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(a) Bellman's principle is:

An optimal policy has the property that whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

It is intuitive, but relies on a Markovian property of the multi-stage decision process. Future decisions depend only upon the current state and not on the previous history of the process. Of course, if necessary, the state can be defined to include relevant properties of the history. (4)

(b) Define $f_i(x)$ as minimum wastage from a roll length x using orders $i, (i+1), \ldots n$. Then

$$f_n(x) = x \qquad x < l_n$$
$$= x - l_n \qquad x \geq l_n$$

and for $i = 1, \ldots, n-1$

$$f_i(x) = \operatorname{Min}\left[f_{i+1}(x),\ f_{i+1}(x - l_i)\right], \quad x \geq l_i$$
$$= f_{i+1}(x), \quad x < l_i$$

These can be solved recursively, first tabulating $f_n(x)$ for $0 \leq x \leq L$ and successively $f_{n-1}(x), \ldots f_2(x)$ and finally $f_1(L)$. $y_i = 1$ if length i included and $y_i = 0$ otherwise, the value of y_i is also recorded for each tabulated $f_i(x)$. (4)

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(b) (cont)

Using the given small problem for illustration :-

x	$f_4(x)$	y_4	$f_3(x)$	y_3	$f_2(x)$	y_2
1	1	0	1	0	1	0
2	2	0	2	0	2	0
3	3	0	3	0	3	0
4	4	0	4	0	4	0
5	5	0	5	0	5	1
6	6	0	6	0	1	1
7	7	0	0	1	0	0
8	0	1	0	0	0	0
9	1	1	1	0	1	0
10	2	1	2	0	2	0
11	3	1	3	0	3	0
12	4	1	4	0	0	1
13	5	1	5	0	0	1
14	6	1	6	0	1	1

Now $f_1(14) = \text{Min}\left[f_2(14), f_2(11) \right] = f_2(14) = 1 \quad y_1 = 0$ (4)

Back tracking, solution is $y_1 = 0, y_2 = 1, y_3 = 0, y_4 = 1$.

producing waste of 1.

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(c) Define $f_i(x)$ as maximal utility over times $t = i, i+1, \ldots, n$ starting time i with amount x of cash.

Recurrence is

$$f_i(x) = \max_{0 \le y_i \le x} \left[\phi(y_i) + f_{i+1}\left[\alpha(x - y_i)\right]\right] \quad i = 1, \ldots n-1$$

$$f_n(x) = \phi(x)$$

Solve recursively for $f_1(C)$. (4)

In the modified problem the 'state' needs to include the amount invested at the previous time point.

Define $f_i(x, v)$ as maximal utility over times $t = i, i+1, \ldots, n$ starting with cash x and investment v.

Recurrence is

$$f_i(x, v) = \max_{0 \le y_i \le x} \left[\phi(y_i) + f_{i+1}\left[\alpha(x - y_i) + \beta v, y\right]\right]$$

$$f_n(x, v) = \phi(x)$$ Solve recursively for $f_1(C, 0)$ (4)

Since each $f_i(x, v)$ must be calculated for each value of the state (x, v) this is much more work than with a univariate state. Bellman calls this "the curse of dimensionality".

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(i) Since $A\underline{x} \le \underline{b}$ and $\underline{w} \ge \underline{0}$

$$\underline{w}^T A \underline{x} \le \underline{w}^T b = b^T \underline{w} \quad -(1)$$

Since $A^T \underline{w} \ge \underline{c}$ and $\underline{x} \ge \underline{0}$

$$\underline{x}^T A^T \underline{w} \ge \underline{x}^T \underline{c} \text{ or transposing } \underline{w}^T A \underline{x} \ge \underline{c}^T \underline{x} \quad -(2)$$

Comparing (1) and (2) gives $\underline{c}^T \underline{x} \le \underline{b}^T \underline{w}$ (4)

(ii) Suppose, for contradiction, dual has feasible solution $\underline{w}$ with $\underline{b}^T \underline{w} = M$. Since primal has no finite maximising solution $\exists\ \underline{x} \in F$ (the set of feasible solns) with

$$\underline{c}^T \underline{x} > M$$

This contradicts result in (i) so dual has no feasible solution. (4)

(iii)

$$(A^T \underline{w}^o - \underline{c})^T \underline{x}^o = 0 \Rightarrow (A^T \underline{w}^o)^T \underline{x}^o = \underline{c}^T \underline{x}^o$$

$$\Rightarrow \underline{w}^{oT} A \underline{x}^o = \underline{c}^T \underline{x}^o$$

$$(A\underline{x}^o - \underline{b})^T \underline{w}^o = 0 \Rightarrow (A\underline{x}^o)^T \underline{w}^o = \underline{b}^T \underline{w}^o$$

$$\Rightarrow \underline{w}^{oT} A x^o = \underline{w}^{oT} \underline{b} = \underline{b}^T \underline{w}^o$$

Thus we have equality between primal and dual objective and by result in part (i) both are optimal. (6)

(cont)

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(iv) Dual problem is

Minimise $3w_1 - 5w_2 + 6w_3$ subject to

$$w_1 - 3w_2 + 2w_3 \geq 3$$
$$w_1 - w_2 + w_3 \geq 5$$
$$2w_1 + w_2 - 3w_3 \geq 12$$

$$w_1, w_2, w_3 \geq 0$$

Using complementary slackness

$$x_1 > 0 \Rightarrow w_1 - 3w_2 + 2w_3 = 3$$
$$x_2 > 0 \Rightarrow w_1 - w_2 + w_3 = 5$$
$$2x_1 + x_2 - 3x_3 < 6 \Rightarrow w_3 = 0$$

Solving $w_1 - 3w_2 = 3$, $w_1 - w_2 = 5$ (6)

gives dual solution

$$w_1 = 6, \; w_2 = 1, \; w_3 = 0$$

with primal objective = dual objective = 13.

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(1) Produce x_1, x_2, x_3 litres of A, B and C respectively.

Problem is Maximise $z = 1.5x_1 + 2.0x_2 + 1.5x_3$

Subject to $0.3x_1 + 0.2x_2 + 0.1x_3 \leq 2500$

$0.1x_1 + 0.2x_2 + 0.2x_3 \leq 1800$ $\quad x_1, x_2, x_3 \geq 0$ (3)

Adding slack variables x_4, x_5 and solving:-

	x_1	x_2	x_3	x_4	x_5	
x_4	0.3	0.2	0.1	1	0	2500
x_5	0.1	0.2	0.2	0	1	1800
z	-1.5	-2.0	-1.5			0
		↑				

	x_1	x_2	x_3	x_4	x_5	
x_4	0.2	0	-0.1	1	-1	700
x_2	0.5	1	1	0	5	9000
z	-0.5	0	0.5	0	10	18000
	↑					

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	-0.5	5	-5	3500
x_2	0	1	1.25	-2.5	7.5	7250
z	0	0	0.25	2.5	7.5	19750

Optimal solution $x_1 = 3500$, $x_2 = 7250$, $x_3 = 0$, $z = 19,750$

i.e. produce 3500 litres of A, 7250 litres of B and none of C. (7)

Total proceeds are £19,750.

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(ii) The dual problem is

Minimise $2500w_1 + 1800w_2$

subject to
$$0.3w_1 + 0.1w_2 \geq 1.5$$
$$0.2w_1 + 0.2w_2 \geq 2.0 \qquad w_1, w_2 \geq 0$$
$$0.1w_1 + 0.2w_2 \geq 1.5$$

with optimal solution $\underline{w_1 = 2.5,\ w_2 = 7.5}$

w_1, w_2 represent the marginal value of 1 litre of orange, blackcurrent concentrates respectively. 3

For feasibility require $B^{-1}\underline{b} \geq \underline{0}$ so

ranging b_1, $\begin{pmatrix} 5 & -5 \\ -2.5 & 7.5 \end{pmatrix}\begin{pmatrix} b_1 \\ 1800 \end{pmatrix} = \begin{pmatrix} 5b_1 - 9000 \\ -2.5b_1 - 13{,}500 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

when $\underline{1800 \leq b_1 \leq 5400}$

Ranging b_2: $\begin{pmatrix} 5 & -5 \\ -2.5 & 7.5 \end{pmatrix}\begin{pmatrix} 2500 \\ b_2 \end{pmatrix} = \begin{pmatrix} 12500 - 5b_2 \\ -6250 + 7.5b_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

when $\underline{833\tfrac{1}{3} \leq b_2 \leq 2500}$ 3

These are the ranges within which the above marginal values apply.

(iii) For optimality require $\sum_{i \in p} c_i y_{ij} - c_j \geq 0 \qquad j = 1, \dots, 5$

giving (a) $-0.5(1.5) + 1.25(c_2) - 1.5 = 1.25c_2 - 2.25 \geq 0$

when $c_2 \geq 1.8$

(b) $5(1.5) - 2.5c_2 \geq 0$ when $c_2 \leq 3$ (cont)

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(iii) cont

(c) $-5(1.5) + 7.5c_2 \geq 0$ when $c_2 \geq 1$

Thus solution is optimal for

$1.8 \leq c_2 \leq 3$ (2)

(iv) If values per litre are b and c then objective should contain terms in x_4, x_5, the slack variables

i.e. $Z = 1.5x_1 + 2.0x_2 + 15x_3 + bx_4 + cx_5$ (2)

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(a) Wish to minimise the total transportation cost in moving commodity between

m sources with availabilities a_i $i=1,\dots,m$

n destinations .. requirements b_j $j=1,\dots,n$

and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Cost per unit of transporting between source i and destination j is c_{ij} $\forall (i,j)$.

Problem is

Minimise $C = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$ subject to

$\sum_{j=1}^{n} x_{ij} = a_i$ $i=1,\dots,m$, $\sum_{i=1}^{m} x_{ij} = b_j$ $j=1,\dots,n$

where x_{ij} is quantity transported between source i and destination j. (4)

Rank of constraint matrix is $m+n-1$ (1)

Suppose solution has k variables +ve and that stated property is false. Then

$k \geq 2m$ and $k \geq 2n$

and so $k \geq m+n$

For any basic solution $k \leq m+n-1$.

Contradiction shows statement is true. (4)

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(b) If x_{ij} items are produced in month i to meet demand in month j problem can be expressed as a transportation problem with following cost / availability / requirement table.

	60	80	85	100	70	80
90	50	54	57	59	60	0
95		55	59	62	64	0
100			60	64	67	0
100	M			60	64	0
90			M		65	0

Final column represents unused capacity.

M - arbitrary large number. (4)

Initial solution best by cheapest route method

	60	80	85	100	70	80	
90	60	30-			+		0
95		50+	45-				1
100			40+			60-	2
100				100	0		1
90					70-	20+	2
	50	54	58	59	63	-2	

(Some difficulty because of degeneracy)

giving optimal solution.

	60	80	85	100	70	80	
90	60				30		0
95		80	15				4
100			70			30	5
100				100	0		4
90					40	50	5
	50	51	55	56	60	-5	

(Cont)

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$u_i + v_j \leq c_{ij} \quad \forall (ij)$ so this is optimal solution.

So produce 90 in month 1 to meet demand of 60 in month 1 and 30 in month 5 etc
Total cost is £22,885

(7)

[This can easily take more iterations to solve if degeneracy is handled differently but above should be attainable]

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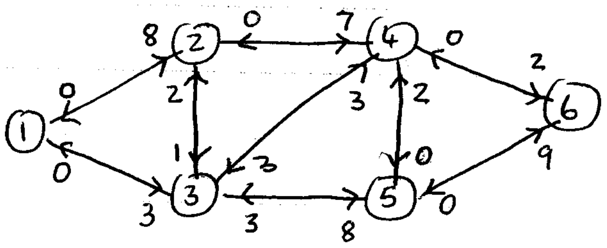
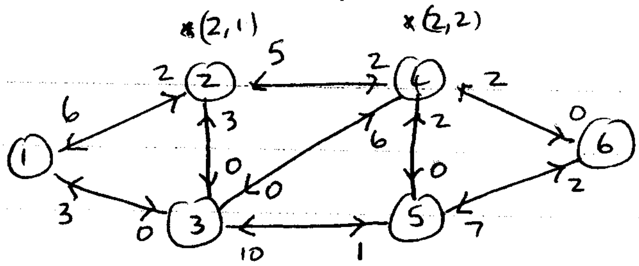
(a) (i) The network is



Alternative ways of introducing augmented flows but possible sequence is

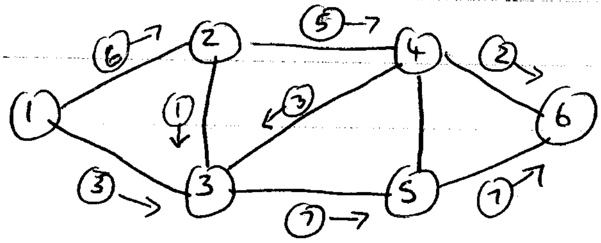
Flow 2 ① → ② → ④ → ⑥
Flow 3 ① → ③ → ⑤ → ⑥
Flow 3 ① → ② → ④ → ③ → ⑤ → ⑥
Flow 1 ① → ② → ③ → ⑤ → ⑥

Final set of residual capacities is



Only nodes ②, ④ can be labelled (with unsaturated path from ①) Flow is optimal.

corresponding to optimal flow pattern



Maximal flow = 9 (8)

Total

DEPARTMENT OF MATHEMATICAL SCIENCES University of Bath

Outline solution to Examination Questions

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(a) (ii) The minimal cut is $(1,2,4)(3,5,6)$

with capacity $\gamma(1,3) + \gamma(2,3) + \gamma(4,3) + \gamma(4,5) + \gamma(4,6)$

$= 3 + 1 + 3 + 0 + 2 = 9$

By virtue of maximal flow / minimal cut theorem no flow from ① to ⑥ can exceed 9, so above flow is maximal. (3)

(b) (i) An assignment with bottleneck

$\min\{r_{ij} \mid x_{ij} = 1\} = 7$ is obvious

$$\begin{pmatrix} 4 & 7 & 10^{\times} & 7 & 2 \\ 5 & 4 & 7 & 9^{\times} & 6 \\ 2 & 9^{\times} & 6 & 5 & 8 \\ 8^{\times} & 10 & 5 & 2 & 3 \\ 5 & 6 & 8 & 4 & 7^{\times} \end{pmatrix}$$

so need to test feasibility of bottleneck 8.

Set $r_{ij}' = r_{ij}$ $r_{ij} \geq 8$

$= 0$ $r_{ij} < 8$

$$\begin{array}{c|ccccc|l} & 6 & 7 & 8 & 9 & 10 & \\ \hline 1 & 0 & 0 & 10^{\times} & 0 & 0 & (\times 8) \\ 2 & 0 & 0 & 0 & 9^{\times} & 0 & \\ 3 & 0 & 9^{\times} & 0 & 0 & 8 & \\ 4 & 8^{\times} & 10 & 0 & 0 & 0 & \\ 5 & 0 & 0 & 8 & 0 & 0 & \times(0) \\ \hline & & & (\times 5) & & & \end{array}$$

Labels show total allocation of 4 is maximal. Problem with bottleneck 8 has no feasible solution. (5)

Above solution with bottleneck 7 is optimal.

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The related network is one with nodes:

Source node 0

nodes 1,...,5 corresponding to rows

nodes 6,...,10 - - - - - columns

and sink node 11.

Capacities $\gamma(i,j)$ are

$\gamma(0,i) = 1$ for $i = 1,\ldots,5$

$\gamma(j,11) = 1$ for $j = 6,\ldots,10$

$\gamma(i,j) = 5$ if $i = 1,\ldots,5$ $j = 6,\ldots,10$ and $r'_{i,j-1} > 0$

All other $\gamma(i,j) = 0$

Making a full allocation is equivalent to sending a flow of 5 through the related network.

Labelling shows the maximal flow is 4 when

$r'_{ij} = 0$ when $r_{ij} < 8$

So there is no solution with bottleneck 8. 4

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