University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA50087: OPTIMISATION METHODS OF OPERATIONAL RESEARCH

Tuesday 25th January 2005, 13.00–15.00

Candidates may use university-supplied calculators.

Full marks will be given for correct answers to THREE questions.
All candidates must do Section A, and two questions from Section B.
Only the best three answers will contribute towards the assessment.

Section A

1. (a) State Bellman's principle of optimality and comment briefly on its meaning and applicability.

(b) There are n orders for lengths l_i $(i = 1, 2, ..., n)$ of a particular type of carpet. Each order must be supplied in one piece. A part roll of this carpet of length L is available and it is required to use as much of this as possible in meeting some or all of the orders. Any remaining orders will be supplied from a full roll. Show how the problem of minimising the wastage in the part roll may be formulated as a dynamic programme.

Explain the computational procedure for finding the optimal solution, using the following small example as an illustration.

$n = 4, L = 14, l_1 = 3, l_2 = 5, l_3 = 7, l_4 = 8.$

(c) A woman has an initial amount, C, of cash and at each time $t(t = 1, 2, ..., n)$ makes a decision on how much to spend and how much to invest. An amount y_t spent at time t produces utility $\Phi(y_t)$; the amount not spent v_t at time t is invested, producing cash αv_t at time $(t + 1)$. She is not allowed to be in debt, but cash received from any investment at time $(t - 1)$ is received before she makes her choice of y_t and v_t. She requires to find the policy which will maximise her overall utility

$$\textstyle\sum_{t=1}^{n} \Phi(y_t)$$

Show how the problem may be formulated as a dynamic programme.

Suppose the situation changes so that an amount v_t invested at any time t now produces cash αv_t at time $(t+1)$ **plus** an amount βv_t at time $(t+2)$. Show how the modified problem may be formulated, but explain briefly why obtaining the solution is considerably more difficult.

Section B

2. Consider the linear programming problem

$$\begin{aligned}&\text{maximise } \mathbf{c}^T\mathbf{x} \text{ subject to:}\\ &A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\end{aligned}$$

and its dual

$$\begin{aligned}&\text{minimise } \mathbf{b}^T\mathbf{w} \text{ subject to:}\\ &A^T\mathbf{w} \geq \mathbf{c}, \mathbf{w} \geq \mathbf{0}.\end{aligned}$$

(i) Show that if $\mathbf{x}$ and $\mathbf{w}$ are feasible solutions to the primal and dual problems respectively, then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{w}$.

(ii) If the primal problem has a feasible solution but no finite maximising solution, show that the dual has no feasible solution.

(iii) If $\mathbf{x}^0$ and $\mathbf{w}^0$ are feasible solutions to the primal and dual respectively and satisfy the complementary slackness conditions:

$$(A^T\mathbf{w}^0 - \mathbf{c})^T\mathbf{x}^0 = 0$$
$$(Ax^0 - \mathbf{b})^T\mathbf{w}^0 = 0$$

show that $\mathbf{x}^0$ and $\mathbf{w}^0$ are optimal solutions.

(iv) The optimal solution to:

$$\begin{aligned}\text{Maximise } & 3x_1 + 5x_2 + 12x_3\\ \text{subject to } & x_1 + x_2 + 2x_3 \leq 3\\ & 3x_1 + x_2 - x_3 \geq 5\\ & 2x_1 + x_2 - 3x_3 \leq 6\\ & x_1, x_2, x_3 \geq 0\end{aligned}$$

is $x_1 = 1, x_2 = 2, x_3 = 0$.
Write down the dual problem and, using the complementary slackness conditions or otherwise, find its optimum solution. Take care over signs in this question.

3. The Sunshine drinks company makes three brands of canned drinks by blending orange and blackcurrant juice concentrates with water before carbonating and canning the mixture. The three brands are called Amaze, Black and Cerise and each uses different concentrations of orange and blackcurrant. The following table shows the number of litres of concentrate required to make one litre of product.

Drink	Orange	Blackcurrant
Amaze	0.3	0.1
Black	0.2	0.2
Cerise	0.1	0.2

On a particular day, there are 2,500 litres of orange juice and 1,800 litres of blackcurrant juice available. It is required to determine how much of each product should be produced in order to maximise total proceeds given that Amaze and Cerise sell for £1.50 per litre while Black sells for £2 per litre.

(i) Formulate the problem of maximising total proceeds as a linear programme and use the Simplex method to obtain the optimal solution.

(ii) Write down the dual problem and solution and explain how the optimal values of the dual variables may be interpreted in this case. Carry out a "right hand side ranging" and explain how this relates to the interpretation of the dual solution.

(iii) Obtain the range of the selling price for Black for which the solution remains optimal.

(iv) How would you amend your formulation of the problem to allow for the fact that a value can be placed on any concentrate unused at the end of the day.

4. (a) Explain what is meant by a transportation problem with m sources and n destinations. State without proof the rank of the constraint matrix. Show that in any basic solution, **either** there is a source which supplies only one destination **or** there is a destination supplied by only one source.

(b) A factory makes a product for which there is a fluctuating but predictable demand. The following table shows the predicted demand for each of the next five months together with the production capacity and unit production costs for each month.

Month	1	2	3	4	5
Demand ('000)	60	80	85	100	70
Production Capacity ('000)	90	95	100	100	90
Unit production cost (pence)	50	55	60	60	65

Items produced in a month are available to meet demand in the same month, but they may also be stored to meet demand 1, 2, 3 or 4 months later at a cost of 4, 7, 9 and 10 pence per item respectively.

It is required to schedule production to meet the anticipated demand at minimum total cost. Formulate this as a transportation problem and obtain the solution.

5. (a) Consider the capacitated network having six nodes and the following capacity matrix.

$$\begin{pmatrix} 0 & 8 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 7 & 0 & 0 \\ 0 & 2 & 0 & 3 & 8 & 0 \\ 0 & 0 & 3 & 0 & 0 & 2 \\ 0 & 0 & 3 & 2 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) Draw the corresponding network and use the Ford-Fulkerson method to determine the maximal flow from node 1 to node 6. Show clearly the required flows along each arc.

(ii) Determine the cut of minimal capacity and, quoting an important theorem in network flow, justify the optimality of your flow.

(b) Consider the bottleneck assignment problem with the following ratings.

$$\begin{pmatrix} 4 & 7 & 10 & 7 & 2 \\ 5 & 4 & 7 & 9 & 6 \\ 2 & 9 & 6 & 5 & 8 \\ 8 & 10 & 5 & 2 & 3 \\ 5 & 6 & 8 & 4 & 7 \end{pmatrix}$$

(i) Solve the problem.

(ii) Explain the argument, involving the maximal flow through a related network, that shows that your solution to the bottleneck problem is optimal.