University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA30087: OPTIMISATION METHODS OF OPERATIONAL RESEARCH

Friday 17th January 2003, 09.30–11.30

Candidates may use university-supplied calculators.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

Examiners will attach importance to the number of well-answered questions.

1. Two fertilizers, A and B, are made by blending together raw materials P and Q. It may be assumed that no material is lost in the blending operation and that 1 ton of P plus 3 tons of Q make 4 tons of fertilizer A, whilst 2 tons of P plus 2 tons of Q make 4 tons of B.

The selling prices of A and B are £80 and £50 per ton respectively, but at most 2000 tons of A can be sold. There is no limit to how much of B may be sold. Given that there are available 3200 tons of P and 4000 tons of Q, it is required to decide how much of A and B to make in order to maximise proceeds.

(i) Formulate the problem as a linear programming problem and obtain the solution by the Simplex method.

(ii) Determine the range of the selling price for B for which your solution remains optimal. What is the new solution when the price is just above the upper limit of your range?

(iii) Suppose that you now discover that more than 2000 tons of A could be sold, but any amount in excess of this figure would sell for only £70 per ton. The first 2000 tons would still sell for £80 per ton. Show how the situation may be formulated by adding an additional variable to the original problem and use post-optimality analysis to examine whether the optimal solution changes.

2. Consider the linear programming problem:

$$\text{Maximise } z = \mathbf{c}^{\mathbf{T}}\mathbf{x} \text{ subject to}$$
$$A\mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}$$

where A is a given $m \times n$ matrix and $\mathbf{b}$ and $\mathbf{c}$ are given vectors. After adding a vector $\mathbf{x_s}$ of slack variables, the constraints in canonical form become:

$$A\mathbf{x} + \mathbf{x_s} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x_s} \geq \mathbf{0}$$

(a) Suppose the above problem has an optimal basic solution. Let B be the corresponding basis matrix and $\mathbf{c_B}$ be the corresponding m vector of coefficients of basic variables in the objective function. Obtain expressions in terms of $A, \mathbf{b}, \mathbf{c}, B$ and $\mathbf{c_B}$ for

(i) the optimal values of the variables and objective function.

(ii) the elements of the objective row of the optimal Simplex tableau.

(b) Write down the problem which is dual to the initial problem above.

(i) State and prove an inequality that holds between the primal and dual objective functions.

(ii) Write down an expression involving $\mathbf{c_B}$ and B for the optimal dual solution. Prove that your answer is both feasible and optimal.

(iii) Prove that if both primal and dual problems possess feasible solutions, then both possess finite optimising solutions.

3. (a) Consider the transportation problem:

$$\text{Minimise } C = \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\textstyle\sum_{j=1}^{n} x_{ij} = a_i \text{ for } i = 1, ..., m$$
$$\textstyle\sum_{i=1}^{m} x_{ij} = b_j \text{ for } j = 1, ..., n.$$

and $x_{ij} \geq 0 \ \ \forall (i, j)$.

(i) Write down the dual problem.

(ii) Obtain the optimal solution to the problem with $m = 3$ and $n = 4$ when $a_i = (4, 10, 5), b_j = (6, 4, 6, 3)$ and

$$c_{ij} = \begin{pmatrix} 5 & 2 & 1 & 7 \\ 4 & 3 & 8 & 2 \\ 2 & 6 & 4 & 8 \end{pmatrix}$$

(iii) Show, by considering a solution to the dual problem, that your answer to part(ii) was indeed optimal.

(iv) Find the range of c_{11} for which your solution remains optimal. Do the same for c_{21}.

(b) A company is planning its production levels over an n month period to meet varying, but known, monthly demands for r_i units, $i = 1, ..., n$. The production capacity in each month i in normal working time is m_i but an extra p_i units may be produced in overtime. The unit production costs vary over the period and in month i are c_i in normal time and d_i in overtime. The product can be stored between production and sale but this incurs a stock holding cost of h per unit per month.

Show the problem of meeting demand at minimum total cost can be formulated as a transportation problem.

4. (a) Use the network flow method to solve the 5×5 assignment problem where it is required to maximise the sum of assigned ratings, the individual ratings being given in the following matrix.

$$\begin{pmatrix} 3 & 1 & 4 & 7 & 8 \\ 5 & 3 & 6 & 4 & 3 \\ 7 & 5 & 5 & 6 & 4 \\ 6 & 1 & 3 & 4 & 2 \\ 5 & 7 & 4 & 5 & 7 \end{pmatrix}.$$

(b) For the algorithm used in part (a), write down the rules by which rows and columns of the matrix are marked at each iteration and explain how this shows whether or not the restricted primal has a feasible solution. Show how this is equivalent to the rules for labelling nodes in the maximal flow algorithm. Sketch the equivalent network.

(c) Explain what is meant by a bottleneck assignment problem and show that your solution in part (a) is also optimal for the corresponding bottleneck problem.

MEB