j=1}^{n} x_{kj} = 0 \; k = 2, ..., n-1 \\
&0 \leq x_{ij} \leq \gamma(i,j) \; i, j = 1, ..., n
\end{aligned}
$$

In that case we may think of the flow (as we have defined it in lectures) as $f(i,j) = x_{ij} - x_{ji}$.