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Part		Mark
(a)	B corresponds to the columns of A which in turn correspond to the non-zero entries of $\underline{x}$. By def^n of x_B being a BFS $\Rightarrow$ cols of B are l.i. $\Rightarrow \exists B^{-1}$	1
(b)	Consider any other BFS y.	

Let $z(\underline{x}) = \underline{c}\cdot x$, $z(y) = \underline{c}\cdot y$.

Then $Ay = \underline{b} \Rightarrow (A^0 \mid B)\begin{pmatrix} y^0 \\ y_B \end{pmatrix} = \underline{b}$

$\Rightarrow A^0 y^0 + B y_B = b \Rightarrow y_B = B^{-1}(\underline{b} - A^0 y^0)$

$\therefore z(y) = \underline{c}^0 \cdot y^0 + \underline{c}_B \cdot y_B$
$= \underline{c}^0 \cdot y^0 + \underline{c}_B \cdot (B^{-1}\underline{b} - B^{-1}A^0 y^0)$

But $A\underline{x} = \underline{b} \Rightarrow B\underline{x}_B = \underline{b} \Rightarrow B^{-1}\underline{b} = \underline{x}_B$

Also $z(\underline{x}) = \underline{c}\cdot\underline{x} = \underline{c}_B \cdot \underline{x}_B$.

$\therefore z(y) = z(x) + (\underline{c}^0 - \underline{c}_B^T B^{-1}A^0) y^0$

Let $\eta_i = (\underline{c}^0 - \underline{c}_B^T B^{-1}A^0)_i$

then if we take y such that the i^{th} non-basic variable is put into basis and one of the basic variables taken out then $z(y)$ increases the value of

	Total

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Part | Mark

the objective $\Leftrightarrow$ $\eta_i > 0$.

Hence $z(\underline{x})$ is optimal $\Leftrightarrow$ $\eta_i \leq 0$ $\forall i = 1, \dots, n-m$

$\Leftrightarrow$ $(\underline{c}^0 - \underline{c}_B^T B^{-1} A^0)\underline{e}_i \leq 0$ $\forall i = 1, \dots, n-m$ (8)

(c) We borrow ideas from the two phase method.

Introduce artificial variables and try to solve:

maximise $z = -u_1 - u_2 - u_3$

subject to

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + x_4 + u_1 &= 3\\
2x_1 + 4x_2 + x_3 + 2x_4 + u_2 &= 12\\
x_1 + 4x_2 + 2x_3 + x_4 + u_3 &= 9
\end{aligned}
$$

(2)

Step 1 $z = -24 + 4x_1 + 10x_2 + 2x_3 + 4x_4$

	x_1	x_2	x_3	x_4	u_1	u_2	u_3	
u_1	1	2	-1	1	1	0	0	3
u_2	2	4	1	2	0	1	0	12
u_3	1	4	2	1	0	0	1	9
	-4	-10	-2	-4	0	0	0	-24

(3)

Total

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Introducing

$x_1 : x_1 \to 3 \quad u_1 \to 0, \ u_2 \to 6, \ u_3 \to 6, \quad z \to -12.$

Step 2. pivot $r_2 - 2r_1$, $r_3 - r_1$.

	x_1	x_2	x_3	x_4	u_1	u_2	u_3	
x_1	1	2	-1	1	1	0	0	3
u_2	0	0	3	0	-2	1	0	6
u_3	0	2	3	0	-1	0	1	6
	0	-2	-6	0	4	0	0	-12

$$z = -24 + 4(3 - 2x_2 + x_3 - x_4 - u_1) + 10x_2 + 2x_3 + 4x_4$$

$$= -12 + 2x_2 + 6x_3 - 4u_1$$

3

Introducing x_3.

$x_3 \to 2, \ x_1 \to 5, \ u_1 \to 0, \ u_3 \to 0, \ z \to 0.$

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(d) contd.

Step 3 $r_1 + \frac{1}{3} r_2$, $r_3 - r_2$, $\frac{1}{3} r_3$

	x_1	x_2	x_3	x_4	u_1	u_2	u_3	
x_1	1	2	0	1	1/3	1/3	0	5
x_3	0	0	1	0	−2/3	1/3	0	2
u_3	0	2	0	0	1	−1	1.	0.
	must be optimal as $z=0$							0.

(3)

hence BFS given by

$(x_1 — x_4) = (5, 0, 2, 0).$

check $5-2=3$
$2 \times 5+2 = 12$
$5 + 2 \times 2 = 9$

Total (20)

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Part		Mark
2(a)	Dual problem:	
	minimise $z = \underline{b}\cdot\underline{y}$ s.t.	
	$A^T\underline{y} \geq \underline{c}$	
	$\underline{y} \geq \underline{0}_m$.	2
(b)	~~Weak dual~~	
	$\underline{x}$ solves primal & $\underline{y} \geq \underline{0}_m \Rightarrow \underline{y}^T A\underline{x} \leq \underline{y}^T\underline{b}$	
	$\underline{y}$ solves dual & $\underline{x} \geq \underline{0}_n \Rightarrow \underline{y}^T A\underline{x} \geq \underline{c}^T\underline{x}$	
	$\therefore$ $\underline{y}^T\underline{b} \geq \underline{c}^T\underline{x}$. ie $\underline{y}\cdot\underline{b} \geq \underline{c}\cdot\underline{x}$.	2
(c)	Suppose feasible region to (P) not empty.	
	and $\forall M > 0$ $\exists$ $x \in$ feasible region s.t. $\underline{c}\cdot\underline{x} \geq M$.	
	$\Rightarrow$ $\forall M>0$, any feasible sol^n to dual, $\underline{y}$, (if it exists)	
	satisfies $\underline{y}\cdot\underline{b} \geq M$	
	As M is arbitrarily large $\Rightarrow$ $\max \underline{y}\cdot\underline{b} = \infty$. for all feas. $\underline{y}$ (if any exist.)	
		Total

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Part		Mark
(c)	contd $\Rightarrow$ no bounded optimal sol^n to (D)	3
(d)	Suppose $\underline{x}, \underline{y}$ feasible & $\underline{c}\cdot\underline{x} = \underline{b}\cdot\underline{y}$. Then for all other feasible $\underline{x}_0$,	
	$\underline{x}_0 \cdot \underline{c} \leq \underline{b}\cdot\underline{y} = \underline{x}\cdot\underline{c}$	
	$\Rightarrow$ $\underline{x}\cdot\underline{c}$ is ~~opt~~ larger than / or equal to any other sol^n	
	$\Rightarrow$ $\underline{x}$ is optimal.	
	Similar argument for y	2
(e)	Complementary Slackness (Symmetric dual).	
	~~Suppose~~ $\underline{x}, \underline{y}$ are optimal feasible solutions for (P) & (D) IFF	
	$y_i > 0 \Rightarrow (A\underline{x})_i = b_i$	
	$x_i > 0 \Rightarrow (\underline{y}^T A)_i = c_i$	3

	Total

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Part		Mark
(f)	Suppose given solⁿ is optimal	

Then $x_2, x_3 > 0 \Rightarrow$.

$$\left.\begin{aligned} y_1 + 2y_2 + 2y_3 &= 6 \\ y_1 + 4y_2 \quad\quad &= 4 \end{aligned}\right\} (*)$$

On the otherhand
"$y_i > 0 \Rightarrow (Ax)_i = b_i$" is equivalent to
"$(Ax)_i < b_i \Rightarrow y_i = 0$".

Note from proposed solⁿ.

$$0 + \frac{15}{2} + \frac{3}{2} = 9 < 10.$$

Hence ~~if~~ if given solⁿ optimal then $\boxed{y_1 = 0}$

Hence in (*) $\boxed{y_2 = 1 \,|\, y_3 = 2}$

However now note that

$$\underline{c} \cdot \underline{x} = (5 \times 0) + (6 \times 15/2) + (4 \times \tfrac{3}{2}) = 51$$

$$\underline{y} \cdot \underline{b} = (1 \times 21) + (2 \times 15) = 51$$

Total

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Part		Mark
	Hence given x & established y are feasible with equal objective	7
	(d) $\Rightarrow$ optimal	
		Total

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Part		Mark
3(a)	Balanced transportation problem:	
	Minimise $z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$	
	subject to $\sum_{i=1}^{m} x_{ij} = d_j \quad j=1,\dots,n$	
(P)	$\sum_{j=1}^{n} x_{ij} = s_i \quad i=1,\dots,m$	
	$x_{ij} \geq 0 \quad \forall i,j$	
	Where $(c_{ij}) \in \mathbb{R}^{m\times n}$, $(s_1,\dots,s_m) \in \mathbb{R}^m$, $(d_1,\dots,d_n) \in \mathbb{R}^n$ are given, and $(x_{ij}) \in \mathbb{R}^{m\times n}$ and $\sum_{i=1}^{n} s_i = \sum_{j=1}^{n} d_j$	4
(b).	Dual: maximise $\sum_{j=1}^{m} v_j d_j - \sum_{i=1}^{n} u_i s_i$	
(D)	subject to $v_j - u_i \leq c_{ij} \quad \forall i=1,\dots,m \quad \forall j=1,\dots,n$	
	Where $u_i, v_j \in \mathbb{R} \quad \forall i,j$.	3
		Total

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Part		Mark

(c) The primal has a finite optimal feasible sol^n $\Longleftrightarrow$ The dual has a finite optimal feasible sol^n in which case their objectives are equal.

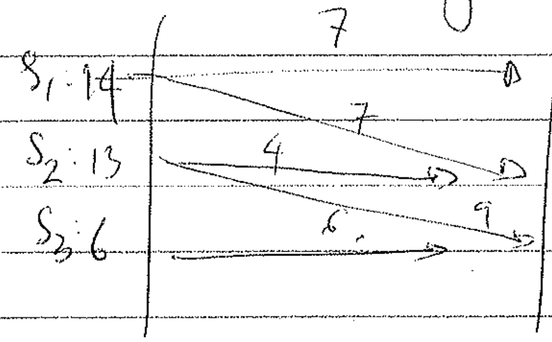
Now consider that

$$x_{ij} = s_i d_j / \sum s_i \quad \text{is a FS to } (P) \quad (*)$$

and $u_i = v_j = 0 \ \forall i,j$ is a FS to (D). $(**)$

Then it must be true that both have a finite feasible sol^n as ~~upper~~ lower bound for OFS for (P) is given by objective value in $(**)$. and ~~lower~~ upper bound for OFS for (D) is given by objective value in $(*)$. (4)

(d) Northwest corner gives initial sol^n



$$(x_{ij}) = \begin{pmatrix} 7 & 7 & 0 \\ 0 & 4 & 9 \\ 0 & 0 & 6 \end{pmatrix}$$

Total

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(d) cont'd

	x_{i1}	x_{i2}	x_{i3}	s_i	c_{i1}	c_{i2}	c_{i3}
	7	$7_{-\eta}$	$0_{+\eta}$	14	14	13	19
	0	$4_{+\eta}$	$9_{-\eta}$	13	15	14	18
	0	0	6	6.	9	11	2.
d_j	7	11	15				

Sol $\boxed{u_1 = 0}$

$x_{11} > 0 \Rightarrow \boxed{v_1 = 14}$

$x_{12} > 0 \Rightarrow v_2 - u_1 = 13 \quad \boxed{v_2 = 13}$

$x_{22} > 0 \Rightarrow v_2 - u_2 = 14 \quad \boxed{u_2 = -1}$

$x_{23} > 0 \Rightarrow v_3 - u_2 = 8 \quad \boxed{v_3 = 7}$

$x_{33} > 0 \Rightarrow v_3 - u_3 = 2 \quad \boxed{u_3 = 5}$

But $x_{13} = 0$ and $v_3 - u_1 = 7 \nless 6$. $\Rightarrow$ sol^n not optimal.

Increase x_{13} by η by incr. x_{22} and x_{23} by η
decr. x_{12}, x_{23} by η.

This yields change in objective $-\eta$.

max $\eta = 7$ with this operation.

	Total

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new tableau

	x_{i1}	x_{i2}	x_{i3}	s_i
	$7-\eta$	0	$7+\eta$	14
	$0+\eta$	11	$2-\eta$	13
	0	0	6	$6.$
d_j	7	11	15	

Set $\boxed{u_1=0}$ $\boxed{v_1=14}$ $\boxed{v_3=6}$ $v_1-u_2=14.$ $v_3-u_2=8$ $\boxed{u_2=-2}$

$\boxed{v_2=12}$ $v_3-u_3=2$ $\boxed{u_3=4}$.

$x_{21}=0$ but $v_1-u_2=16\not\leq 15. \Rightarrow$ not optimal.

introduce $x_{21}=\eta$ max $\eta=2$

new table

	x_{i1}	x_{i2}	x_{i3}	
	$5_{-\eta}$	0	$9_{+\eta}$	14
	2	11	0	13
	$0_{+\eta}$	0	$6_{-\eta}$	6
	7	11	$15.$	

$\boxed{u_1=0}$ $\boxed{v_1=14}$ $\boxed{v_3=6}$ $v_1-u_2=15$ $\boxed{u_2=-1}$ $\boxed{v_2-u_2=14}$ $\boxed{v_2=13}$ $v_3-u_3=2$ $\boxed{u_3=4}$

$x_{31}=0$, $v_1-u_3=14-4=10\not\leq 9$

$\Rightarrow$ not optimal

Total

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Part | Mark

Cost already increased x_{21} hence introduce x_{31} at cost of x_{11}.

$x_{31} = \eta$ $\max \eta = 5$.

x_{i1}	x_{i2}	x_{i3}	
0	0	14	14
2	11	0	13
5	0	1	6.
7	11	15	

$\boxed{u_1 = 0}$ $\boxed{v_3 = 6}$

$v_1 - u_2 = 15$ $\boxed{u_2 = -2}$

$v_2 - u_2 = 14$ $\boxed{v_2 = 12}$

$v_1 - u_3 = 9$ $\boxed{v_1 = 13}$

$v_3 - u_3 = 2$ $\boxed{u_3 = 4}$

$x_{11} = 0$ & $v_1 - u_1 = 13 \le 14$ ✓

$x_{12} = 0$ & $v_2 - u_1 = 12 < 13$ ✓

$x_{23} = 0$ & $v_3 - u_2 = 8 \le 8$ ✓

$x_{32} = 0$ & $v_2 - u_3 = 8 < 11$ ✓

Hence optimal by complementary slackness. ✓ 9

Total

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Part		Mark
(a)	A cut is an ordered pair of subsets of N satisfying	
	(i) $s \in A$ (ii) $d \in B$ (iii) $A \cup B = N$	
	(iv) $A \cap B = \emptyset$.	2
(b)	(i) First show $\sum_j f(s,j)$ = "flow" = $f(A,B)$ for any cut.	
	(ii) Then show $f(A,B) \leq \gamma(A,B)$ for any (A,B)	
	Hence $\sum_j f(s,j) \leq \min_{(A,B)} \gamma(A,B)$	
	To show (i) Note defn of flow $\Rightarrow f(i,i) = 0$.	
	$\sum_{i \in N \setminus \{s,d\}} \sum_{j \in N} f(i,j) = 0$	
	Also total flow $= \sum_{j \in N \setminus \{s\}} f(s,j) = \sum_{j \in N} f(s,j) + 0$	
	$= \sum_{i \in A} \sum_{j \in N} f(i,j)$.	
	$= \sum_{i \in A} \sum_{j \in A} f(i,j) + \sum_{i \in A} \sum_{j \in B} f(i,j)$	
		Total

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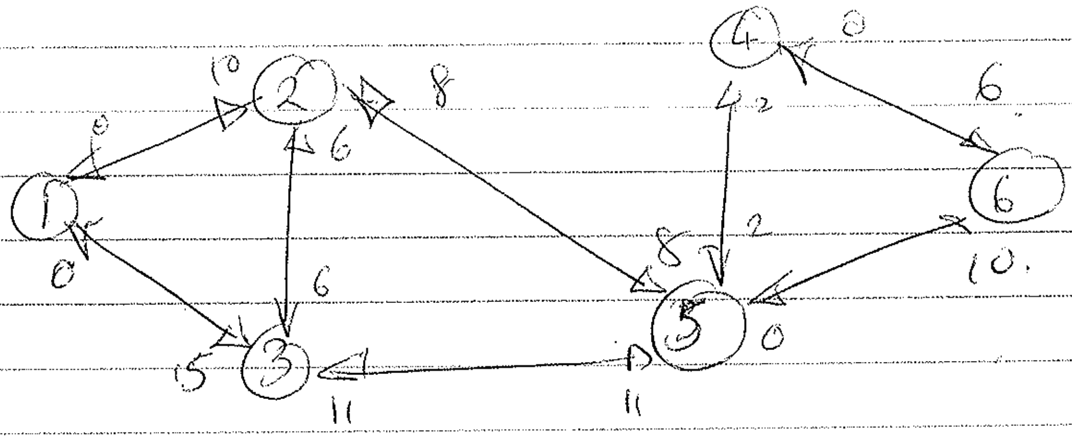
But if $i, j \in A$ then $f(i,j)$ and $f(j,i)$ occur in sum & cancel.

$\therefore$ Total flow $= \sum_{i \in A} \sum_{j \in B} f(i,j) = f(A,B)$.

(ii) To show 2nd part. This is trivial as $f(i,j) \leq \gamma(i,j)$.

6

(c) (i)



Construct initial flow:

$1 \to 2 \to 5 \to 6$: flow 8
$1 \to 3 \to 5 \to 6$: flow 2
$1 \to 3 \to 5 \to 4 \to 6$: flow 2 (*)

Total

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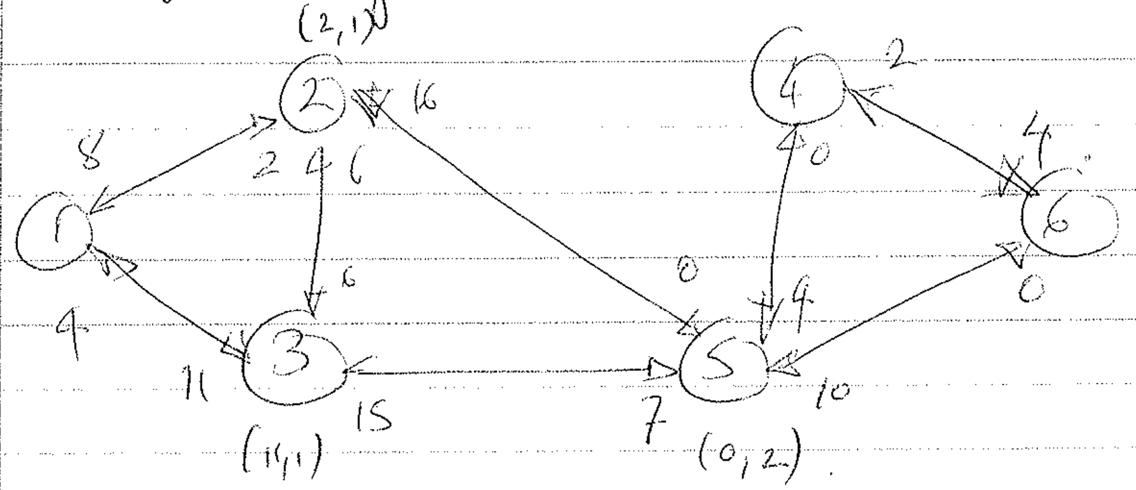
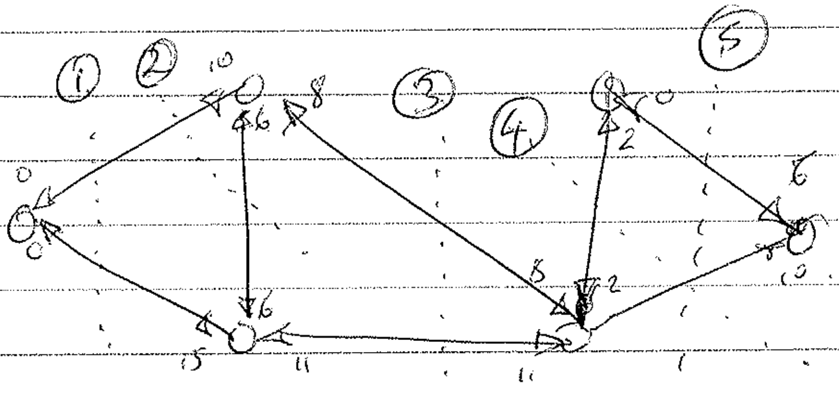
Residual capacities



Max flow = 12. (7)

(c)(ii) optimal flow – min cut.



① capacity = 10 + 15 = 25
② capacity = 10 + 6 + 11 = 27
③ ——— = 8 + 11 = 19
$\boxed{④ ——— = 2 + 10 = 12}$ ← Minimum. (3)
⑤ ——— 10 + 6 = 16

Total

DEPARTMENT OF MATHEMATICAL SCIENCES University of Bath
Outline solution to Examination Questions

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Part		Mark
(d)(iii)	Increasing $\gamma(4,6)$ to 7 would change nothing.	
	eg: minimum cut is still 12.	
	eg: in the last diagram of (c) the sink would still end up not being labelled.	
	eg: initial flow construction ⓔ would still have flow 2.	2

Total	

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