## Solutions: Week 6

**Solution 19** The problem was

$$\begin{aligned}
&\text{maximise } z = 3x_1 + 6x_2 + 2x_3\\
&\text{subject to:}\\
&3x_1 + 4x_2 + 2x_3 + x_4 \qquad\quad = 200\\
&x_1 + 3x_2 + 2x_3 \qquad\qquad + x_5 = 100\\
&x_1, \cdots, x_5 \geq 0.
\end{aligned}$$

and the solution was found to be $(x_1, x_2, x_3) = (40, 20, 0)$. Complementary slackness requires that $x_i[(\mathbf{A}^{\mathrm{T}}\mathbf{y})_i - \mathbf{c}_i] = 0$ for $i = 1, 2, 3$ if the previous solution is to be optimal. Since $x_1, x_2 > 0$ this implies that for the dual variables $(y_1, y_2)$,

$$3y_1 + y_2 = 3 \text{ and } 4y_1 + 3y_2 = 6$$

which implies that $y_1 = 3/5$ and $y_2 = 6/5$. It is now easily see that $\mathbf{c} \cdot \mathbf{x} = (3 \cdot 40 + 6 \cdot 20) = 240 = (200 \cdot 3/5 + 100 \cdot 6/5) = \mathbf{b} \cdot \mathbf{y}$.

**Solution 20** The primal problem was given as

$$\begin{aligned}
&\text{maximise } z = 9x_1 + 14x_2 + 7x_3\\
&\text{subject to:}\\
&2x_1 + x_2 + 3x_3 \leq 6\\
&5x_1 + 4x_2 + x_3 \leq 12\\
&2x_2 \leq 5\\
&x_1, x_2, x_3 \text{ unrestricted.}
\end{aligned}$$

with candidate optimal solution

$$\mathbf{x}^{\mathrm{T}} = (\frac{5}{26}, \frac{5}{2}, \frac{27}{26}).$$

Since all entries of $\mathbf{x}$ are strictly positive, complementary slackness would imply that $\mathbf{A}\mathbf{y} = \mathbf{c}$. That is to say

$$\begin{aligned}
&2y_1 + 5y_2 = 9\\
&y_1 + 4y_2 + 2y_3 = 14\\
&3y_1 + y_2 = 7.
\end{aligned}$$

Solving this system one finds $(y_1, y_2, y_3) = (2, 1, 4)$. Now check that $\mathbf{c} \cdot \mathbf{x} = 44 = \mathbf{b} \cdot \mathbf{y}$.

**Solution 21** Here are two possible solutions.

a) solve by the simplex algorithm or graphically to deduce that the optimal solution is such that the first constraint is slack. In that case, the complementary slackness theorem implies that $y_1 = 0$.

b) We want to deduce that the optimal solution is such that the first constraint is slack as the Complementary Slackness Theorem would then imply that $y_1 = 0$. We thus argue for contradiction that the optimal solution $(x_1, x_2)$ is such that $x_1 + 2x_2 = 10$. In that case, making use of the equality to change each of the constraints (including positivity), the problem becomes to

$$\begin{array}{l} \text{maximise } z = 20 + x_1 \\ \text{subject to:} \\ x_1 \leq 6 \\ x_1 \leq 2 \\ x_1 \geq 0 \\ x_1 \leq 10. \end{array}$$

This suggests we should take $x_1 = 2$ to give the optimal value $z = 22$ (and $x_2 = 4$). On the other hand, here is another feasible solution which gives a greater value of $z$: $(x_1, x_2) = (8, 0)$. Thus $(2, 4)$ is not optimal and hence the first constraint must be slack for the optimal solution implying that $y_1 = 0$.