University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA30087/50087: OPTIMISATION METHODS OF OPERATIONAL RESEARCH

Friday 20th January 2006, 09.30–11.30

Candidates may use university-supplied calculators.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

1. (a) Show that the set of feasible solutions to the linear programming problem:

$$\text{maximise } \mathbf{c}^{\mathbf{T}}\mathbf{x} \quad \text{subject to:}$$
$$A\mathbf{x} = \mathbf{b}, \qquad \mathbf{x} \geq \mathbf{0}$$

is convex. Prove that if $\mathbf{x}^{\mathbf{0}}$ is an extreme point of the set, then it is a basic feasible solution.

(b) Use the two-phase method to solve the following linear programming problem.

$$\text{Maximise } z = x_1 + 2x_2 + 4x_3 \text{ subject to}$$
$$x_1 + x_2 + x_3 \leq 12$$
$$x_1 - 2x_2 - x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0.$$

2. A company assembles three different products from parts bought from outside suppliers. Each product requires both skilled and unskilled work to complete. The following table shows for each product, the number of minutes of skilled and unskilled work to assemble one unit.

Product	K	L	M
Skilled time	4	8	2
Unskilled time	8	10	9

Each week there are 40,000 minutes of skilled time and 100,000 minutes of unskilled time available. The unit profit on products K, L and M is £3.00, £5.00 and £2.00 respectively and there is no limit to how many of each product can be sold.

(i) Formulate the problem of deciding how much of each product to make each week in order to maximise total weekly profit as a linear programming problem.

(ii) Given that the optimal solution involves producing products K and M, but none of L, write down the basis matrix, B, corresponding to the optimal solution. Calculate B^{-1} and hence obtain the simplex tableau for the optimal solution.

(iii) Examine the ranges (separately) of the unit profit values of the products for which the solution remains optimal.

(iv) How much should the company be prepared to pay for an extra minute of skilled time? Up to how many minutes would this value apply? Answer the same questions with respect to unskilled time.

3. (a) Show how the standard balanced transportation problem of minimising the total cost of sending given amounts of a product from m sources to n destinations may be formulated as a special structure linear programme. Write down the dual problem and explain how it is derived from the primal.

(b) Stone is quarried at three sites, A, B and C, the maximum weekly tonnages being 8000 at A, 6000 at B and 8000 at C. It must be transported to three destinations, L, M and N, each requiring 6000 tons per week. The costs, in pounds per ton, of transporting between the quarries and destinations are given below.

	L	M	N
A	2	4	3
B	6	5	12
C	7	3	8

(i) Formulate the problem of moving the stone at minimum total cost as a transportation problem.

(ii) Obtain an optimal solution to the problem

(iii) Show that if the amount available at A and amount required at M are both increased by the same small amount, k tons, then the optimal transportation cost is reduced. Comment on this paradox, relating it to the dual solution.

4. (a) Consider a capacitated network (N, γ) having a single source $s \in N$ and a single sink $d \in N$. Any flow f in the network satisfies the usual conditions

$$f(i,j) \le \gamma(i,j) \quad \forall i,j \in N$$
$$f(i,j) = -f(j,i) \quad \forall i,j \in N$$
$$\textstyle\sum_{j \in N} f(i,j) = \sum_{j \in N} f(j,i) \quad \forall i \in N \setminus \{s,d\}$$

Define a **cut**, (A, B) and show that the total flow from the source equals the flow across any cut (A, B) i.e.

$$\textstyle\sum_{j \in N} f(s,j) = \sum_{i \in A} \sum_{j \in B} f(i,j)$$

(b) State without proof the maximal flow, minimal cut theorem

(c) Draw the 6 node network whose capacities are given in the following matrix and find the maximal flow. State the minimal cut and verify that the theorem in (b) holds.

$$\begin{pmatrix} 0 & 7 & 12 & 0 & 0 & 0 \\ 0 & 0 & 3 & 10 & 0 & 0 \\ 0 & 3 & 0 & 0 & 4 & 4 \\ 0 & 10 & 0 & 0 & 3 & 6 \\ 0 & 0 & 4 & 3 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$