Solutions: Week 5

Solution 15 Suppose that $\mathbf{x}$ and $\mathbf{y}$ are feasible solutions to the asymmetric primal and dual respectively. Note that since $\mathbf{Ax} = \mathbf{b}$ we have that $\mathbf{y}^{\mathrm{T}}\mathbf{Ax} = \mathbf{y} \cdot \mathbf{b}$. On the other hand, since $\mathbf{x} \geq \mathbf{0}_n$ and $\mathbf{y}^{\mathrm{T}}\mathbf{A} \geq \mathbf{c}^{\mathrm{T}}$ it follows that $\mathbf{y}^{\mathrm{T}}\mathbf{Ax} \geq \mathbf{c} \cdot \mathbf{x}$. In conclusion $\mathbf{c} \cdot \mathbf{x} \leq \mathbf{y} \cdot \mathbf{b}$ as required. If $\mathbf{c} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{b}$ then the same argument as in the symmetric case tells us that both $\mathbf{x}$ and $\mathbf{y}$ are optimal for primal and dual respectively.

Solution 16 We are given a linear programming problem in canonical form

$$\begin{array}{l} \text{maximise } z = \mathbf{c} \cdot \mathbf{x} \\ \text{subject to:} \\ \mathbf{Ax} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0}_n \end{array}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ are given. A basic feasible solution may, without loss of generality, be written in the form

$$\mathbf{x} = \begin{pmatrix} \mathbf{0}_{n-m} \\ \mathbf{x_B} \end{pmatrix}$$

and partitioning $\mathbf{A}$ accordingly as $(\mathbf{A}^o|\mathbf{B})$ and similarly with $\mathbf{c}$ then the tableau associated with the next step of the simplex algorithm may be rearranged as follows:

$\mathbf{B}^{-1}\mathbf{A}^0$	$\vdots$	$\mathbf{I}_m$	$\mathbf{x_B}$
$(\mathbf{c_B}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}^0 - (\mathbf{c}^0)^{\mathrm{T}})$	$\vdots$	$\mathbf{0}_m^{\mathrm{T}}$	$\mathbf{c_B}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b}$

Note in particular that $\mathbf{Ax} = \mathbf{b}$ implies that $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$.

(a) If $\mathbf{x}$ is optimal then when the objective is written in terms of non-basic variables, then all the coefficients are non-positive. In terms of the tableau above this means simply that

$$\mathbf{c_B}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}^0 - (\mathbf{c}^0)^{\mathrm{T}} \geq \mathbf{0}_{n-m}^{\mathrm{T}}$$

which is the required inequality.

(b) We have

$$\mathbf{w}^{\mathrm{T}}\mathbf{A} = \mathbf{c_B}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A} = \mathbf{c_B}^{\mathrm{T}}\mathbf{B}^{-1}(\mathbf{A}^o|\mathbf{B}) = (\mathbf{c_B}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}^o, \mathbf{c_B}^{\mathrm{T}}) \geq ((\mathbf{c}^o)^{\mathrm{T}}, \mathbf{c_B}^{\mathrm{T}}) = \mathbf{c}^{\mathrm{T}}$$

where we have used part (a) for the inequality.

(c) Compute

$$\mathbf{b} \cdot \mathbf{w} = \mathbf{w}^{\mathrm{T}}\mathbf{b} = \mathbf{c_B}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} = \mathbf{c_B}^{\mathrm{T}}\mathbf{x_B} = \mathbf{c} \cdot \mathbf{x}$$

where we have used the fact that $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$ and that $\mathbf{c} \cdot \mathbf{x} = \mathbf{c_B} \cdot \mathbf{x_B}$.

Finally we appeal the the Weak Duality Theorem for the asymmetric dual to deduce that $\mathbf{w}$ is optimal.

Solution 17 Add in slack variable to convert the problem to canonical form:

$$\begin{array}{l} \text{maximise } z = 3x_1 + 6x_2 + 2x_3 \\ \text{subject to:} \\ 3x_1 + 4x_2 + 2x_3 + x_4 \qquad\quad = 200 \\ x_1 + 3x_2 + 2x_3 \qquad\quad + x_5 = 100 \\ x_1, \cdots, x_5 \geq 0. \end{array}$$

If we are to take x_1, x_2 basic then we have

$$\mathbf{B} = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix} \text{ and hence } \mathbf{B}^{-1} = \begin{pmatrix} 3/5 & -4/5 \\ -1/5 & 3/5 \end{pmatrix}.$$

Moreover we have $\mathbf{c_B}^{\mathrm{T}} = (3, 6)$. Hence constructing the tableau as mentioned at the beginning of the previous exercise we have

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	-2/5	3/5	-4/5	40
x_2	0	1	4/5	-1/5	3/5	20
	0	0	8/5	3/5	6/5	240

Note that the Dual problem is

$$\begin{array}{l} \text{minimise } z' = 200w_1 + 100w_2 \\ \text{subject to:} \\ 3w_1 + w_2 \geq 3 \\ 4w_1 + 3w_2 \geq 6 \\ 2w_1 + 2w_2 \geq 2 \\ w_1, w_2 \geq 0. \end{array}$$

According to the conclusion of the previous exercise we have that $(x_1, x_2, x_3) = (40, 20, 0)$ is the optimal solution to the primal implies that $\mathbf{c_B}^{\mathrm{T}}\mathbf{B}^{-1} = (w_1, w_2) = (3/5, 6/5)$ is the optimal solution to the dual with common optimal values $\mathbf{c} \cdot \mathbf{x} = 240 = \mathbf{b} \cdot \mathbf{w}$.

Solution 18 The original problem was

$$\begin{aligned} &\text{maximise } z = 250x_1 + 300x_2 + 400x_3 \\ &\text{subject to:} \\ &5x_1 + 3x_2 + 5x_3 \leq 100 \\ &3x_1 + 5x_2 + 5x_3 \leq 80 \\ &x_1, x_2, x_3 \geq 0. \end{aligned}$$

After adding slack variables u_1, u_2 the terminal tableau was

	x_1	x_2	x_3	u_1	u_2	
x_1	1	-1	0	$1/2$	$-1/2$	10
x_3	0	$8/5$	1	$-3/10$	$1/2$	10
	0	90	0	5	75	6500

Note that

$$\mathbf{B} = \begin{pmatrix} 5 & 5 \\ 3 & 5 \end{pmatrix} \text{ and } \mathbf{B}^{-1} = \frac{1}{10}\begin{pmatrix} 5 & -5 \\ -3 & 5 \end{pmatrix}.$$

Moreover $\mathbf{c_B}^{\mathrm{T}} = (250, 400)$.

The dual problem is written

$$\begin{aligned} &\text{minimise } z' = 100w_1 + 80w_2 \\ &\text{subject to:} \\ &5w_1 + 3w_2 \geq 250 \\ &3w_1 + 5w_2 \geq 300 \\ &5w_1 + 5w_2 \geq 400 \\ &w_1, w_2 \geq 0. \end{aligned}$$

We are told in the solution to Exercise 16 that $\mathbf{w}^{\mathrm{T}} = \mathbf{c_B}^{\mathrm{T}}\mathbf{B}^{-1}$ and a little algebra shows that this is equal to $(5, 75)$ and the optimal values to primal and dual objectives agree at 6500.

The total 'valuation' of red ore is 5 units of currency per ton and of black ore is 75 units per ton. Total valuation of resources equals total profit made. Valuation of ingredients needed for 1 ton of hard alloy is $(3\times5)+(5\times75) = 390$ units of currently. None is made in optimal plan as profit per ton is only 300 units of currency. Only economic to produce hard alloy if profit per ton increased to 390 units of currency.