University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES
EXAMINATION

MA30087/50087: OPTIMISATION METHODS OF OPERATIONAL RESEARCH

2008-2009

Candidates may use university-supplied calculators.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

1. Below is a standard linear programming problem and its *symmetric dual.*

$$(\mathrm{P}): \begin{array}{l} \text{maximise } \mathbf{c}\cdot\mathbf{x} \\ \text{subject to} \\ \mathbf{A}\mathbf{x} \le \mathbf{b} \\ \mathbf{x} \ge \mathbf{0}_n \end{array} \quad \text{and} \quad (\mathrm{D}): \begin{array}{l} \text{minimise } \mathbf{b}\cdot\mathbf{y} \\ \text{subject to} \\ \mathbf{A}^{\mathrm{T}}\mathbf{y} \ge \mathbf{c} \\ \mathbf{y} \ge \mathbf{0}_m \end{array}$$

where $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$, $\mathbf{y}, \mathbf{b} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m\times n}$.

(a) Show that if $\mathbf{x}$ is feasible for (P), $\mathbf{y}$ is feasible for (D) and $\mathbf{c}\cdot\mathbf{x} = \mathbf{b}\cdot\mathbf{y}$ then $\mathbf{x}$ and $\mathbf{y}$ are optimal for (P) and (D). [7]

(b) Show that if the objective in (P) can be made arbitrarily large then (D) has no feasible solution. [5]

(c) Consider now the primal linear programming problem in linear form.

$$(\mathrm{P})^*: \begin{array}{l} \text{maximise } \mathbf{c}\cdot\mathbf{x} \\ \text{subject to} \\ \mathbf{A}\mathbf{x} = \mathbf{b} \\ \mathbf{x} \ge \mathbf{0}_n \end{array}$$

by writing the constraint $\mathbf{A}\mathbf{x} = \mathbf{b}$ as two inequalities derive its *asymmetric dual* from (D). [6]

(d) The Duality theorem states that (P) has a finite optimal solution if and only if (D) does. Using the same trick as in part (c) deduce that $(\mathrm{P})^*$ has a finite optimal solution if and only if the asymmetric dual does. [2]

2. Consider the following linear programming problems:

$$\begin{aligned}
&\text{minimise } z = -3x_1 + 9x_2 + 2x_3 + 4x_4 - x_5 - x_6\\
&\text{subject to:}\\
&x_1 - 8x_4 + 10x_6 = 1\\
&x_2 + x_3 - 11x_4 + 13x_6 = 7\\
&5x_2 + \alpha x_4 + x_5 + 2x_6 = 12\\
&x_1, \cdots, x_6 \geq 0
\end{aligned}$$

where $\alpha \in \mathbb{R}$.

(a) With the help of a simplex tableau, explain in detail why the objective, z, can be made arbitrarily large when $\alpha < -2$. [10]

(b) Show that when $\alpha > -2$ show that there is an optimal solution stating clearly the associated values of $x_1, \cdots, x_6$ and the optimal value of the objective. [4]

(c) Use the complementary slackness theorem in conjunction with your answer to part (b) to derive the solution to the dual problem, again stating clearly the associated values of the dual variables. [4]

(d) What happens when $\alpha = -2$? Comment briefly with justification. [2]

3. (a) (i) Consider the transportation problem in equilibrium:

$$\begin{aligned}
&\text{minimise } z = \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
&\text{subject to:} \\
&\textstyle\sum_{j=1}^{n} x_{ij} = s_i \quad i = 1, ..., m \\
&\textstyle\sum_{i=1}^{m} x_{ij} = d_j \quad j = 1, ..., n \\
&x_{ij} \geq 0 \quad \forall i = 1, ..., m \text{ and } j = 1, ..., n.
\end{aligned}$$

where $\{s_i : i = 1, \cdots, m\}$ are the supplies, $\{d_j : j = 1, \cdots, n\}$ are the demands, $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$ and c_{ij} is the cost of shipping from $i = 1, \cdots, m$ to $j = 1, \cdots, n$.

Write the above problem in the form of a canonical linear programming problem, as is described in part (c) of Question 1, specifically identifying the vectors $\mathbf{c}, \mathbf{b}$ and the matrix $\mathbf{A}$. [5]

Use the representation in part (i) to derive the dual to the transportation problem with the help of the asymmetric dual to the canonical linear programming problem. [4]

(ii) Starting with the North West Corner Method to produce an initial basic feasible solution, solve the transportation problem with supplies $(s_1, s_2, s_3) = (3, 4, 5)$ and demands $(d_1, d_2, d_3, d_4) = (2, 3, 4, 3)$ and cost matrix

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 7 & 6 & 2 & 3 \\ 8 & 8 & 4 & 4 \end{pmatrix}.$$

In your calculations, state clearly which theorem you are using to justify that an optimal solution has been obtained. [5]

(b) Find a basic feasible solution to the following linear programming problem

$$\begin{aligned}
&\text{minimise } 3x_1 + \tfrac{7}{2}x_2 + x_3 + x_4 - x_5 \\
&\text{subject to:} \\
&x_1 + \tfrac{5}{2}x_2 + x_4 - x_5 = 3 \\
&\tfrac{3}{2}x_2 + x_3 + x_4 - x_5 = 3 \\
&x_2 + 2x_4 - 2x_5 = 2 \\
&x_1, \cdots, x_5 \geq 0.
\end{aligned}$$

[6]

4. Consider a capacitated network (N, γ) having a single source $s \in N$ and single sink $d \in N$. Any flow f in the network satisfies the usual constraints

$$\begin{aligned}
&f(i,j) \leq \gamma(i,j) \ \forall i,j \in N \\
&f(i,j) = -f(j,i) \ \forall i,j \in N \\
&\textstyle\sum_{j\in N} f(i,j) = \sum_{j \in N} f(j,i) \ \forall i \in N \backslash \{s,d\}.
\end{aligned}$$

We are interested in maximising the flow from the orign,

$$A(f) = \sum_{j \in N} f(s,j).$$

(a) Define a **cut** (A, B) and the capacity of a cut $\gamma(A, B)$. [3]

(b) Prove that f is a maximal flow **if and only if** there exists a cut (A, B) such that $A(f) = \gamma(A, B)$. You may use in your calculations the fact that for any cut (A, B) and any flow f,

$$A(f) = f(A,B) := \sum_{i \in A} \sum_{j \in B} f(i,j)$$

[9]

(c) Find the optimal flow through the nextwork whose capacities are given by the matrix

$$\begin{pmatrix}
0 & 8 & 3 & 4 & 0 & 0 & 0 \\
4 & 0 & 3 & 0 & 3 & 4 & 0 \\
0 & 0 & 0 & 3 & \infty & 0 & 0 \\
4 & 0 & 3 & 0 & 7 & 0 & 2 \\
0 & 0 & 0 & 5 & 0 & 0 & 9 \\
0 & 4 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 3 & 6 & 0
\end{pmatrix}.$$

[8]