

University of Bath

**DEPARTMENT OF MATHEMATICAL SCIENCES
EXAMINATION**

**MA30087: OPTIMISATION METHODS OF OPERATIONAL
RESEARCH**

Tuesday 25th January 2005, 13.00–15.00

Candidates may use university-supplied calculators.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

Examiners will attach importance to the number of
well-answered questions.

1. Consider the linear programming problem

$$\begin{aligned} &\text{maximise } \mathbf{c}^T \mathbf{x} \text{ subject to:} \\ &A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0} \end{aligned}$$

and its dual

$$\begin{aligned} &\text{minimise } \mathbf{b}^T \mathbf{w} \text{ subject to:} \\ &A^T \mathbf{w} \geq \mathbf{c}, \mathbf{w} \geq \mathbf{0}. \end{aligned}$$

- (i) Show that if \mathbf{x} and \mathbf{w} are feasible solutions to the primal and dual problems respectively, then $\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{w}$.
- (ii) If the primal problem has a feasible solution but no finite maximising solution, show that the dual has no feasible solution.
- (iii) If \mathbf{x}^0 and \mathbf{w}^0 are feasible solutions to the primal and dual respectively and satisfy the complementary slackness conditions:

$$\begin{aligned} (A^T \mathbf{w}^0 - \mathbf{c})^T \mathbf{x}^0 &= 0 \\ (A\mathbf{x}^0 - \mathbf{b})^T \mathbf{w}^0 &= 0 \end{aligned}$$

show that \mathbf{x}^0 and \mathbf{w}^0 are optimal solutions.

- (iv) The optimal solution to:

$$\begin{aligned} &\text{Maximise } 3x_1 + 5x_2 + 12x_3 \\ &\text{subject to } x_1 + x_2 + 2x_3 \leq 3 \\ &\quad 3x_1 + x_2 - x_3 \geq 5 \\ &\quad 2x_1 + x_2 - 3x_3 \leq 6 \\ &\quad x_1, x_2, x_3 \geq 0 \end{aligned}$$

is $x_1 = 1, x_2 = 2, x_3 = 0$.

Write down the dual problem and, using the complementary slackness conditions or otherwise, find its optimum solution. Take care over signs in this question.

2. The Sunshine drinks company makes three brands of canned drinks by blending orange and blackcurrant juice concentrates with water before carbonating and canning the mixture. The three brands are called Amaze, Black and Cerise and each uses different concentrations of orange and blackcurrant. The following table shows the number of litres of concentrate required to make one litre of product.

Drink	Orange	Blackcurrant
Amaze	0.3	0.1
Black	0.2	0.2
Cerise	0.1	0.2

On a particular day, there are 2,500 litres of orange juice and 1,800 litres of blackcurrant juice available. It is required to determine how much of each product should be produced in order to maximise total proceeds given that Amaze and Cerise sell for £1.50 per litre while Black sells for £2 per litre.

- (i) Formulate the problem of maximising total proceeds as a linear programme and use the Simplex method to obtain the optimal solution.
- (ii) Write down the dual problem and solution and explain how the optimal values of the dual variables may be interpreted in this case. Carry out a “right hand side ranging” and explain how this relates to the interpretation of the dual solution.
- (iii) Obtain the range of the selling price for Black for which the solution remains optimal.
- (iv) How would you amend your formulation of the problem to allow for the fact that a value can be placed on any concentrate unused at the end of the day.

3. (a) Explain what is meant by a transportation problem with m sources and n destinations. State without proof the rank of the constraint matrix. Show that in any basic solution, **either** there is a source which supplies only one destination **or** there is a destination supplied by only one source.
- (b) A factory makes a product for which there is a fluctuating but predictable demand. The following table shows the predicted demand for each of the next five months together with the production capacity and unit production costs for each month.

Month	1	2	3	4	5
Demand ('000)	60	80	85	100	70
Production Capacity ('000)	90	95	100	100	90
Unit production cost (pence)	50	55	60	60	65

Items produced in a month are available to meet demand in the same month, but they may also be stored to meet demand 1, 2, 3 or 4 months later at a cost of 4, 7, 9 and 10 pence per item respectively.

It is required to schedule production to meet the anticipated demand at minimum total cost. Formulate this as a transportation problem and obtain the solution.

4. (a) Consider the capacitated network having six nodes and the following capacity matrix.

$$\begin{pmatrix} 0 & 8 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 7 & 0 & 0 \\ 0 & 2 & 0 & 3 & 8 & 0 \\ 0 & 0 & 3 & 0 & 0 & 2 \\ 0 & 0 & 3 & 2 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- (i) Draw the corresponding network and use the Ford-Fulkerson method to determine the maximal flow from node 1 to node 6. Show clearly the required flows along each arc.
 - (ii) Determine the cut of minimal capacity and, quoting an important theorem in network flow, justify the optimality of your flow.
- (b) Consider the bottleneck assignment problem with the following ratings.

$$\begin{pmatrix} 4 & 7 & 10 & 7 & 2 \\ 5 & 4 & 7 & 9 & 6 \\ 2 & 9 & 6 & 5 & 8 \\ 8 & 10 & 5 & 2 & 3 \\ 5 & 6 & 8 & 4 & 7 \end{pmatrix}$$

- (i) Solve the problem.
- (ii) Explain the argument, involving the maximal flow through a related network, that shows that your solution to the bottleneck problem is optimal.