Solutions: Week 3

Solution 8 Let x_1 be number of tons of Soft alloy, x_2 the number of tons of Hard alloy and x_3 the number of tons of Strong alloy. The problem is:

$$\begin{aligned}
&\text{maximise } z = 250x_1 + 300x_2 + 400x_3\\
&\text{subject to:}\\
&5x_1 + 3x_2 + 5x_3 \leq 100\\
&3x_1 + 5x_2 + 5x_3 \leq 80\\
&x_1, x_2, x_3 \geq 0.
\end{aligned}$$

Adding slack variables, we have the following sequence of tableaus when performing the simplex algorithm.

We start with the initial solution $(x_1, x_2, x_3, u_1, u_2) = (0, 0, 0, 100, 80)$.

	x_1	x_2	$\mathbf{x_3}$	u_1	u_2	
u_1	5	3	5	1	0	100
$\mathbf{u_2}$	3	5	**5**	0	1	80
	-250	-300	-400	0	0	0

Variable x_3 replaces u_2, row 1 $\rightarrow$ row 1 - row 2 and then row2 $\rightarrow \frac{1}{5}$row 2.

	$\mathbf{x_1}$	x_2	x_3	u_1	u_2	
$\mathbf{u_1}$	**2**	-2	0	1	-1	20
x_3	3/5	1	1	0	1/5	16
	-10	100	0	0	80	6400

Now x_1 enters and u_1 leaves the basis, row 2 $\rightarrow$ row 2 - $\frac{3}{10}$ row 1 and then row 1 $\rightarrow \frac{1}{2}$row1.

	x_1	x_2	x_3	u_1	u_2	
x_1	1	-1	0	1/2	$-1/2$	10
x_3	0	8/5	1	$-3/10$	1/2	10
	0	90	0	5	75	6500

Optimal solution $(x_1, x_2, x_3, u_1, u_2) = (10, 0, 10, 0, 0)$ and $z = 6500$.

Solution 9 Suppose at any stage of the simplex algorithm, the current tableau is represented in the form

$\mathbf{A}$	$\vdots$	$\mathbf{I}_m$	$\mathbf{x_B}$
$-\mathbf{o}^{\mathrm{T}}$	$\vdots$	$\mathbf{0}_m^{\mathrm{T}}$	z

then if β is the set of indicies in the basis, then for each $i \in \beta$ we have

$$\sum_{j \notin \beta} a_{ij} x_j + x_i = (\mathbf{x_B})_i$$

(in particular $x_j = 0$ for all $j \notin \beta$) and the current value of the objective is

$$z + \sum_{j \notin \beta} o_j x_j$$

(in particular the sum gives zero contribution). Now set $x_l = \theta > 0$ where $l \notin \beta$ and set $x_i = (\mathbf{x_B})_i - a_{il}\theta$ for $i \in \beta$. Thanks to the fact that $a_{il} \leq 0$ for all i it follows that for any $\theta > 0$ we have created a feasible solution. Moreover, the associated objective is given by $z + o_l\theta$. Hence since $o_j > 0$ we may choose θ arbitrarily large so that the objective becomes arbitrarily large.

Solution 10 We set the problem out immediately in tableau form.

	x_1	$\mathbf{x_2}$	x_3	u_1	u_2	u_3	
$\mathbf{u_1}$	3	**1**	-4	1	0	0	4
u_2	1	-1	-1	0	1	0	10
u_3	1	-2	6	0	0	1	9
	1	-2	-1	0	0	0	0

Introducing x_2 into the basis in place of u_1 gives us (row 2 + row1, row3 + 2×row1)

	x_1	x_2	x_3	u_1	u_2	u_3	
x_2	3	1	-4	1	0	0	4
u_2	4	0	-5	1	1	0	14
u_3	7	0	-2	2	0	1	17
	7	0	-9	2	0	0	8

Now note that $o_3 > 0$ and $a_{i3} \leq 0$ for all i. Hence the previous question indicates the objective function can be made arbitrarily large. Setting for example $x_3 = 111$ gives $x_2 = 448$, $x_3 = 0$, $u_1 = 0$, $u_2 = 569$, $u_3 = 239$ and $z = 1007$.