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Part | | Mark

(a) Let $F = \{\underline{x} \mid A\underline{x} = \underline{b}, \underline{x} \geq \underline{0}\}$. If F has fewer than 2 members it is convex by convention. Else suppose $\underline{x}^1, \underline{x}^2 \in F$

Let $0 \leq \lambda \leq 1$ and $\underline{x} = \lambda \underline{x}^1 + (1-\lambda)\underline{x}^2$ then

$A\underline{x} = \lambda A\underline{x}^1 + (1-\lambda) A\underline{x}^2 = \underline{b}$, $\underline{x} \geq \underline{0}$ so $\underline{x} \in F$

Thus F is convex. (3)

Can write $\underline{x}^0 = (x_1^0, x_2^0, \ldots x_e^0, 0, \ldots, 0)^T$ wlog by renumbering variables if necessary. Since $\underline{x}^0 \in F$

(1) — $\sum_{i=1}^{e} x_i^0 \underline{a}_i = \underline{b}$ where $A = [\underline{a}_i]$

<u>If $\underline{x}^0$ is not basic</u>, $\underline{a}_1, \ldots, \underline{a}_e$ are lin dep and $\exists\, t_i\ i = 1, \ldots, e$ not all zero s.t. $\sum_{i=1}^{e} t_i \underline{a}_i = \underline{0}$ — (2)

(1) + ε(2) gives $\sum_{i=1}^{e} (x_i^0 + \varepsilon t_i)\, \underline{a}_i = \underline{b}$

(1) − ε(2) — $\sum_{i=1}^{e} (x_i^0 - \varepsilon t_i)\, \underline{a}_i = \underline{b}$

and for ε sufficiently small $\underline{x}^1 = (x_1^0 + \varepsilon t_1, \ldots x_e^0 + \varepsilon t_e, 0, \ldots 0)^T$ and $\underline{x}^2 = (x_1^0 - \varepsilon t_1, \ldots x_e^0 - \varepsilon t_e, 0, \ldots, 0)^T$ are feasible. $\underline{x}^1 \neq \underline{x}^2$ (7)

But $\underline{x}^0 = \frac{1}{2}\underline{x}^1 + \frac{1}{2}\underline{x}^2$ contradicting $\underline{x}^0$ being an extreme point. $\therefore$ $\underline{x}^0$ is basic.

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(b) Introducing slack variable x_4, surplus variable x_5 and artificial variable x_6, initial tableau is as follows
(min

	x_1	x_2	x_3	x_4	x_5	x_6	W	
x_4	1	1	1	1	0	0	0	12
x_6	1	−2	−1	0	−1	1	0	8
W	−1	2	1	0	1	0	1	−8

↑

(3)

x_1 enters and x_6 leaves basic set

	x_1	x_2	x_3	x_4	x_5	x_6	W		
x_4	0	3	2	1	1		0	4	End of
x_1	1	−2	−1	0	−1		0	8	Phase 1
W	0	0	0	0	0		1	0	$W = 0$
Z	0	−4	−5	0	−1			8	

↑

(4)

x_3 enters and x_4 leaves basic set

	x_1	x_2	x_3	x_4	x_5	Z	
x_3	0	$3/2$	1	$1/2$	$1/2$	0	2
x_1	1	$-1/2$	0	$1/2$	$-1/2$	0	10
Z	0	$7/2$	0	$5/2$	$3/2$	1	18

(3)

Optimal solution is $x_1 = 10,\ x_2 = 0,\ x_3 = 2\ (x_4 = x_5 = 0)$

$Z = 18$

Total 20

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(i) Make x_1, x_2 and x_3 units of K, L and M respectively.

Problem is: Maximise $3x_1 + 5x_2 + 2x_3$ subject to

$$4x_1 + 8x_2 + 2x_3 \le 40{,}000$$

$$8x_1 + 10x_2 + 9x_3 \le 100{,}000 \qquad x_1, x_2, x_3 \geq 0$$ (3)

(ii) From information given basis matrix B corresponding to the optimal solution is

$$B = \begin{pmatrix} 4 & 2 \\ 8 & 9 \end{pmatrix} \text{ and hence } B^{-1} = \begin{pmatrix} 9/20 & -1/10 \\ -2/5 & 1/5 \end{pmatrix}$$

Main body of simplex tableau contains $B^{-1}A \mid B^{-1}\underline{b}$

so tableau is (x_4 and x_5 slack variables)

	x_1	x_2	x_3	x_4	x_5	Z	
x_1	1	$13/5$	0	$9/20$	$-1/10$	0	8000
x_3	0	$-6/5$	1	$-2/5$	$1/5$	0	4000
	0	$2/5$	0	$11/20$	$1/10$	1	32 000

(6)

Solution is $x_1 = 8000,\ x_2 = 0,\ x_3 = 4000,\ Z = 32{,}000$

(iii) <u>Ranging c_1</u>. Require: $\frac{13}{5}c_1 - \frac{12}{5} - 5 \geq 0 \qquad c_1 \geq \frac{37}{13}$

$$\frac{9}{20}c_1 - \frac{4}{5} \geq 0 \qquad c_1 \geq \frac{16}{9}$$

$$-\frac{1}{10}c_1 + \frac{2}{5} \geq 0 \qquad c_1 \leq 4$$

so optimal for

$$\underline{37/13 \leq c_1 \leq 4}$$

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Part (iii) cont

Ranging c_2. $\quad \frac{39}{5} - \frac{12}{5} - c_2 \geq 0$

$$\underline{c_2 \leq \frac{27}{5}}$$

Ranging c_3.
$$\frac{39}{5} - \frac{6}{5}c_3 - 5 \geq 0 \qquad c_3 \leq \frac{7}{3}$$
$$\frac{27}{20} - \frac{2}{5}c_3 \geq 0 \qquad c_3 \leq \frac{27}{8}$$
$$-\frac{3}{10} + \frac{1}{5}c_3 \geq 0 \qquad c_3 \geq \frac{3}{2}$$

solution optimal for
$$\underline{\frac{3}{2} \leq c_3 \leq \frac{7}{3}}$$ (6)

Skilled time. From dual solution can pay up to £$\frac{11}{20}$ for extra minute. This applies for

$$\begin{pmatrix} \frac{9}{20} & -\frac{1}{10} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} b_1 \\ 100000 \end{pmatrix} = \begin{pmatrix} \frac{9}{20}b_1 - 10,000 \\ -\frac{2}{5}b_1 + 20000 \end{pmatrix} \geq 0$$

i.e. $\quad \frac{200,000}{9} \leq b_1 \leq 50000$

Up to an extra 10,000 mins.

Unskilled time. From dual can pay up to £$\frac{1}{10}$ per minute.

Applies for $\begin{pmatrix} \frac{9}{20} & -\frac{1}{10} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 40000 \\ b_2 \end{pmatrix} \geq 0$

ie $\quad 80,000 \leq b_2 \leq 180,000$ (5)

Up to an extra 80,000 minutes

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(a)	Suppose m sources have a_i $i=1,\dots,m$ available and n destinations require amounts b_j $j=1,\dots,n$ with $\sum a_i = \sum b_j$. Let c_{ij} be unit cost of transportation between source i and destination j.	
	If x_{ij} is amount transported from source i to destination j, the problem becomes:	
	Minimise $\sum_i^m \sum_j^n c_{ij} x_{ij}$ subject to	
	$\sum_{j=1}^n x_{ij} = a_i \quad i=1,\dots,m$	
	$\sum_{i=1}^m x_{ij} = b_j \quad j=1,\dots,n \qquad x_{ij} \geq 0 \quad \forall (i,j)$	3
	Using dual variables u_i $i=1,\dots,m$ corresponding to first m constraints and v_j $j=1,\dots,n$ corresponding to last n constraints and noting that the column of x_{ij} has just two entries of '1' in rows i and $(m+j)$, the dual problem is	
	Maximise $\sum_{i=1}^m a_i u_i + \sum_{j=1}^n b_j v_j$	
	subject to $u_i + v_j \leq c_{ij} \quad i=1,\dots,m,\ j=1,\dots,n$	
	with dual variables not sign restricted (since primal constraints are equations).	3

	Total

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(b)

(i) After adding a dummy destination with requirement of 4, problem is :-

(Units of 1000 tons)

		L	M	N	Dummy
		6	6	6	4
A	8	2	4	3	0
B	6	6	5	12	0
C	8	12	3	8	0

and cheapest route method gives following initial basic feasible solution.

(ii)

	6	6	6	4	
8	6-		2+		0
6	+		2-	4	9
8		6	2		5
	2	-2	3	-9	

x_{21} becomes basic
x_{23} – non basic
$\theta = 2$

	6	6	6	4	
8	4-		4+		0
6	2+			4-	4
8		6	2-	+	5
	2	-2	3	-4	

x_{34} becomes basic
x_{33} – non basic
$\theta = 2$

	6	6	6	4	
8	2		6		0
6	4			2	4
8		6		2	4
	2	-1	3	-4	

Optimal solution

2 from A to L (Units 1000 tons)
6 – A to N
etc.

Cost is 64K pounds

(8)

Total

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(iii) Modifying problem and using the same set of basic variables gives

	6	6+k	6	4	
8+k	2+k		6		0
6	4−k			2+k	4
8		6+k		2−k	5
	2	−1	3	−4	

which has cost (64−k) thousand pounds and is feasible for $k \leq 2$. (3)

The paradox can be explained

(i) for the primal $\sum_{(i,j)\in k+} c_{ij} < \sum_{(i,j)\in k-} c_{ij}$

where k+ and k− are set of routes increasing and decreasing by k.

(ii) for the dual and corresponding dual variables have $u_1 + v_2 = -1 < 0$ (3)

so objective decreases when a_1 and b_2 are increased simultaneously.

Total 20

Part | Ma

(a) A cut is an ordered pair (A,B) of subsets of N satisfying (i) $s \in A$ (ii) $d \in B$, (iii) $A \cup B = N$ (iv) $A \cap B = \emptyset$ (2)

Denote $\sum_{i \in C} \sum_{j \in D} f(i,j)$ by $f(C,D)$ for $C,D \subseteq N$

Then $\sum_{j \in N} f(s,j) \equiv f(s,N) = f(A,N) - f((A \setminus s), N)$

$= f(A,A) + f(A,B) - f((A\setminus s),N) \quad — (1)$

Since $f(i,j) = -f(j,i) \;\; \forall\, i,j \in N$ $f(A,A) = -f(A,A)$
so $f(A,A) = 0$.

For $i \neq s,d$ $\sum_{j \in N} f(i,j) = \sum_{j \in N} f(j,i)$

but $f(i,j) = -f(j,i)$ so both sums must be zero.
i.e. $f(i,N) = 0$ for $i \neq s,d$.
Summing over $i \in A \setminus s$ gives
$f(A \setminus s, N) = 0$
Thus from (1) above $f(s,N) = f(A,B)$
as required (7)

(b) Let f be a flow that maximises $f(s,N)$ then for any cut (A,B)

$$f(s,N) = f(A,B) \le \sum_{i \in A} \sum_{j \in B} \gamma(i,j) \equiv \gamma(A,B)$$

Total

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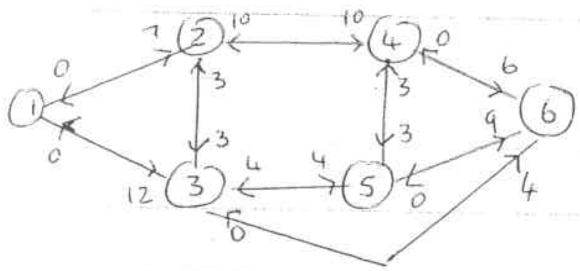
(b) (cont)	The theorem states that there is a cut (A, B) such that $f(s,N) = \gamma(A,B)$ i.e. maximal flow equals minimal cut.	2
(c)	Network is as follows	



Source is node 1 and sink is node 6 — 2

By inspection make initial flows

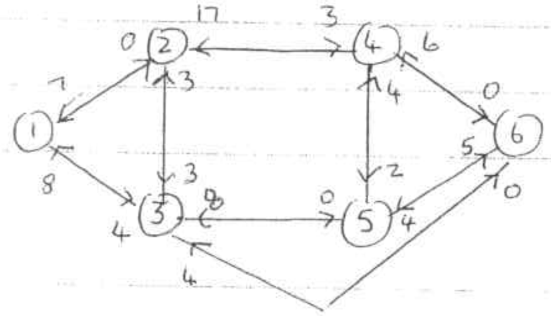
$1 \to 2 \to 4 \to 6$ Flow 6

$1 \to 2 \to 4 \to 5 \to 6$ Flow 1

$1 \to 3 \to 5 \to 6$ Flow 4

$1 \to 3 \to 6$ Flow 4

Residual capacities are :-



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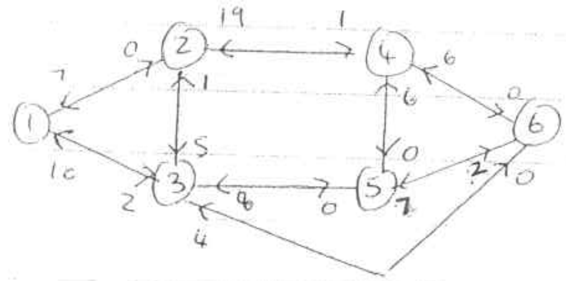
(c) cont

Labelling nodes (γ, i) where γ is residual capacity of path from source and i is preceding node in path.

Node 3 (4, 1), Node 2 (3, 3), Node 4 (3, 2)
Node 5 (2, 4), Node 6 (2, 5)

There is unsaturated path $1 \to 3 \to 2 \to 4 \to 5 \to 6$ with residual capacity 2. Increasing flow by 2 along this path gives new residual capacities as follows.



(5)

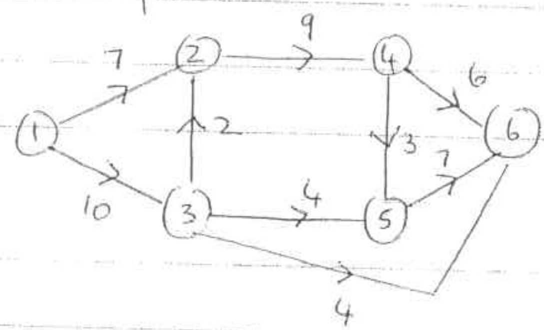
Labelling now gives:

Node 3 (2, 1), Node 2 (1, 3), Node 4 (1, 2)

So there is no unsaturated path from $1 \to 6$. Flow is maximal.

Minimal cut is [(1, 2, 3, 4) (5, 6)] with capacity
$4 + 4 + 3 + 6 = 17$

Maximal flow is :-



Amount of flow is 17 which equals capacity of cut above and is therefore maximal.

(2)

Total 20