University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA30087/50087: OPTIMISATION METHODS OF OPERATIONAL RESEARCH

Monday 14th January 2008 16:30–18:30

Candidates may use university-supplied calculators.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

1. Below is a linear programming problem in standard and canonical form respectively.

$$\text{(S)}: \begin{array}{l} \text{maximise } z(\mathbf{x}) = \mathbf{c}\cdot\mathbf{x} \\ \text{subject to:} \\ \mathbf{A}\mathbf{x} \le \mathbf{b} \\ \mathbf{x} \ge \mathbf{0}_n \end{array} \quad \text{and} \quad \text{(C)}: \begin{array}{l} \text{maximise } z'(\mathbf{y}) = \mathbf{c}'\cdot\mathbf{y} \\ \text{subject to:} \\ \mathbf{A}'\mathbf{y} = \mathbf{b} \\ \mathbf{y} \ge \mathbf{0}_{n+m} \end{array}$$

where $\mathbf{x}, \mathbf{c} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{A}' = (\mathbf{A}\vdots\mathbf{I}_m)$, $\mathbf{I}_m$ is the m-dimensional identity matrix, $\mathbf{y} \in \mathbb{R}^{m+n}$ and

$$\mathbf{c}' = \begin{pmatrix} \mathbf{c} \\ \mathbf{0}_m \end{pmatrix}.$$

Write $F(S)$ for the feasible region of (S) and $F(C)$ for the feasible region of (C).

(a) Suppose that $F(C) \neq \emptyset$. Show that if $\mathbf{y} \in F(C)$ and $\mathbf{y}$ is written in the form

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}^o \\ \mathbf{y}_\mathbf{B} \end{pmatrix}$$

where $\mathbf{y}^o \in \mathbb{R}^n$, then $\mathbf{y}^o \in F(S)$.

Show moreover that

$$\sup_{\mathbf{x}\in F(S)} z(\mathbf{x}) \ge \sup_{\mathbf{y}\in F(C)} z'(\mathbf{y}).$$

[6]

(b) Now suppose that $F(S) \neq \emptyset$. Let $\mathbf{x} \in F(S)$. Show that there exists $\mathbf{u} \ge \mathbf{0}_m$ such that

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} \in F(C).$$

Show moreover that

$$\sup_{\mathbf{x}\in F(S)} z(\mathbf{x}) \le \sup_{\mathbf{y}\in F(C)} z'(\mathbf{y}).$$

[6]

(c) Use parts (a) and (b) of the question to deduce that (S) has an optimal solution if and only if (C) has an optimal solution in which case their objectives are equal. [8]

2. (a) Consider the (primal) linear programming problem

$$\begin{aligned}&\text{maximise } z = \mathbf{c}\cdot\mathbf{x}\\ &\text{subject to:}\\ &\mathbf{A}\mathbf{x} \le \mathbf{b}\\ &\mathbf{x} \ge \mathbf{0}_n\end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$.

(i) Write down the dual linear programming problem. [2]

(ii) State without proof the Duality Theorem. [2]

(iii) Suppose that the primal has an optimal solution and that the dual problem has a strictly positive optimal solution. Prove from first principles, making use of the Duality Theorem, that if $\mathbf{x}_0$ is an optimal solution to the primal, then $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$. [6]

(b) Use complementary slackness to deduce whether or not $(x_1, x_2, x_3, x_4, x_5) = (1, 4, 0, 0, 0)$ is an optimal solution to the following linear programming problem

$$\begin{aligned}&\text{minimise } z = x_1 - x_3 + 5x_4 - x_5\\ &\text{subject to:}\\ &x_1 - x_3 + 3x_4 - x_5 = 1\\ &x_2 + x_3 + 4x_4 + 2x_5 = 4\\ &-x_4 + 3x_5 = 0\\ &x_1, \ldots, x_5 \ge 0\end{aligned}$$

[5]

(c) Use complementary slackness to deduce whether or not $(x_1, x_2, x_3) = (1, 1, 0)$ is an optimal solution to the following linear programming problem

$$\begin{aligned}&\text{maximise } z = x_2\\ &\text{subject to:}\\ &x_1 + x_2 + x_3 \le 2\\ &x_1 \le 1\\ &x_1, x_2, x_3 \ge 0\end{aligned}$$

[5]

3. Consider the transportation problem in balanced form

$$\begin{aligned}&\text{minimise } z = \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ &\text{subject to:} \\ &\textstyle\sum_{j=1}^{n} x_{ij} = s_i \quad i = 1, ..., m \\ &\textstyle\sum_{i=1}^{m} x_{ij} = d_j \quad j = 1, ..., n \\ &x_{ij} \geq 0 \quad \forall i = 1, ..., m \text{ and } j = 1, ..., n\end{aligned}$$

where $c_{ij} > 0$, $s_i > 0$, $d_j > 0$ are given with $\sum_i s_i = \sum_j d_j$.

(a) Describe the matrix $\mathbf{A} \in \mathbb{R}^{(m+n)\times mn}$ and the vector $\mathbf{b}$ if the constraints are written in the form $\mathbf{A}\mathbf{x} = \mathbf{b}$. [3]

(b) Formulate the dual. [3]

(c) Suppose that $\{x_{ij} : i = 1, ..., m \text{ and } j = 1, ..., n\}$ is a non-degenerate basic feasible solution to the transportation problem and the variables $\{u_i : i = 1, ..., m\}$ and $\{v_j : j = 1, ..., n\}$ solve the system of equations $u_1 = 0$ and

$$u_i + v_j = c_{ij} \text{ for pairs } (i, j) \text{ satisfying } x_{ij} > 0.$$

Assume moreover that there exist indices (k, l) such that

$$c_{kl} - u_k - v_l < 0.$$

(i) Explain why $\{x_{ij} : i = 1, ..., m \text{ and } j = 1, ..., n\}$ is not optimal. [1]

(ii) Prove that introducing the variable x_{kl} into a new basic feasible solution will reduce the value of the objective. [4]

(d) Solve the transportation problem with supplies $(s_1, s_2, s_3) = (8, 11, 16)$, demands $(d_1, d_2, d_3, d_4) = (4, 9, 9, 13)$ and cost matrix

$$(c_{ij}) = \begin{pmatrix} 4 & 3 & 3 & 1 \\ 3 & 2 & 4 & 8 \\ 5 & 4 & 6 & 3 \end{pmatrix}$$

using the matrix method to produce the first basic feasible solution. [9]

4. (a) Solve, using the simplex algorithm, the following linear programming problem

$$\begin{aligned}&\text{maximise } z = 2x_1 - 3x_2 + x_3\\ &\text{subject to:}\\ &3x_1 + 6x_2 + x_3 \leq 6\\ &4x_1 + 2x_2 + x_3 \leq 4\\ &x_1 - x_2 + x_3 \leq 3\\ &x_1, x_2, x_3 \geq 0.\end{aligned}$$

[8]

(b) Draw the directed network whose capacities are given by the matrix

$$\begin{pmatrix} 0 & 8 & 8 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 & 3+K \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $K \geq 0$. [1]

(c) Suppose that $K = 0$. Find the maximum flow through this network from source to sink. [6]

(d) Find a minimal cut when $K = 0$ and verify that the capacity across this cut equals the maximal flow. [2]

(e) Describe how the maximal flow changes as K increases through the non-negative numbers. [3]