DEPARTMENT OF MATHEMATICAL SCIENCES University of Bath
Outline solution to Examination Questions

Unit Code	Unit Title	
MA30087	OPTIMISATION METHODS OF OPERATIONAL RESEARCH	
Academic Year 2003/4	Examiner M E BRIGDEN	
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Part | | Mark

(a) Suppose $\underline{x}^0$ is an optimal solution with exactly p variables positive. Without loss of generality take $\underline{x}^0$ to be of form

$$\underline{x}^0 = (x_1^0, x_2^0, \dots x_p^0, 0, \dots, 0)^T$$

with $x_i^0 > 0 \quad i = 1, \dots, p$.

Either (i) $\underline{a}_1, \underline{a}_2, \dots \underline{a}_p$ are l.i. when $\underline{x}^0$ is basic

or (ii) can write $\quad t_1\underline{a}_1 + t_2\underline{a}_2 + \dots t_p\underline{a}_p = \underline{0} \quad$ — ①

with at least one $t_i > 0$.

Since $\underline{x}^0$ satisfies constraints,

$$x_1^0\underline{a}_1 + x_2^0\underline{a}_2 + \dots x_p^0\underline{a}_p = \underline{b}. \quad \text{— ②}$$

② − ε① gives $\quad \displaystyle\sum_{i=1}^{p}(x_i^0 - \varepsilon t_i)\underline{a}_i = \underline{b} \quad$ — ③

Define $\quad \delta_1 = \max\limits_{t_i<0}\left(\dfrac{x_i^0}{t_i}\right) \quad$ or $= -\infty$ if all $t_i \geq 0$

$$\delta_2 = \min_{t_i>0}\left(\frac{x_i^0}{t_i}\right)$$

then for $\delta_1 \leq \varepsilon \leq \delta_2$ the new solution

$$\left.\begin{aligned} x_i &= x_i^0 - \varepsilon t_i \quad & i = 1, \dots, p \\ &= 0 & i = p+1, \dots n \end{aligned}\right\}$$

is feasible.

By optimality of $\underline{x}^0$

$$\sum_{i=1}^{p} c_i x_i^0 \geq \sum_{i=1}^{p} c_i(x_i^0 - \varepsilon t_i) \quad \forall \varepsilon \in [\delta_1, \delta_2]$$

i.e. $\quad \varepsilon\displaystyle\sum_{i=1}^{p} c_i t_i \geq 0 \quad \forall \varepsilon \in [\delta_1, \delta_2]$

Since $\delta_1 < 0$ and $\delta_2 > 0$ this implies $\displaystyle\sum_{i=1}^{p} c_i t_i = 0$

Total

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	(Cont)	
	and $\sum_{i=1}^{p} c_i x_i^0 = \sum_{i=1}^{p} c_i (x_i^0 - \varepsilon t_i)$	
	so the new solution is also optimal.	
	Thus setting $\varepsilon = \delta_2$ gives an optimal solution	
	$x_i = x_i^0 - \delta_2 t_i \quad i = 1, \dots, p$	
	$= 0 \quad i = (p+1), \dots n$	
	and by virtue of the definition of δ_2, this has at most $(p-1)$ variables that are positive.	
	The whole argument can be repeated until case (i) at start occurs when an optimal basic feasible solution will have been created.	14
	Bookwork but hard to get correct in exam conditions	
(b)	Create 2 new variables x_1^+ and x_1^- such that	
	$x_1 = x_1^+ - x_1^-, \quad x_1^+, x_1^- \geq 0 \quad x_1^+ x_1^- = 0$	
	Then the problem becomes the L.P. :-	
	Maximise $c_1 x_1^+ + c_1 x_1^- + \sum_{j=2}^{n} c_j x_j$	
	subject to $a_{i1}(x_1^+ - x_1^-) + \sum_{j=2}^{n} a_{ij} x_j = b_i \quad i = 1, \dots, m$	
	$x_1^+, x_1^- \geq 0, \quad x_j \geq 0 \quad j = 2, \dots, n.$	3
	If $\underline{a}_1^+$ and $\underline{a}_1^-$ are columns of x_1^+ and x_1^- in the constraint system then $\underline{a}_1^+ = -\underline{a}_1^-$. No basic solution can have both x_1^+ and x_1^- as basic variables so in any basic solution the non linear	
	$x_1^+ x_1^- = 0$ is automatically satisfied.	3
	Not routine. Tests understanding.	
	Total	20

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(i)	Make x_i units of Product i per week $i = 1, 2, 3$.	

Problem is to maximise $Z = 8x_1 + 3x_2 + 9x_3$

subject to

$$3x_1 + 2x_2 + 2x_3 \le 2000$$
$$x_1 + 2x_2 + 3x_3 \le 1500$$
$$x_1 + x_2 \ge 500$$
$$x_1, x_2, x_3 \ge 0$$

3

Add slack variables x_4, x_5, surplus variable x_6 and artificial variable x_7.

Phase I objective is to Maximise $W = -x_7$ subject to augmented system.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	w	
x_4	3	2	2	1					2000
x_5	1	2	3		1				1500
x_7	1	1				-1	1		500
w	-1	-1						1	500
	↑								

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	w	z	
x_4	0	-1	2	1		3				500
x_5	0	1	3		1	1				1000
x_1	1	1				-1				500
w	0	0	0	0	0	0		1		0
Z	0	5	-9	0	0	-8				4000
			↑							

End of Phase 1

	x_1	x_2	x_3	x_4	x_5	x_6	z	
x_3	0	$-1/2$	1	$1/2$	0	$3/2$		250
x_5	0	$5/2$	0	$-3/2$	1	$-7/2$		250
x_1	1	1	0	0	0	-1		500
z	0	$1/2$	0	$9/2$	0	$11/2$		6250

Standard type of exercise

End of Phase 2

7

Total

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(i)	(cont) Optimal solution is to produce 500 of Product 1, 250 of Product 3 with weekly profit £6250	1
(ii)	Writing original problem in form of standard maximisation problem: Maximise $8x_1 + 3x_2 + 9x_3$ subject to $3x_1 + 2x_2 + 2x_3 \le 2000$, $x_1 + 2x_2 + 3x_3 \le 1500$, $-x_1 - x_2 \le -500$, $x_1, x_2, x_3 \geq 0$	
	So dual problem is Minimise $2000w_1 + 1500w_2 - 500w_3$ subject to $3w_1 + w_2 - w_3 \geq 8$, $2w_1 + 2w_2 - w_3 \geq 3$, $2w_1 + 3w_2 \geq 9$, $w_1, w_2 w_3 \geq 0$	2
	From simplex tableau optimal dual solution is $w_1 = 9/2$, $w_2 = 0$, $w_3 = 11/2$.	
	By inspection this is dual feasible and gives a dual objective of 6250. This equals the primal objective so by the Duality Theorem, solutions are optimal.	2
	w_1 represents the marginal benefit (£) from extra (mins) machining time w_2 - - - - - - hand finishing time w_3 - - - - - - of reducing (unit) the contracted quantity.	2

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(iii)	The basis inverse, B^{-1}, in this case is	
	$$B^{-1} = (\underline{y}_4, \underline{y}_5, -\underline{y}_6) = \begin{pmatrix} 1/2 & 0 & -3/2 \\ -3/2 & 1 & 7/2 \\ 0 & 0 & 1 \end{pmatrix}$$	
	With new right hand side vector $\underline{b}' = \begin{pmatrix} 2000 \\ 1500 \\ 600 \end{pmatrix}$	
	values of variables are given by	
	$$B^{-1}\underline{b}' = \begin{pmatrix} 100 \\ 600 \\ 600 \end{pmatrix}$$	
	i.e. new solution is $x_1 = 600,\ x_2 = 0,\ x_3 = 100,\ x_4 = 0,\ x_5 = 600$	3
	with $z = 5700$	
	(ii) and (iii) standard type of questions but made more than usually difficult because of mixed constraints in primal. Meaning of u_3 and obtaining B^{-1} require insight.	
	Total	**20**

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Problem can be formulated as following balanced transportation problem.

	30	60	20	15	20
25	120	70	200	60	100
50	80	130	100	130	140
30	190	130	70	110	160
40	130	90	100	180	150

Initial (degenerate) basic feasible solution from cheapest route method

	30	60	20	15	20	u_i
25		10+		15−		0
50	30	0			20	60
30		10−	20	+		60
40		40				20
v_j	20	70	10	60	80	

Not optimal. x_{34} becomes basic $\theta_{max} = 10$ x_{32} becomes non basic

	30	60	20	15	20	u_i
25		20		5		−60
50	30	0			20	0
30			20	10		−10
40		40				−40
v_j	80	130	80	120	140	

$u_i + v_j \leq c_{ij}$ $\forall (i,j)$

Solution is optimal

Take 20,000 tons A to L, 5,000 tons A to N etc

Objective 13,000 or 13,000,000 tons/metres. Standard problem 10

Total

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(a) Problem obtained from previous by reducing a_1 and b_1 by 20

	10	60	20	15	20
5	120	70	200	60	100
50	80	130	100	130	140
30	190	130	70	110	160
40	130	90	100	180	150

	10	60	20	15	20	U_i
5		0		5		-60
50	10	20			20	0
30			20	10		-10
40		40				-40
V_j	80	130	80	120	140	

Solution plus 20 A to K.

Objective 15,000 or 15,000,000 tons/metres

(b) Formulate by splitting source D into D_1 and D_2 "pricing out" route $D_2 \to L$.

	30	60	20	15	20
25	120	70	200	60	100
50	80	130	100	130	140
30	190	130	70	110	160
D_1 20	130	90	100	180	150
D_2 20	130	M	100	180	150

	30	60	20	15	20	U_i
25		20		5		-60
50	30	20			0	0
30			20	10		-10
D_1 20		20				-40
D_2 20					20	10
V_j	80	130	80	120	140	

Optimal solution 20,000 tons A to L etc

Objective 14,000 or 14,000,000 tons/metres

Not straightforward to formulate modified problems. Solutions will cause time problem unless candidate understands well how to obtain good initial solution.

Total 2

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(a)	A cut is an ordered pair (A, B) of subsets of N satisfying (i) $s \in A$ (ii) $d \in B$ (iii) $A \cup B = N$ (iv) $A \cap B = \Phi$	2
	Denote $\sum_{i \in C} \sum_{j \in D} f(i,j)$ by $f(C, D)$ for $C \subseteq N$, $D \subseteq N$	
	Then $\sum_{j \in N} f(s,j) = f(s, N) = f(A, N) - f((A \setminus s), N)$	
	$= f(A,A) + f(A,B) - f(A \setminus s, N)$ — ①	
	Since $f(i,j) = -f(j,i)$ $\forall$ $i,j \in N$ $f(A,A) = -f(A,A)$	
	Bookwork but not at all easy. so $f(A,A) = 0$	
	For $i \neq s, d$ $\sum_{j \in N} f(i,j) = \sum_{j \in N} f(j,i)$	
	but $f(i,j) = -f(j,i)$ so both sums must be zero	
	i.e. $f(i, N) = 0$ and summing over $i \in A \setminus s$	
	gives $f(A \setminus s, N) = 0$.	
	Thus from ① above $f(s, N) = f(A, B)$ as required.	7
(b)	Let f be a flow that maximises $f(s, N)$ then for any cut (A, B)	
	$f(s, N) = f(A, B) \leq \sum_{i \in A} \sum_{j \in B} \gamma(i,j) = \gamma(A, B)$	
	The theorem states that there exists a cut (A, B) such that $f(s, N) = \gamma(A, B)$ Bookwork	
	i.e. maximal flow equals minimal cut.	3
	Total	

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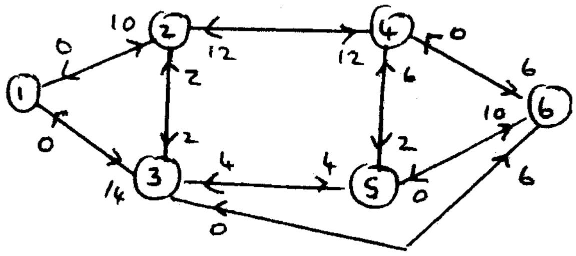
(c) Network is as follows



By inspection make initial flows

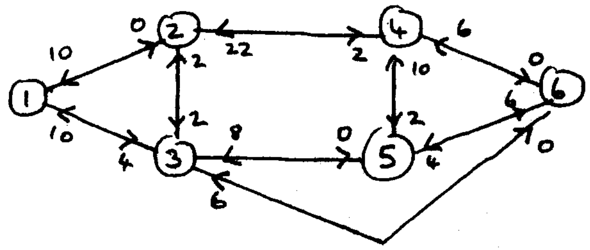
$1 \to 2 \to 4 \to 6$ Flow 6

$1 \to 2 \to 4 \to 5 \to 6$ Flow 4

$1 \to 3 \to 5 \to 6$ Flow 4

$1 \to 3 \to 6$ Flow 6

Residual capacities are



Labelling nodes (γ, i) where γ is residual capacity from source, i is previous node of unsaturated path

Node 3 (4,1), Node 2 (2,3), Node 4 (2,2)

Node 5 (2,4) Node 6 (2,5)

identifies unsaturated path $1 \to 3 \to 2 \to 4 \to 5 \to 6$ with residual capacity 2.

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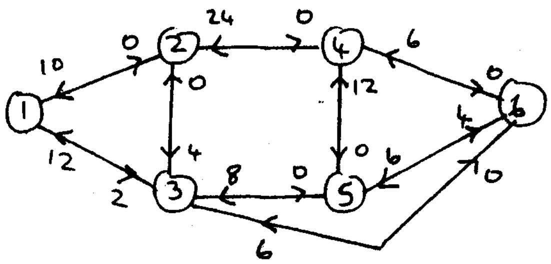
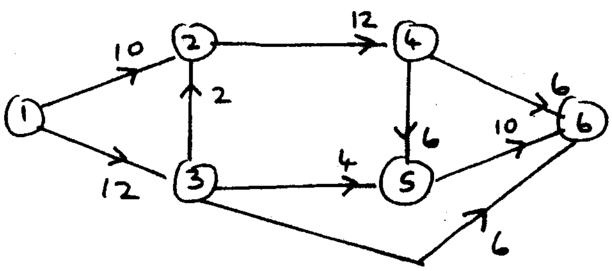
(c) (cont)

Increasing flow by 2 along above path gives new residual capacities as follows.



Labelling now gives just Node 3 (2, 1) and there is no unsaturated path 1 → 6.

Maximal flow is



6

Maximal flow is 22

Minimal cut is [(1,3) (2,4,5,6)]

with capacity 10 + 2 + 4 + 6 = 22

Standard problem type but more difficult than usual.

2

Total 20