University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MATH0087: OPTIMISATION METHODS OF OPERATIONAL RESEARCH

Thursday 24th January 2002, 09.30–11.30

Candidates may use university-supplied calculators.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

Examiners will attach importance to the number of well-answered questions.

1. Consider the linear programming problem:

$$\text{Maximise } z = \mathbf{c}^{\mathbf{T}}\mathbf{x} \text{ subject to}$$
$$A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$$

where A is a given $m \times n$ matrix with $m < n$ and $\mathbf{b}$ and $\mathbf{c}$ are given vectors.

(a) Write down the **unsymmetric** dual problem. Prove an inequality that holds between primal and dual objectives for any feasible solutions to the **unsymmetric** dual pair of problems.

(b) Suppose the primal problem has a basic feasible solution corresponding to basis matrix B with $\mathbf{c_B}$ being the corresponding vector of objective function coefficients. Define the $m-$vector π by

$$\pi^{\mathbf{T}} = \mathbf{c}_{\mathbf{B}}^{\mathbf{T}} B^{-1}$$

(i) Explain how, in the revised Simplex algorithm, the vector π can be used to calculate objective row elements using **original** data without having to generate the transformed column. Comment briefly on why this is an advantage.

(ii) If the basic feasible solution corresponding to the basis matrix B is optimal, the objective row elements will all be non-negative. Show in this case that π is an optimal solution to the dual.

(iii) Explain the argument that relates the elements of π to the effect on the optimal value of the objective function brought about by small changes in the primal problem. Mention whether the objective function increases or decreases.

2. A nutritionist is planning a meal from four foods, A, B, C and D. He wishes the meal to contain at most 10 units of fat but at least 20 units of carbohydrate. Within these restrictions he wishes to maximise the protein content. The following table shows the number of units of fat, carbohydrate and protein contained in one ounce of the foods.

Food	A	B	C	D
Fat	2	1	1	2
Carbohydrate	3	3	4	2
Protein	2	1	1	3

(i) Formulate the problem of planning the nutritionist's meal as a linear programming problem. Use the Two-Phase Simplex method to find the optimal solution.

(ii) Food D is a natural product whose protein content is variable. Determine the range of the number of units of protein per ounce for which the solution remains optimal.

(iii) Determine the effect of increasing the carbohydrate requirement to 22 units.

(iv) A new food, E, is being considered. It contains 1 unit of fat, 2 units of carbohydrate and 1 unit of protein per ounce. Determine the effect on the previous optimal tableau of including food E in the problem and say whether the optimal meal is changed.

3. Stone is quarried at three sites, A, B and C the maximum weekly tonnages being 7000 at A, 6000 at B and 8000 at C. It must be transported to three destinations, L, M and N each requiring 5000 tons per week. The cost per ton of transporting between the mines and destinations are given below.

	L	M	N
A	2	6	3
B	6	14	12
C	12	3	8

(i) Formulate the problem of moving the stone at minimum total cost as a transportation problem.

(ii) Obtain an optimal solution to the problem.

(iii) Write down the dual problem and an optimal solution to the dual. Use the duality theorem to demonstrate that the solution you obtained to the transportation problem is indeed optimal.

(iv) Show that if the amount available at A and amount required at M are both increased by 1000 tons, then the optimal transportation cost is **reduced**. Comment briefly on this paradox. Up to what size of increase in these quantities does the total cost continue to fall?

4. (a) Consider the capacitated network having six nodes and the following capacity matrix.

$$\begin{pmatrix} 0 & 7 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Draw the corresponding network and describe it briefly in terms of sources, sinks and directionality of the capacities. Use the Ford-Fulkerson method to determine the maximal flow from node 1 to node 6. Show clearly the required flows along each arc.

(b) Define a cut in a capacitated network and state without proof a theorem relating the maximal flow to the properties of a cut. Use this to demonstrate that your answer in part (a) was really a maximal flow.

(c) What is the greatest change possible in the maximal flow that can be brought about by increasing the capacity of **only one** arc? Which are the possible arcs that could bring this increase about?