Solutions: Week 7

Solution 22 The given data for the transportation problem is given by

	55	70	35	40
80	13	11	18	17
100	2	14	10	1
20	5	8	18	11

We begin with the matrix method to generate a solution and obtain the following sequence of tables (not that the choice $u_1 = 0$ is always made):

	55	70	35	40	$u_i \downarrow$
80	0 $_{[13]}$	$50^{+\eta}$ $_{[11]}$	$30^{-\eta}$ $_{[18]}$	0 $_{[17]}$	0
100	$55^{-\eta}$ $_{[2]}$	0 $_{[14]}$	$5^{+\eta}$ $_{[10]}$	40 $_{[1]}$	-8
20	$0^{+\eta}$ $_{[5]}$	$20^{-\eta}$ $_{[8]}$	0 $_{[18]}$	0 $_{[11]}$	-3
$v_j \rightarrow$	10	11	18	9	$\cdot$

Necessarily must take $\eta = 20$.

	55	70	35	40	$u_i \downarrow$
80	0 $_{[13]}$	70 $_{[11]}$	10 $_{[18]}$	0 $_{[17]}$	0
100	35 $_{[2]}$	0 $_{[14]}$	25 $_{[10]}$	40 $_{[1]}$	-8
20	20 $_{[5]}$	0 $_{[8]}$	0 $_{[18]}$	0 $_{[11]}$	-5
$v_j \rightarrow$	10	11	18	9	$z = 1410$

in this final table $x_{ij}(u_i + v_j - c_{ij}) = 0$ for all i, j and moreover $x_{ij} \geq 0$ and $u_i + v_j \leq c_{ij}$ for all i, j. Hence, solution is optimal and one may check that

$$\sum_{i,j} x_{ij} c_{ij} = 1410.$$

Solution 23 The given data for the transportation problem is not balanced and hence it is necessary to introduce a dump to consume the residual 4 units of supply over demand so that our cost matrix should actually look like

	X	Y	Z	Dump
A	50	60	30	0
B	60	40	20	0
C	40	70	30	0

Starting with the North-West Corner method, we have the following sequence of tables (always choosing $u_1 = 0$):

	5	4	3	4	$u_i \downarrow$
8	$5_{[50]}$	$3^{-\eta}{}_{[60]}$	$0_{[30]}$	$0^{+\eta}{}_{[0]}$	0
5	$0_{[60]}$	$1^{+\eta}{}_{[40]}$	$3_{[20]}$	$1^{-\eta}{}_{[0]}$	-20
3	$0_{[40]}$	$0_{[70]}$	$0_{[30]}$	$3_{[0]}$	-20
$v_j \rightarrow$	50	60	40	20	$\cdot$

We must take $\eta = 1$.

	5	4	3	4	$u_i \downarrow$
8	$5^{-\eta}{}_{[50]}$	$2_{[60]}$	$0_{[30]}$	$1^{+\eta}{}_{[0]}$	0
5	$0_{[60]}$	$2_{[40]}$	$3_{[20]}$	$0_{[0]}$	-20
3	$0^{+\eta}{}_{[40]}$	$0_{[70]}$	$0_{[30]}$	$3^{-\eta}{}_{[0]}$	0
$v_j \rightarrow$	50	60	40	0	$\cdot$

Note that there are multiple 'worst offenders' and we pick one arbitrarily. We necessarily must take $\eta = 3$.

	5	4	3	4	$u_i \downarrow$
8	$2_{[50]}$	$2^{-\eta}{}_{[60]}$	$0^{+\eta}{}_{[30]}$	$4_{[0]}$	0
5	$0_{[60]}$	$2^{+\eta}{}_{[40]}$	$3^{-\eta}{}_{[20]}$	$0_{[0]}$	-20
3	$3_{[40]}$	$0_{[70]}$	$0_{[30]}$	$0_{[0]}$	-10
$v_j \rightarrow$	50	60	40	0	$\cdot$

We must take $\eta = 2$.

	5	4	3	4	$u_i \downarrow$
8	$2_{[50]}$	$0_{[60]}$	$2_{[30]}$	$4_{[0]}$	0
5	$0_{[60]}$	$4_{[40]}$	$1_{[20]}$	$0_{[0]}$	-10
3	$3_{[40]}$	$0_{[70]}$	$0_{[30]}$	$0_{[0]}$	-10
$v_j \rightarrow$	50	50	30	0	$z = 460$

We have that $x_{ij}(u_i + v_j - c_{ij}) = 0$ and $x_{ij} \geq 0$ and $u_i + v_j \leq c_{ij}$ for all i, j. One easily checks that

$$z = \sum_{i,j} x_{ij} c_{ij} = 460.$$

Solution 24 The idea is to split the source B into two subsidiary sources B_1 (the first three) and B_2 (the last two). Then we write out the cost matrix as follows (recalling that a dump is necessary):

	X	Y	Z	Dump
A	50	60	30	0
B_1	60	40	20	0
B_2	70	50	30	0
C	40	70	30	0

Note that the shipping on the last two items of B (ie B_2) have increased by 10 units as stipulated in the questions whilst shipping t the dump from B_2 is still cost free. Note that this method of splitting only works because the algorithm looks for the **cheapest** assignment of goods and increasing the prices in B_2 will naturally force preference on shipping B_1 before B_2.

The tables progress as follows from the north-west corner solution.

	5	4	3	4	$u_i \downarrow$
8	5 [50]	$3^{-\eta}$ [60]	$0^{+\eta}$ [30]	0 [0]	0
3	0 [60]	$1^{+\eta}$ [40]	$2^{-\eta}$ [20]	0 [0]	-20
2	0 [70]	0 [50]	1 [30]	1 [0]	-10
3	0 [40]	0 [70]	0 [30]	3 [0]	-10
$v_j \rightarrow$	50	60	40	10	·

Take $\eta = 2$

	5	4	3	4	$u_i \downarrow$
8	$5^{-\eta}$ [50]	1 [60]	$2^{+\eta}$ [30]	0 [0]	0
3	0 [60]	3 [40]	0 [20]	0 [0]	-20
2	0 [70]	0 [50]	$1^{-\eta}$ [30]	$1^{+\eta}$ [0]	0
3	$0^{+\eta}$ [40]	0 [70]	0 [30]	$3^{-\eta}$ [0]	0
$v_j \rightarrow$	50	60	30	0	·

Take $\eta = 1$.

	5	4	3	4	$u_i \downarrow$
8	$4^{-\eta}$ [50]	1 [60]	3 [30]	$0^{+\eta}$ [0]	0
3	0 [60]	3 [40]	0 [20]	0 [0]	-20
2	0 [70]	0 [50]	0 [30]	2 [0]	-10
3	$1^{+\eta}$ [40]	0 [70]	0 [30]	$2^{-\eta}$ [0]	-10
$v_j \rightarrow$	50	60	30	10	·

Take $\eta = 2$.

	5	4	3	4	$u_i \downarrow$
8	2 [50]	$1^{-\eta}$ [60]	3 [30]	$2^{+\eta}$ [0]	0
3	0 [60]	3 [40]	0 [20]	0 [0]	-20
2	0 [70]	$0^{+\eta}$ [50]	0 [30]	$2^{-\eta}$ [0]	0
3	3 [40]	0 [70]	0 [30]	0 [0]	-10
$v_j \rightarrow$	50	60	30	0	.

Take $\eta = 1$ to reach the final solution

	5	4	3	4	$u_i \downarrow$
8	2 [50]	0 [60]	3 [30]	3 [0]	0
3	0 [60]	3 [40]	0 [20]	0 [0]	-10
2	0 [70]	1 [50]	0 [30]	1 [0]	0
3	3 [40]	0 [70]	0 [30]	0 [0]	-10
$v_j \rightarrow$	50	50	30	0	$z = 480$

Note, here is another solution which shows that solutions need not necessarily be unique.

	5	4	3	4	$u_i \downarrow$
8	2 [50]	0 [60]	2 [30]	4 [0]	0
3	0 [60]	3 [40]	0 [20]	0 [0]	-10
2	0 [70]	1 [50]	1 [30]	0 [0]	0
3	3 [40]	0 [70]	0 [30]	0 [0]	-10
$v_j \rightarrow$	50	50	30	0	$z = 480$