M

1. Solution

(i) Make x_1 tons of A and x_2 tons of B. Problem is:

$$\text{Maximise } z = 80x_1 + 50x_2 \quad \text{subject to}$$

$$\tfrac{1}{4}x_1 + \tfrac{1}{2}x_2 \le 3200$$

$$\tfrac{3}{4}x_1 + \tfrac{1}{2}x_2 \le 4000 \qquad x_1, x_2 \ge 0$$

$$x_1 \le 2000$$

(3)

After adding slack variables x_3, x_4, x_5, the initial tableau is

	x_1	x_2	x_3	x_4	x_5	
x_3	1/4	1/2	1	0	0	3200
x_4	3/4	1/2	0	1	0	4000
x_5	1	0	0	0	1	2000
z	−80	−50	0	0	0	0

$\xrightarrow{x_1 \text{ replaces } x_5}$

	x_1	x_2	x_3	x_4	x_5	
x_3	0	1/2	1	0	−1/4	2700
x_4	0	1/2	0	1	−3/4	2500
x_1	1	0	0	0	1	2000
z	0	−50	0	0	80	160,000

x_2 replaces x_4

	x_1	x_2	x_3	x_4	x_5	
x_3	0	0	1	−1	1/2	200
x_2	0	1	0	2	−3/2	5000
x_1	1	0	0	0	1	2000
z	0	0	0	100	5	410,000

Optimal solution $x_1 = 2000$, $x_2 = 5000$ ($x_3 = 200$, $x_4 = x_5 = 0$), $z = 410,000$

i.e. produce 2000 tons A, 5000 tons B giving proceeds £410,000 (6)

(ii) In terms of c_2, the above objective row coefficients are

$$o_4 = 2c_2 \ge 0 \quad \text{when } c_2 \ge 0$$

$$o_5 = -\tfrac{3}{2}c_2 + 80 \ge 0 \quad \text{when } c_2 \le 160/3$$

(3)

Optimal for $\underline{0 \le c_2 \le 160/3}$

For c_2 just above $160/3$, o_5 becomes negative. New optimal solution found with x_5 replacing x_3 as a basic variable. This solution is $x_1 = 1600$, $x_2 = 5600$, ($x_3 = x_4 = 0$, $x_5 = 400$) (2)

(iii) Introduce a new variable x_6 representing tonnage of A produced in <u>excess</u> of 2000 tons. Column $\underline{a}_6$ of x_6 in constraints is $\underline{a}_6 = \begin{pmatrix} 1/4 \\ 3/4 \\ 0 \end{pmatrix}$

with objective function coefficient $c_6 = 70$.

In final tableau $\underline{y}_6 = B^{-1}\underline{a}_6 = \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 2 & -3/2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/4 \\ 3/4 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 3/2 \\ 0 \end{pmatrix}$ (6)

So objective row element $o_6 = (0, 50, 80)\,\underline{y}_6 - c_6 = 75 - 70 = 5 \ge 0$

2. Solution

(a)(i) The final tableau consists of the original constraint system, in canonical form, multiplied by B^{-1}. Thus the main elements are

$$B^{-1}A \;\vdots\; B^{-1} \;\Big|\; B^{-1}\underline{b}$$

So the optimal values of the basic variables are $\underline{x}_B^0 = B^{-1}\underline{b}$ with other variables zero. The optimal value of the objective is

$$Z^* = \underline{c}_B^T \underline{x}_B^0 = \underline{c}_B^T B^{-1}\underline{b}$$

(2)

(ii) The elements of the objective row are obtained from multiplying final tableau by $\underline{c}_B^T$ and subtracting objective function coefficients. For original main variables this gives

$$\underline{c}_B^T B^{-1} A - \underline{c}^T$$

and for the slacks

$$\underline{c}_B^T B^{-1}$$

(3)

(b) The dual problem is Minimise $\underline{b}^T\underline{w}$ subject to

$$A^T\underline{w} \geq \underline{c}, \quad \underline{w} \geq \underline{0}.$$

(2)

(i) For any feasible solutions $\underline{x}, \underline{w}$ to primal and dual problems $\underline{c}^T\underline{x} \leq \underline{b}^T\underline{w}$

Since $A\underline{x} \leq \underline{b}$, $\underline{w} \geq \underline{0}$ $\quad \underline{w}^T A \underline{x} \leq \underline{w}^T \underline{b} = \underline{b}^T\underline{w}$

Since $A^T\underline{w} \geq \underline{c}$ or $\underline{w}^T A \geq \underline{c}^T$ and $\underline{x} \geq 0$ $\quad \underline{w}^T A \underline{x} \geq \underline{c}^T\underline{x}$ (4)

Comparing obtain $\underline{b}^T\underline{w} \geq \underline{c}^T\underline{x}$.

(ii) The optimal dual solution $\underline{w}^*$ is given by $\underline{w}^* = (\underline{c}_B^T B^{-1})^T$

Since at optimality objective row elements are non negative, we have from part (ii)

$$\underline{c}_B^T B^{-1} A \geq \underline{c}^T \qquad \text{ie} \qquad A^T\underline{w}^* \geq \underline{c}$$

and $\quad \underline{c}_B^T B^{-1} \geq \underline{0}^T \qquad \text{ie} \qquad \underline{w}^* \geq \underline{0}$ (3)

thus $\underline{w}^*$ is feasible. (cont)

2. Solution (cont)

Value of dual objective is $\underline{b}^T \underline{w}^* = \underline{c}_B^T B^{-1} \underline{b} = z^*$

z^* being the optimal value of the primal objective.

In view of the inequality in (i) both solutions are optimal. (2)

(iii) Suppose $\underline{w}$ is dual feasible and $\underline{b}^T \underline{w} = k$

Then primal has feasible solutions and for any feasible solution $\underline{c}^T \underline{x} \le k$. Therefore primal must have a finite maximising solution.

Similarly if $\underline{x}$ is feasible with $\underline{c}^T \underline{x} = m$ then dual has feasible solutions with objective bounded below by m. It has a finite minimising solution. (4)

Total 20

3. Solution

(a) (i) Dual is to find u_i $i=1,\dots,m$, v_j $j=1,\dots,n$ to maximise $\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$ subject to $u_i + v_j \le c_{ij}$ $\forall (i,j)$. (2)

(ii) Using cheapest route method to obtain initial basic feasible solution and solving iteratively:

	6	4	6	3	u_i
4			4		−7
10	1⁺	4	2⁻	3	0
5	5⁻		+		−2
v_j	4	3	8	2	

$\theta_{max} = 2 \longrightarrow$

	6	4	6	3	u_i
4			4		−5
10	3	4		3	0
5	3		2		−2
v_j	4	3	6	2	

Have $u_i + v_j \le c_{ij}$ $\forall (i,j)$ so solution is optimal

i.e. $x_{13} = 4$, $x_{21} = 3$, $x_{22} = 4$ etc with $C = 48$ (5)

(iii) Shadow costs u_i, v_j satisfy dual constraints with dual objective $= \sum a_i u_i + \sum b_j v_j = -30 + 78 = 48$ (2)

This equals primal objective so both solutions must be optimal.

(iv) Solution optimal provided $u_1 + v_1 \le c_{11}$ i.e. $\underline{c_{11} \geq -1}$ (1)

Dealing with c_{21} is more difficult as x_{21} is basic.

Calculating shadow costs in terms of c_{21} gives:

$u_1 = -1$, $u_2 = c_{21}$, $u_3 = 2$, $v_1 = 0$, $v_2 = 3 - c_{21}$, $v_3 = 2$, $v_4 = 2 - c_{21}$

Solution is optimal provided:

$u_1 + v_1 \le c_{11}$ i.e. $-1 \le 0$, $u_1 + v_2 \le c_{12}$ i.e. $2 - c_{21} \le 2$ $\underline{c_{21} \geq 0}$

$u_1 + v_4 \le c_{14}$ i.e. $1 - c_{21} \le 7$ $\underline{c_{21} \geq -6}$,

$u_2 + v_3 \le c_{23}$ i.e. $c_{21} + 2 \le 8$ $\underline{c_{21} \le 6}$

$u_3 + v_2 \le c_{32}$ i.e. $5 - c_{21} \le 6$ $\underline{c_{21} \geq -1}$

$u_3 + v_4 \le c_{34}$ i.e. $4 - c_{21} \le 8$ $\underline{c_{21} \geq -4}$

Solution is optimal for $\underline{0 \le c_{21} \le 6}$ (4)

(cont)

3 Solution (cont)

(b) Assuming $\sum m_i + \sum p_i > \sum r_i$ else there is no feasible solution or, in the case of equality no problem, the formulation is as follows.

Dummy destination

$$\begin{array}{c|cccccc}
 & r_1 & r_2 & r_3 & \cdots & r_n & \begin{pmatrix}\sum m + \sum p \\ -\sum r_i\end{pmatrix} \\
\hline
m_1 & c_1 & c_1+h & c_1+2h & \cdots & c_1+(n-1)h & 0 \\
p_1 & d_1 & d_1+h & d_1+2h & \cdots & d_1+(n-1)h & 0 \\
m_2 & M & c_2 & c_2+h & \cdots & c_2+(n-2)h & 0 \\
p_2 & M & d_2 & d_2+h & \cdots & & 0 \\
m_3 & M & M & c_3 & \cdots & & 0 \\
p_3 & M & M & d_3 & \cdots & & 0 \\
 & \vdots & \vdots & & & & \vdots \\
m_n & M & M & \cdots & M & c_n & 0 \\
p_n & M & \cdots & \cdots & M & d_n & 0
\end{array}$$

M large cost to "price out" impossible "routes".

(6)

Total 20

4. Solution

(a) $\max\{r_{ij}\} = 8$ so first convert to a minimisation problem by setting $c_{ij} = 8 - r_{ij} \quad \forall (i,j)$ and minimising $\sum_{i=1}^{5}\sum_{j=1}^{5} c_{ij} x_{ij}$

$$\{c_{ij}\} = \begin{pmatrix} 5 & 7 & 4 & 1 & 0 \\ 3 & 5 & 2 & 4 & 5 \\ 1 & 3 & 3 & 2 & 4 \\ 2 & 7 & 5 & 4 & 6 \\ 3 & 1 & 4 & 3 & 1 \end{pmatrix}$$

Initial dual solution $u_i = 0 \quad i = 1,\dots,5$

$v_1 = 1, \; v_2 = 1, \; v_3 = 2, \; v_4 = 1, \; v_5 = 0$

For algorithm work with

$w_{ij} = c_{ij} - u_i - v_j$

$$\{w_{ij}\} = \begin{array}{cc} \begin{pmatrix} 4 & 6 & 2 & 0^* & 0 \\ 2 & 4 & 0^* & 3 & 5 \\ 0^* & 2 & 1 & 1 & 4 \\ 1 & 6 & 3 & 3 & 6 \\ 2 & 0^* & 2 & 2 & 1 \end{pmatrix} & \begin{matrix} \\ \\ \\ * \\ \\ \end{matrix} \end{array}$$

Initial assignment shown by *

Not feasible row 4 marked

$t = \min\{w_{ij}\}$ over i marked, j unmarked

$= 1$

$w_{ij}' = w_{ij} - 1 \qquad i$ marked

$w_{ij}' = w_{ij} + 1 \qquad j$ marked

$$\{w_{ij}'\} = \begin{array}{cc} \begin{pmatrix} 4 & 6 & 2 & 0^* & 0 \\ 2 & 4 & 0^* & 3 & 5 \\ 0 & 2 & 1 & 1 & 4 \\ 0^* & 5 & 2 & 2 & 5 \\ 2 & 0^* & 2 & 2 & 1 \end{pmatrix} & \begin{matrix} \\ \\ * \\ * \\ \\ \end{matrix} \\ \begin{matrix} * & & & & \end{matrix} & \end{array}$$

Not feasible

Rows 3, 4 + col 1 marked

$t = 1$

$$w_{ij}' = \begin{pmatrix} 5 & 6 & 2 & 0 & 0^* \\ 3 & 4 & 0^* & 3 & 5 \\ 0 & 1 & 0 & 0^* & 3 \\ 0^* & 4 & 1 & 1 & 4 \\ 3 & 0^* & 2 & 2 & 1 \end{pmatrix}$$

Feasible and hence optimal.

Assign 1 to 5, 2 to 3, 1 to 4, 4 to 1 and 5 to 2.

Sum of ratings $= 33$.

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(cont)