Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH		
Academic Year 2007-2008	Examiner A. KYPRIANOU		
Semester 1	Question No. 1	Page 1 of 3	

Part		Mark

(a) Suppose $y \in F(C)$ then $A'y = b$

hence $(A \,\vdots\, I_M)\begin{pmatrix} y^0 \\ y_B \end{pmatrix} = b$

or $Ay^0 + y_B = b$ (*)

hence $y \in F(C)$ it also follows that

$$y^0 \geq 0_n \qquad (**)$$

and $y_B \geq 0_M$. Hence from (*)

$$Ay^0 \leq b \qquad (***)$$

(**) and (***) $\Rightarrow$ $y^0 \in F(S)$

We now have that $\left\{ y^0 : \begin{pmatrix} y^0 \\ y_B \end{pmatrix} \in F(C) \right\} \subseteq F(S)$

Thus $\sup_{y \in F(C)} y \cdot c' = \sup_{y \in F(C)} y^0 \cdot c$

$$\leq \sup_{x \in F(S)} x \cdot c$$

6

Total

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH		
Academic Year 2007-2008	Examiner A. KYPRIANOU		
Semester 1	Question No. 1	Page 2 of 3	

Part | Mark

(b) Now suppose $\underline{x} \in F(S)$ and note that

$$\underline{u} = \underline{b} - (A\underline{x}) \geq \underline{0}_m. \quad \text{as } \underline{x} \in F(S) \; (*)$$

Let $y^* = \begin{pmatrix} \underline{x} \\ \underline{u} \end{pmatrix}$ and note that

$$A' y^* = (A \,\vdots\, I_m) \begin{pmatrix} \underline{x} \\ \underline{u} \end{pmatrix} = A\underline{x} + \underline{u}$$

$$= A\underline{x} + (\underline{b} - A\underline{x}) = \underline{b}.$$

moreover $y^* \geq \underline{0}_{n+m}$ since $\underline{x} \geq \underline{0}_n$ (as $\underline{x} \in F(S)$)

and $\underline{u} \geq \underline{0}_m$ (see $(*)$ above).

Hence $y^* \in F(C)$.

Now note that $\left\{ \begin{pmatrix} \underline{x} \\ \underline{u} \end{pmatrix} : \underline{x} \in F(S), \underline{u} = \underline{b} - (A\underline{x}) \right\} \subseteq F(C)$. $(+)$

Hence $\sup_{\underline{x} \in F(S)} \underline{c} \cdot \underline{x} = \sup_{\underline{x} \in F(S)} \begin{pmatrix} \underline{x} \\ \underline{u} \end{pmatrix} \cdot \underline{c}'$

$$\overset{(+)}{\leq} \sup_{y \in F(C)} y \cdot \underline{c}'$$

6

Total

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH		
Academic Year 2007-2008	Examiner A. KYPRIANOU		
Semester 1	Question No. 1	Page 3	of 3

Part		Mark
(c)	Now suppose (S) has an optimal sol^n, say $\underline{x}^* \in F(S)$.	
	Then by (b) with $\underline{u}^* = \underline{b} - A\underline{x}^*$ $\begin{pmatrix} \underline{x}^* \\ \underline{u}^* \end{pmatrix} \in F(C)$ and hence $F(C) \neq \emptyset$.	
	Moreover $\sup_{\underline{x} \in F(S)} \underline{c} \cdot \underline{x} \leq \sup_{\underline{y} \in F(C)} \underline{y} \cdot \underline{c}'$	
	But then, since $F(C) \neq \emptyset$ we have by (a).	
	$\sup_{\underline{x} \in F(S)} \underline{c} \cdot \underline{x} \geq \sup_{\underline{y} \in F(C)} \underline{y} \cdot \underline{c}'$	
	Thus $\sup_{F(S)} \underline{c} \cdot \underline{x} = \sup_{F(C)} \underline{y} \cdot \underline{c}'$.	
	Note ~~that~~ also that since $\underline{x}^* \cdot \underline{c} = \sup_{F(S)} \underline{c} \cdot \underline{x}$ ~~then~~ and $\underline{x}^* \cdot \underline{c} = \begin{pmatrix} \underline{x}^* \\ \underline{u}^* \end{pmatrix} \cdot \underline{c}'$	
	it follows that $\begin{pmatrix} \underline{x}^* \\ \underline{u}^* \end{pmatrix}$ is ~~the~~ optimal sol^n for (C).	8
		Total

Similar argument for converse

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH	
Academic Year 2007-2008	Examiner A. KYPRIANOU	
Semester 1	Question No. 2	Page 1 of 4

(a) (i) Dual: minimise $z' = \underline{b} \cdot \underline{y}$

subject to:

$$A^T \underline{y} \geq \underline{c} \qquad \underline{y} \in \mathbb{R}^M$$

$$\underline{y} \geq \underline{0}_M.$$

(2)

(ii) Duality Theorem:

The primal problem (P) has a finite optimal solution if and only if the dual (D) does too in which case the objectives ~~values~~ are optimally equal. (2)

(iii) say $\underline{x}_0$ is optimal for (P) and $\underline{y}_0$ is optimal for (D).

Since $A\underline{x}_0 \leq \underline{b}$ and $\underline{y}_0 \geq \underline{0}_M$

we have $\underline{y}_0^T A \underline{x}_0 \leq \underline{y}_0^T \underline{b} = \underline{y}_0 \cdot \underline{b}$.

Similarly, since $\underline{y}_0^T A \geq \underline{c}^T$ and $\underline{x}_0 \geq \underline{0}_N$, it follows that

$$\underline{y}_0^T A \underline{x}_0 \geq \underline{c}^T \underline{x}_0 = \underline{c} \cdot \underline{x}_0.$$

But since $\underline{x}_0, \underline{y}_0$ are optimal, it follows that

Total

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH	
Academic Year 2007-2008	Examiner A. KYPRIANOU	
Semester 1	Question No. 2	Page 2 of 4

$\underline{c} \cdot \underline{x}_0 = \underline{b} \cdot y_0$ from the ~~Weak~~ Duality Theorem.

Hence
$$\underline{b} \cdot y_0 \geq y_0^T A \underline{x}_0 \geq \underline{c} \cdot \underline{x}_0 = \underline{b} \cdot y_0 .$$

So all inequalities are equalities and in particular
$$y_0^T A \underline{x}_0 = y_0^T \underline{b} .$$

or $y_0^T (A \underline{x}_0 - \underline{b}) = 0 .$

or $\sum_{i=1}^{M} (y_0)_i \left[(A \underline{x}_0)_i - b_i \right] = 0 .$

since $(A \underline{x}_0)_i - b_i \leq 0$. and $(y_0)_i > 0 \ \forall i$

it follows that all summands are non-positive and hence, since sum $= 0 \Rightarrow$ all terms are zero

$\Rightarrow (y_0)_i \left[(A \underline{x}_0)_i - b_i \right] = 0 \quad \forall i$

$\Rightarrow (A \underline{x}_0)_i - b_i = 0 \quad \forall i \quad$ as $(y_0)_i > 0 \ \forall i$

ie $\quad A \underline{x}_0 = \underline{b} .$

Mark: 6

Total

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH	
Academic Year 2007-2008	Examiner A. KYPRIANOU	
Semester 1	Question No. 2	Page 3 of 4

Part		Mark

(b) Suppose $\underline{x} = (1, 4, 0, 0, 0)$ is optimal. The dual problem is:

minimise $z' = y_1 + 4y_2$

subject to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 4 & -1 \\ -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} -1 \\ 0 \\ 1 \\ -5 \\ 1 \end{pmatrix} \qquad y_i \in \mathbb{R}, \; i = 1, 2, 3$$

(Note we have converted primal problem to a maximisation problem before writing down the dual).

$x_1 > 0 \Rightarrow y_1 = -1$

$x_2 > 0 \Rightarrow y_2 = 0.$

also remaining constraints require

$\left.\begin{matrix} \cancel{-3y_3} -3 - y_3 \geq -5 \\ 1 + 3y_3 \geq 1 \end{matrix}\right\} y_3 \in [0, 2]$

now note that $\underline{y} \cdot \underline{b} = -1 = \underline{c} \cdot \underline{x}$

Hence by ~~weak~~ duality theorem, since $(-1, 0, y_3)$ is feasible (for any $y_3 \in [0, 2]$) and the objective agrees with the objective of the primal for given $\underline{x}$ then $\underline{x}$ is optimal. (5)

	Total	

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH	
Academic Year 2007-2008	Examiner A. KYPRIANOU	
Semester 1	Question No. 2	Page 4 of 4

Part		Mark
(c)	$\underline{x} = (1,1,0)$. Assume $\underline{x}$ is optimal.	

Dual problem minimise $2y_1 + y_2$.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{s.t. } y_1, y_2 \geq 0.$$

$$\left.\begin{array}{lll} x_1 > 0 \Rightarrow & y_1 + y_2 = 0. \\ x_2 > 0 \Rightarrow & y_1 = 1. \end{array}\right\} \Rightarrow y_2 = -1$$

Which is a contradiction as need positivity for a feasible sol^n to dual.

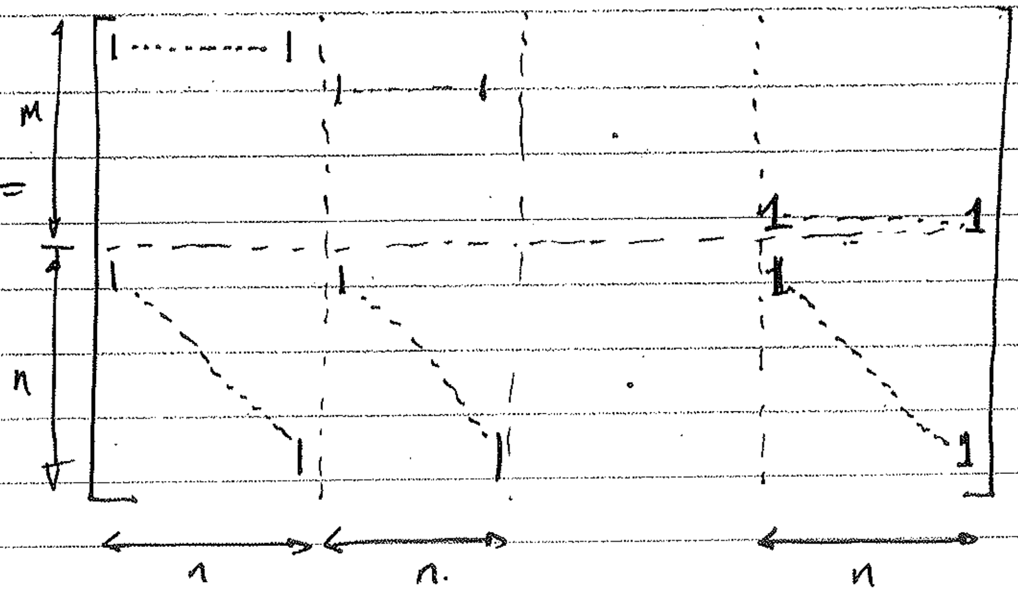
Hence $\underline{x}$ is not optimal. (5)

Total	

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH		
Academic Year 2007-2008	Examiner A. KYPRIANOU		
Semester 1	Question No. 3	Page 1 of 4	

Part | **Mark**

(a) $A =$



$\underline{b}^T = (s_1, \ldots, s_m, d_1, \ldots, d_n)$. (3)

(b) Dual: Maximise $z' = \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j$

subject to $u_i + v_j \le c_{ij} \quad \forall i, j$

$u_i, v_j \in \mathbb{R} \quad \forall i, j$. (3)

(c) (i) (x_{ij}) is not optimal because if it were then the system (u_i, v_j) would constitute a solution to the dual. However (u_i, v_j) cannot be feasible for the dual since

Total

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH	
Academic Year 2007-2008	Examiner A. KYPRIANOU	
Semester 1	Question No. 3	Page 2 of 4

Part		Mark
	$u_k + v_\ell > c_{k\ell}$.	1
(ii)	Suppose that introducing $x_{k\ell}$ contributes perturbing the table via a loop of variables through existing basic variables	
	$x_{k\ell} \to x_{kp}, \to x_{rp}, \to x_{r\circ} \to \cdots \to x_{s\circ} \to x_{s\ell} \to x_{k\ell}$.	
	Then the resulting change in the objective function is (taking $x_{k\ell} = \eta$)	
	$\eta(c_{k\ell} - c_{kp} + c_{rp} - c_{r\circ} + \cdots + c_{s\circ} - c_{s\ell})$	
	$= \eta[c_{k\ell} - (u_k + v_p) + (u_r + v_p) - (u_r + v_\circ) + \cdots + (u_s + v_\circ) - (u_s + v_\ell)]$	
	where we have used the fact that all x_{ij} in the loop are basic and hence for such (i,j) $u_i + v_j = c_{ij}$	
	Thus increase in objective is equal to	
	$\eta(c_{k\ell} - u_k - v_\ell) < 0$. ie a decrease.	4
		Total

Unit Code	Unit Title
MA30087/MA50087	OPTIMIZATION METHODS OF OPR. RESEARCH
Academic Year	**Examiner**
2007-2008	A.KYPRIANOU
Semester 1	**Question No.** 3 — **Page** 3 **of** 4

Part | Mark

(d) Transportation problem data:

	4	9	9	13
8	4	3	3	1
11	3	2	4	8
16	5	4	6	3.

Matrix method now gives first BFS.

	4	9	9	13	
8	0_4	0_3	0_3	8_1	0 ← choose $u_1=0$.
11	2_3	9_2	0_4	0_8	0
16	2_8	0_4	9_6	5_3	2
	3	2	4	1	

Note $x_{13}=0$ but $u_1+v_3=0+4>3$.
hence above sol^n not optimal. ↖ the only such (i,j)

Introduce x_{13} and remove x_{14} via the loop shown.

Total

DEPARTMENT OF MATHEMATICAL SCIENCES University of Bath

Outline solution to Examination Questions

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH	
Academic Year 2007-2008	Examiner A. KYPRIANOU	
Semester 1	Question No. 3	Page 4 of 4

	4	9	9	13	
8	0_4	0_3	8_3	0_1	0 ← choose $u_1=0$.
11	2_3	9_2	0_4	0_8	1
16	2_5	0_4	1_6	13_3	3
	2	1	3	0.	

note for all (i,j) such that $x_{ij}=0$

we have $u_i+v_j \le c_{ij}$

specifically:

$0+2 \le 4$ $(1,1)$

$0+1 \le 3$ $(1,2)$.

$0+0 \le 1$ $(1,4)$.

$1+3 \le 4$. $(2,3)$

$1+0 \le 8$ $(2,4)$.

$3+1 \le 4$ $(3,2)$.

Hence sol^n optimal

$$Z = \underbrace{\underbrace{24+6}_{30}+\underbrace{18+10}_{28}}_{58}+\underbrace{6+39}_{19.} = 103$$

Mark: 9

Total

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH		
Academic Year 2007-2008	Examiner A. KYPRIANOU		
Semester 1	Question No. 4	Page 1	of 5

Part | Mark

(a) maximise $z = 2x_1 - 3x_2 + x_3$

subject to:
$$3x_1 + 6x_2 + x_3 \le 6$$
$$4x_1 + 2x_2 + x_3 \le 4$$
$$x_1 - x_2 + x_3 \le 3$$
$$x_1, x_2, x_3 \ge 0.$$

Introduce slack variables $x_4 \to x_6$

$\theta = b_i/a_{ij}$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	$3^{\theta=2}$	6	1^{6}	1	0	0	6
x_5	4^{1}	2	1^{4}	0	1	0	4
x_6	1^{3}	-1	1^{3}	0	0	1	3
	-2	3	-1	0	0	0	

$\downarrow$ z increase by 2 $\qquad$ $\downarrow$ z increase by 3.

pivoting at circled entry:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	$2^{3/2}$	7	0	1	0	-1	3
x_5	$3^{1/3}$	3	0	0	1	-1	1
x_3	1^{3}	-1	1	0	0	1	3
	-1	2	0	0	0	1	3

Total

DEPARTMENT OF MATHEMATICAL SCIENCES

University of Bath

Outline solution to Examination Questions

Unit Code: MA30087/MA50087	Unit Title: OPTIMIZATION METHODS OF OPR. RESEARCH
Academic Year: 2007-2008	Examiner: A. KYPRIANOU
Semester: 1	Question No. 4 — Page 2 of 5

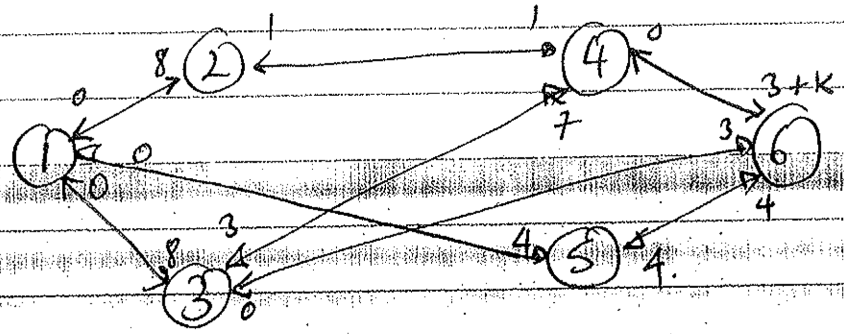
	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	5	0	1	$-2/3$	$-1/3$	$7/3$
x_1	1	1	0	0	$1/3$	$-1/3$	$1/3$
x_3	0	-2	1	0	$-1/3$	$4/3$	$8/3$
	0	3	0	0	$1/3$	$2/3$	$10/3$

This is optimal as all coefficients in final row are positive. (8)

(3)

$$\begin{bmatrix} 0 & 8 & 8 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 & 3+k \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

can be represented as



(1)

Total

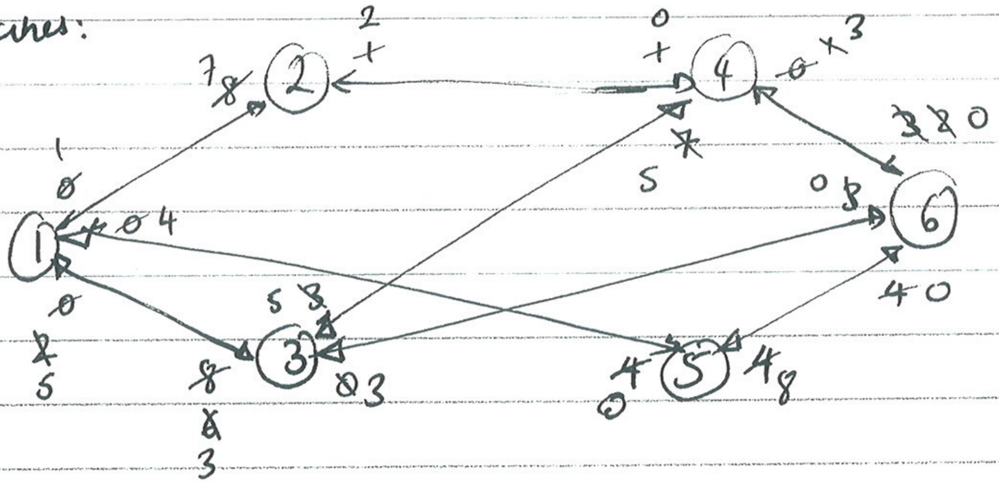
Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH		
Academic Year 2007-2008	Examiner A. KYPRIANOU		
Semester 1	Question No. 4	Page 3	of 5

Part | Mark

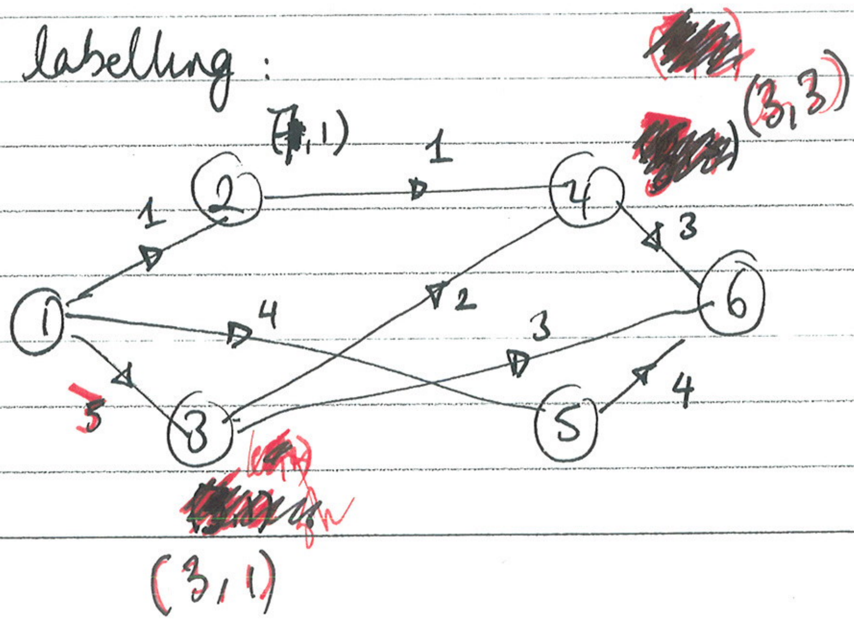
(c) Suppose $k=0$. Then introduce the following flows

$1 \to 2 \to 4 \to 6$: flow 1
$1 \to 5 \to 6$. : flow 4
$1 \to 3 \to 4 \to 6$: flow 2
$1 \to 3 \to 6$: flow 3

residual capacities:



Now labelling:



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Total

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH	
Academic Year 2007-2008	Examiner A. KYPRIANOU	
Semester 1	Question No. 4	Page 4 of 5

Part		Mark
	Sink is not labelled hence saturation.	
	max flow $= 1+4+2+3 = 10$.	6
(d)	Max-flow min-cut theorem states that if f is a maximum flow, then there exists a cut (A^*, B^*) such that	
	flow of f = capacity across (A,B).	
	and the capacity of the cut (A,B) is minimal across all cuts.	
	Cuts: $\{1\}\ \{2,\dots,6\}$ capacity $= 8+4+8=20$.	
	$\{1,2\}\ \{3,\dots,6\}$ — " — $= 8+4+1=13$	
	$\{1,2,3\}\ \{4,5,6\}$ — " — $= 1+7+4+3=15$	
	* $\{1,2,3,4\}\ \{5,6\}$ — " — $= 3+3+4=10$	
	$\{1,\dots,5\}\ \{6\}$ — " — $= 3+3+4=10$	
	minimal cut not unique — one of last two cuts.	
	*Note in proof of Max flow-Min cut theorem, the existence of a min cut is shown by looking at the labelled and non-labelled nodes as providing the cut.	2
		Total

Unit Code MA30087/MA50087	Unit Title OPTIMIZATION METHODS OF OPR. RESEARCH	
Academic Year 2007-2008	Examiner A. KYPRIANOU	
Semester 1	Question No. 4	Page 5. of 5

Part		Mark
(e)	If K is increased from zero then from (*) we can increase the flow by K along $1 \to 3 \to 4 \to 6$ for $K \in (0, 3]$.	
	for $K > 3$, can increase the flow along $1 \to 3 \to 4 \to 6$ by 3.	
	Maximal flow is thus $10 + (K \wedge 3)$.	3
		Total