Solutions: Week 2

Solution 5 The associated problem in canonical form (C) is to

$$\begin{aligned}&\text{maximise } z'(\mathbf{y}) = \mathbf{c}' \cdot \mathbf{y}\\&\text{subject to:}\\&\mathbf{A}'\mathbf{y} = \mathbf{b}\\&\mathbf{y} \geq \mathbf{0}_{n+m}\end{aligned}$$

where $\mathbf{A} = (\mathbf{A}\vdots\mathbf{I}_m)$, $\mathbf{c}' = \binom{\mathbf{c}}{\mathbf{0}_m}$, $\mathbf{y} = \binom{\mathbf{x}}{\mathbf{u}}$ and $\mathbf{u} \in \mathbb{R}^m$.

We present two lines of reasoning which will be used for the remainder of the proof.

(i) Suppose now that (C) feasible solution. Let $\mathbf{y}^* \in \mathbb{R}^{n+m}$ be any solution in the feasible region of (C). Write it in the form $\mathbf{y}^* = \binom{\mathbf{x}^*}{\mathbf{u}^*}$ where $\mathbf{x}^* \in \mathbb{R}^n$. As $\mathbf{A}'\mathbf{y} = \mathbf{b}$ it follows that $\mathbf{A}\mathbf{x}^* + \mathbf{u}^* = \mathbf{b}$. Moreover, since $\mathbf{y}^* \geq \mathbf{0}_{n+m}$ it follows that $\mathbf{A}\mathbf{x}^* \leq \mathbf{b}$. Hence the restriction of $\mathbf{y}^*$ to $\mathbb{R}^n$ is a solution to (S). Since $z'(\mathbf{y}^*) = \mathbf{c} \cdot \mathbf{x}^* = z(\mathbf{x}^*)$ and since $\mathbf{y}^*$ is arbitrary, we have

$$z'_{max} := \sup_{\mathbf{y}^*} z'(\mathbf{y}^*) = \sup_{\mathbf{x}^*} z'(\mathbf{x}^*) \leq z_{max}.$$

where z_{max} is the maximal value of the objective of (S) (which may be infinite).

(ii) Now suppose there exists a feasible solution $\mathbf{x}^{**}$ to (S). From this solution, define $\mathbf{y}^{**} = \binom{\mathbf{x}^{**}}{\mathbf{u}^{**}}$ where $\mathbf{u}_i^{**} = b_i - (\mathbf{A}\mathbf{x}^{**})_i$ for $i = 1, ..., m$. Note then that $\mathbf{A}'\mathbf{y}^{**} = \mathbf{b}$ and $\mathbf{y}^{**} \geq \mathbf{0}_{n+m}$. Moreover $z'(\mathbf{y}^{**}) = \mathbf{c} \cdot \mathbf{x}^{**} = z(\mathbf{x}^{**})$. Since $\mathbf{x}^{**}$ is arbitrarily chosen in the feasible region, it follows that

$$z'_{max} \geq \sup_{\mathbf{y}^{**}} z'(\mathbf{y}^{**}) = \sup_{\mathbf{x}^{**}} z(\mathbf{x}^{**}) = z_{max}.$$

(Note z'_{max} may be infinite).

Now suppose that (C) has an optimal solution. It follows by (i) that a solution to (S) exists and $z'_{max} \leq z_{max}$ (the latter may be infinite). However, since a solution to (S) exists, the argument in (ii) shows that $z_{max} \leq z'_{max}$ and hence $z_{max} = z'_{max}$. Moreover, the restriction of the optimal solution to (C) is an optimal solution to (S). To see this note that in (i) if $\mathbf{y}^*$ is the optimal solution then with just established fact that $z_{max} = z'_{max}$ we have

$$z_{max} = z'_{max} = \mathbf{c}' \cdot \mathbf{y}^* = \mathbf{c} \cdot \mathbf{x}^*.$$

That is to say, the restriction of the optimal solution $\mathbf{y}^*$ to (C) is optimal for (S).

The converse statement can be handled similarly.

Solution 6 In canonical form:

$$\begin{array}{l}
\text{maximise } z = 3x_1 + 2x_2 \\
\text{subject to:} \\
x_1 + x_2 + u_1 \qquad\quad = 4 \\
x_1 + 2x_2 \qquad + u_2 = 6 \\
x_1, x_2, u_1, u_2 \geq 0.
\end{array}$$

Basic solutions are $(x_1, x_2, u_1, u_2) =$

$$\begin{array}{l}
(2, 2, 0, 0)\ z = 10 \\
(6, 0, -2, 0) \text{ not feasible} \\
(4, 0, 0, 2)\ z = 12 \\
(0, 3, 1, 0)\ z = 6 \\
(0, 4, 0, -2) \text{ not feasible} \\
(0, 0, 4, 6)\ z = 0.
\end{array}$$

Optimal solution $(4, 0, 0, 2)$ with $z = 12$.

Solution 7 Look back to the proof of the fundamental theorem and note that reducing an optimal solution to an optimal basic feasible solution does not use the fact that the solution is optimal. It shows in fact that any feasible solution (and in particular the optimal solution) may be reduced to a basic feasible solution with the same objective value (and hence an optimal solution is reduced to a basic optimal solution).