University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA30087/50087: OPTIMISATION METHODS OF OPERATIONAL RESEARCH

Tuesday 16th January 2007, 09.30–11.30

Candidates may use university-supplied calculators.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

1. Consider the linear programming problem

$$\begin{aligned}&\text{maximise } z = \mathbf{c}\cdot\mathbf{x}\\&\text{subject to:}\\&\mathbf{A}\mathbf{x} = \mathbf{b}\\&\mathbf{x} \geq \mathbf{0}_n\end{aligned}$$

where $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ are given and $\mathbf{x} \in \mathbb{R}^n$. You may assume (as usual) that $\text{rank}(\mathbf{A}) = m$ and $m \leq n$.

Suppose that there exists a basic feasible solution, $\mathbf{x}$, which may be written as

$$\mathbf{x} = \begin{pmatrix}\mathbf{0}_{n-m}\\ \mathbf{x_B}\end{pmatrix}$$

where $\mathbf{x_B} \in \mathbb{R}^m$ and $\mathbf{x_B} > \mathbf{0}_m$. Suppose that the columns of $\mathbf{A}$ are partitioned accordingly with the non-zero entries of $\mathbf{x}$ so that $\mathbf{A} = (\mathbf{A}^0|\mathbf{B})$ where $\mathbf{A}^0 \in \mathbb{R}^{m\times(n-m)}$ and $\mathbf{B} \in \mathbb{R}^{m\times m}$.

(a) Explain why the matrix $\mathbf{B}$ is invertible. [1]

Now partition $\mathbf{c}$ accordingly with the partition of $\mathbf{x}$ and write it as

$$\mathbf{c} = \begin{pmatrix}\mathbf{c}^0\\ \mathbf{c_B}\end{pmatrix}$$

so that $\mathbf{c}^0 \in \mathbb{R}^{n-m}$ and $\mathbf{c_B} \in \mathbb{R}^m$. For each $i = 1, ..., n-m$ let $\mathbf{e}_i$ be the unit vector in $\mathbb{R}^{n-m}$ which has a unit entry in the i-th position and all other entries equal to zero.

(b) Show that $\mathbf{x}$ is optimal if and only if

$$((\mathbf{c}^0)^{\mathrm{T}} - \mathbf{c_B}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}^0)\mathbf{e}_i \leq 0 \text{ for all } i = 1, ..., n-m.$$

[8]

(c) Find a basic feasible solution to the following system of constraints

$$\begin{aligned}&x_1 + 2x_2 - x_3 + x_4 = 3\\&2x_1 + 4x_2 + x_3 + 2x_4 = 12\\&x_1 + 4x_2 + 2x_3 + x_4 = 9\\&x_1, x_2, x_3, x_4 \geq 0.\end{aligned}$$

(Hint: use artificial variables). [11]

2. Consider the following linear programming problem in standard form,

$$\begin{aligned}&\text{maximise } z = \mathbf{c}\cdot\mathbf{x}\\&\text{subject to:}\\&\mathbf{A}\mathbf{x} \le \mathbf{b}\\&\mathbf{x} \ge \mathbf{0}_n\end{aligned}$$

where $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ are given and $\mathbf{x} \in \mathbb{R}^n$.

(a) Write down the dual problem to the above primal. [2]

(b) Suppose that $\mathbf{x}$ and $\mathbf{y}$ are feasible solutions to the primal and dual respectively. Show that

$$\mathbf{c}\cdot\mathbf{x} \le \mathbf{b}\cdot\mathbf{y}.$$

[2]

(c) Deduce from part (b) that if the feasible region of the primal is not empty but there is no bounded optimal solution to the primal, then the dual problem has no feasible solution. [3]

(d) Show that if $\mathbf{x}$ and $\mathbf{y}$ are feasible solutions to the primal and dual problems respectively such that $\mathbf{c}\cdot\mathbf{x} = \mathbf{b}\cdot\mathbf{y}$ then they must be optimal solutions for the primal and dual problems respectively. [3]

(e) State without proof the Symmetric Complementary Slackness Theorem. [3]

(f) Consider the linear programming problem

$$\begin{aligned}&\text{maximise } z = 5x_1 + 6x_2 + 4x_3\\&\text{subject to:}\\&x_1 + x_2 + x_3 \le 10\\&3x_1 + 2x_2 + 4x_3 \le 21\\&3x_1 + 2x_2 \le 15\\&x_1, x_2, x_3 \ge 0.\end{aligned}$$

Use the Symmetric Complementary Slackness Theorem to verify that

$$(x_1, x_2, x_3) = \left(0, \frac{15}{2}, \frac{3}{2}\right)$$

is the optimal solution to the above problem. [7]

3. (a) Formulate as a linear programming problem the balanced transportation problem of minimising the total cost of shipping quantities of goods $s_1, ..., s_m$ from m sources to n destinations with demands $d_1, ..., d_n$ where the cost of shipping from location i to location j is c_{ij}. [4]

(b) Formulate its dual. [3]

(c) State without proof the Duality Theorem and use it to prove that a bounded optimal solution to the balanced transportation problem must exist. [4]

(d) Solve the balanced transportation problem in which the cost matrix (c_{ij}) for moving goods from $i = 1, 2, 3$ to $j = 1, 2, 3$ is given by

$$\begin{pmatrix} 14 & 13 & 6 \\ 15 & 14 & 8 \\ 9 & 11 & 2 \end{pmatrix},$$

the supplies are $s_1 = 14, s_2 = 13, s_3 = 6$ and the demands are $d_1 = 7, d_2 = 11, d_3 = 15$. [9]

4. Consider a capacitated network (N, γ) having a single source $s \in N$ and single sink $d \in N$. Any flow f in the network satisfies the usual conditions

$$\begin{aligned} &f(i,j) \leq \gamma(i,j) \ \forall i,j \in N \\ &f(i,j) = -f(j,i) \ \forall i,j \in N \\ &\textstyle\sum_{j \in N} f(i,j) = \sum_{j \in N} f(j,i) \ \forall i \in N \backslash \{s,d\}. \end{aligned}$$

(a) Define a **cut** (A, B). [2]

(b) Show that for any flow f

$$\sum_{j \in N} f(s,j) \leq \min_{\text{all cuts } (A,B)} \sum_{i \in A} \sum_{j \in B} \gamma(i,j)$$

and hence if there exists a flow f' and a cut (A', B') such that

$$\sum_{j \in N} f'(s,j) = \sum_{i \in A'} \sum_{j \in B'} \gamma(i,j). \qquad (*)$$

then f' is optimal. [8]

(c) (i) Draw the six node network whose capacities are given in the matrix $\gamma(i,j)$ below and find the maximal flow.

$$\begin{pmatrix} 0 & 10 & 15 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 8 & 0 \\ 0 & 6 & 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 8 & 11 & 2 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[5]

(ii) Show that, consistently with part (b), if f' is the maximal flow there exists a cut (A', B') such that $(*)$ holds. [3]

(iii) What would happen to the solution if the entry $\gamma(4, 6)$ were changed to 7? Justify carefully your answer. [2]