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Part

(i) Since $A\underline{x} \le \underline{b}$ and $\underline{w} \geq \underline{0}$

$$\underline{w}^T A \underline{x} \le \underline{w}^T b = b^T \underline{w} \quad -(1)$$

Since $A^T \underline{w} \geq \underline{c}$ and $\underline{x} \geq \underline{0}$

$$\underline{x}^T A^T \underline{w} \geq \underline{x}^T \underline{c} \text{ or transposing } \underline{w}^T A \underline{x} \geq \underline{c}^T \underline{x} \quad -(2)$$

Comparing (1) and (2) gives $\underline{c}^T \underline{x} \le \underline{b}^T \underline{w}$

(ii) Suppose, for contradiction, dual has feasible solution $\underline{w}$ with $\underline{b}^T \underline{w} = M$. Since primal has no finite maximising solution $\exists\, \underline{x} \in F$ (the set of feasible solns) with

$$\underline{c}^T \underline{x} > M$$

This contradicts result in (i) so dual has no feasible solution.

(iii)

$$(A^T \underline{w}^o - \underline{c})^T \underline{x}^o = 0 \Rightarrow (A^T w^o)^T \underline{x}^o = \underline{c}^T \underline{x}^o$$

$$\Rightarrow \underline{w}^{oT} A \underline{x}^o = \underline{c}^T \underline{x}^o$$

$$(A \underline{x}^o - \underline{b})^T \underline{w}^o = 0 \Rightarrow (A \underline{x}^o)^T \underline{w}^o = \underline{b}^T \underline{w}^o$$

$$\Rightarrow w^{oT} A x^o = \underline{w}^{oT} \underline{b} = \underline{b}^T \underline{w}^o$$

Thus we have equality between primal and dual objective and by result in part (i) both are optimal.

(cont)

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(iv) Dual problem is

Minimise $3w_1 - 5w_2 + 6w_3$ subject to

$$w_1 - 3w_2 + 2w_3 \geq 3$$
$$w_1 - w_2 + w_3 \geq 5$$
$$2w_1 + w_2 - 3w_3 \geq 12$$
$$w_1, w_2, w_3 \geq 0$$

Using complementary slackness

$$x_1 > 0 \Rightarrow w_1 - 3w_2 + 2w_3 = 3$$
$$x_2 > 0 \Rightarrow w_1 - w_2 + w_3 = 5$$
$$2x_1 + x_2 - 3x_3 < 6 \Rightarrow w_3 = 0$$

Solving $w_1 - 3w_2 = 3$; $w_1 - w_2 = 5$

gives dual solution

$$w_1 = 6, \; w_2 = 1, \; w_3 = 0$$

with primal objective = dual objective = 13.

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(1) Produce x_1, x_2, x_3 litres of A, B and C respectively.

Problem is Maximise $Z = 1.5x_1 + 2.0x_2 + 1.5x_3$

Subject to $0.3x_1 + 0.2x_2 + 0.1x_3 \le 2500$

$0.1x_1 + 0.2x_2 + 0.2x_3 \le 1800$ $\quad x_1, x_2, x_3 \geq 0$

Adding slack variables x_4, x_5 and solving:-

	x_1	x_2	x_3	x_4	x_5	
x_4	0.3	0.2	0.1	1	0	2500
x_5	0.1	0.2	0.2	0	1	1800
Z	-1.5	-2.0	-1.5			0

↑

	x_1	x_2	x_3	x_4	x_5	
x_4	0.2	0	-0.1	1	-1	700
x_2	0.5	1	1	0	5	9000
Z	-0.5	0	0.5	0	10	18000

↑

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	-0.5	5	-5	3500
x_2	0	1	1.25	-2.5	7.5	7250
Z	0	0	0.25	2.5	7.5	19750

Optimal solution $x_1 = 3500$, $x_2 = 7250$, $x_3 = 0$, $Z = 19{,}750$

i.e. produce 3500 litres of A, 7250 litres of B and none of C. (7)

Total proceeds are £19,750.

Total

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(ii) The dual problem is

Minimise $2500w_1 + 1800w_2$

subject to
$$0.3w_1 + 0.1w_2 \geq 1.5$$
$$0.2w_1 + 0.2w_2 \geq 2.0 \qquad w_1, w_2 \geq 0$$
$$0.1w_1 + 0.2w_2 \geq 1.5$$

with optimal solution $\underline{w_1 = 2.5,\ w_2 = 7.5}$

w_1, w_2 represent the marginal value of 1 litre of orange, blackcurrant concentrates respectively.

For feasibility require $B^{-1}\underline{b} \geq \underline{0}$ so

ranging b_1, $\begin{pmatrix} 5 & -5 \\ -2.5 & 7.5 \end{pmatrix}\begin{pmatrix} b_1 \\ 1800 \end{pmatrix} = \begin{pmatrix} 5b_1 - 9000 \\ -2.5b_1 - 13{,}500 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

when $\underline{1800 \leq b_1 \leq 5400}$

Ranging b_2: $\begin{pmatrix} 5 & -5 \\ -2.5 & 7.5 \end{pmatrix}\begin{pmatrix} 2500 \\ b_2 \end{pmatrix} = \begin{pmatrix} 12500 - 5b_2 \\ -6250 + 7.5b_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

when $\underline{833\tfrac{1}{3} \leq b_2 \leq 2500}$

These are the ranges within which the above marginal values apply.

(iii) For optimality require $\sum_{i \in B} c_i y_{ij} - c_j \geq 0 \quad j = 1, \ldots, 5$

giving (a) $-0.5(1.5) + 1.25(c_2) - 1.5 = 1.25c_2 - 2.25 \geq 0$

when $c_2 \geq 1.8$

(b) $5(1.5) - 2.5c_2 \geq 0$ when $c_2 \leq 3$ (cont)

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(iii) cont

(c) $-5(1.5) + 7.5c_2 \geq 0$ when $c_2 \geq 1$

Thus solution is optimal for

$1.8 \leq c_2 \leq 3$

(iv) If values per litre are b and c then objective should contain terms in x_4, x_5, the slack variables

i.e. $Z = 1.5x_1 + 2.0x_2 + 15x_3 + bx_4 + cx_5$

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(a) Wish to minimise the total transportation cost in moving commodity between

m sources with availabilities a_i $i=1,\dots,m$

n destinations " requirements b_j $j=1,\dots,n$

and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Cost per unit of transporting between source i and destination j is c_{ij} $\forall (i,j)$.

Problem is

$$\text{Minimise } C = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \quad \text{subject to}$$

$$\sum_{j=1}^{n} x_{ij} = a_i \quad i=1,\dots,m, \qquad \sum_{i=1}^{m} x_{ij} = b_j \quad j=1,\dots,n$$

where x_{ij} is quantity transported between source i and destination j. $x_{ij} \geq 0$ $\forall i,j$.

Rank of constraint matrix is $m+n-1$

Suppose solution has k variables +ve and that stated property is false. Then

$$k \geq 2m \quad \text{and} \quad k \geq 2n$$

and so $k \geq m+n$

For any basic solution $k \leq m+n-1$.

Contradiction shows statement is true.

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(b) If x_{ij} items are produced in month i to meet demand in month j problem can be expressed as a transportation problem with following cost / availability / requirement table.

	60	80	85	100	70	80
90	50	54	57	59	60	0
95		55	59	62	64	0
100			60	64	67	0
100	M			60	64	0
90		M			65	0

Final column represents unused capacity.

M - arbitrary large number.

Initial solution best by cheapest route method

	60	80	85	100	70	80	
90	60	30-			+		0
95		50+	45-				1
100			40+			60-	2
100				100	0		1
90					70-	20+	2
	50	54	58	59	63	-2	

(Some difficulty because of degeneracy)

giving optimal solution.

	60	80	85	100	70	80	
90	60				30		0
95		80	15				4
100			70			30	5
100				100	0		4
90					40	50	5
	50	51	55	56	60	-5	

(Cont)

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$u_i + v_j \le c_{ij} \quad \forall (ij)$ so this is optimal solution.

So produce 90 in month 1 to meet demand of 60 in month 1 and 30 in month 5 etc

Total cost is £22,850

[This can easily take more iterations to solve if degeneracy is handled differently but above should be alternatives]

(7)

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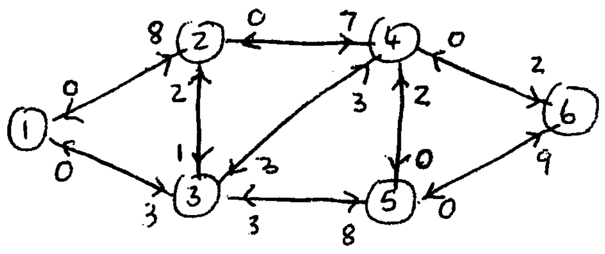
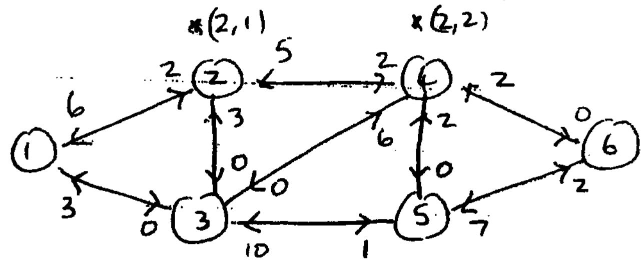
(a) (i) The network is



Alternative ways of introducing augmented flows but possible sequence is

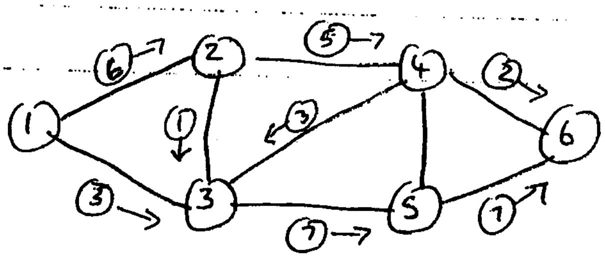
Flow 2 (1)→(2)→(4)→(6)
Flow 3 (1)→(3)→(5)→(6)
Flow 3 (1)→(2)→(4)→(3)→(5)→(6)
Flow 1 (1)→(2)→(3)→(5)→(6)

Final set of residual capacities is



Only nodes (2),(4) can be labelled (with unsaturated path from (1)) Flow is optimal.

corresponding to optimal flow pattern



Maximal flow = 9

(8

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(a) (ii) The minimal cut is $(1,2,4)(3,5,6)$

with capacity $\gamma(1,3) + \gamma(2,3) + \gamma(4,3) + \gamma(4,5) + \gamma(4,6)$

$= 3 + 1 + 3 + 0 + 2 = 9$

By virtue of maximal flow / minimal cut theorem no flow from ① to ⑥ can exceed 9, so above flow is maximal

(b) (i) An assignment with bottleneck

$\min\{r_{ij} \mid x_{ij} = 1\} = 7$ is obvious

$$\begin{pmatrix} 4 & 7 & 10^x & 7 & 2 \\ 5 & 4 & 7 & 9^x & 6 \\ 2 & 9^x & 6 & 5 & 8 \\ 8^x & 10 & 5 & 2 & 3 \\ 5 & 6 & 8 & 4 & 7^x \end{pmatrix}$$

so need to test feasibility of bottleneck 8

Set $r_{ij}' = r_{ij} \quad r_{ij} \geq 8$

$= 0 \quad r_{ij} < 8$

$$\begin{array}{c|c|c} & \begin{matrix} 6 & 7 & 8 & 9 & 10 \end{matrix} & \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 0 & 0 & 10^x & 0 & 0 \\ 0 & 0 & 0 & 9^x & 0 \\ 0 & 9^x & 0 & 0 & 8 \\ 8^x & 10 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \end{pmatrix} & \begin{matrix} (\times 8) \\ \\ \\ \\ \times(0) \end{matrix} \\ & (\times 5) & \end{array}$$

Labels show total allocation of 4 is maximal. Problem with bottleneck 8 has no feasible solution.

Above solution with bottleneck 7 is optimal.

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The related network is one with nodes:

Source node 0

nodes 1, ..., 5 corresponding to rows

nodes 6, ..., 10 — — — — columns

and sink node 11.

Capacities $\gamma(i,j)$ are

$$\gamma(0,i) = 1 \quad \text{for } i = 1, \ldots, 5$$

$$\gamma(j,11) = 1 \quad \text{for } j = 6, \ldots, 10$$

$$\gamma(i,j) = 5 \quad \text{if } i = 1, \ldots, 5 \quad j = 6, \ldots, 10$$
$$\text{and } r'_{i,j-1} > 0$$

All other $\gamma(i,j) = 0$

Making a full allocation is equivalent to sending a flow of 5 through the related network.

Labelling shows the maximal flow is 4 when

$$r'_{ij} = 0 \quad \text{when } r_{ij} < 8$$

So there is no solution with bottleneck 8.

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