

WHAT IS WRONG WITH ANALYTICITY IN GENTZEN'S PROOF THEORY AND HOW TO FIX IT

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Joint work with Paola Bruscoli.
Work in progress with Benjamin Ralph and Michel Parigot - CNRS, Paris } Both Bath

Talk available from my home page and at <http://cs.bath.ac.uk/ag/t/WIW.pdf>

All about deep inference at <http://alessio.guglielmi.name/res/cos>

cut-freeness & subformula property

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AND HOW TO FIX IT

cut-freeness & subformula property

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poor complexity
unnatural semantics

cut-freeness & subformula property

WHAT IS WRONG WITH ANALYTICITY IN GENTZEN'S PROOF THEORY

AND HOW TO FIX IT

poor complexity
unnatural semantics

refound proof theory
around a new notion of
proof composition

ANALYTICITY IN GENTZEN'S PROOF THEORY

Find an analytic proof of: 'There are two irrationals x and y such that x^y is rational'.

ANALYTICITY IN GENTZEN'S PROOF THEORY

Rx^y

Find an analytic proof of: 'There are two irrationals x and y such that x^y is rational'.

ANALYTICITY IN GENTZEN'S PROOF THEORY

$$\exists x \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)$$

Find an analytic proof of: 'There are two irrationals x and y such that x^y is rational'.

ANALYTICITY IN GENTZEN'S PROOF THEORY

$$\bar{R}\sqrt{2}, R(\bar{x}^{\sqrt{2}})^{\bar{y}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)$$

Find an analytic proof of: 'There are two irrationals x and y such that x^y is rational'.

ANALYTICITY IN GENTZEN'S PROOF THEORY

There are two irrationals x and y such that x^y is rational.

ANALYTICITY IN GENTZEN'S PROOF THEORY

There are two irrationals x and y such that x^y is rational.
 Proof: if $\sqrt{2}^{\sqrt{2}}$ is rational, fine, otherwise take $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$.

WHAT IS WRONG WITH ANALYTICITY IN GENTZEN'S PROOF THEORY

$$\begin{array}{c}
 \frac{\neg_R \quad R\bar{J}_2^{\bar{J}_2} \vdash R\bar{J}_2^{\bar{J}_2}}{\vdash R\bar{J}_2^{\bar{J}_2}, \bar{R}\bar{J}_2^{\bar{J}_2}} \quad \frac{w_R \quad \bar{R}\bar{J}_2 \vdash \bar{R}\bar{J}_2}{\bar{R}\bar{J}_2 \vdash R\bar{J}_2^{\bar{J}_2}, \bar{R}\bar{J}_2} \\
 \frac{2w_L \quad \vdash R\bar{J}_2^{\bar{J}_2}, \bar{R}\bar{J}_2^{\bar{J}_2}}{\wedge_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash R\bar{J}_2^{\bar{J}_2}, \bar{R}\bar{J}_2^{\bar{J}_2}} \quad \frac{w_L \quad \bar{R}\bar{J}_2 \vdash R\bar{J}_2^{\bar{J}_2}, \bar{R}\bar{J}_2}{\bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash R\bar{J}_2^{\bar{J}_2}, \bar{R}\bar{J}_2} \\
 \frac{}{\wedge_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash R\bar{J}_2^{\bar{J}_2}, \bar{R}\bar{J}_2^{\bar{J}_2} \wedge \bar{R}\bar{J}_2} \quad \frac{}{\wedge_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash R\bar{J}_2^{\bar{J}_2}, \bar{R}\bar{J}_2^{\bar{J}_2} \wedge \bar{R}\bar{J}_2 \wedge R(\bar{J}_2^{\bar{J}_2})^{\bar{J}_2}} \\
 \frac{w_R \quad \bar{R}\bar{J}_2 \vdash \bar{R}\bar{J}_2 \wedge \bar{R}\bar{J}_2, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\wedge_L \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \bar{R}\bar{J}_2 \wedge \bar{R}\bar{J}_2, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 \frac{w_L \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \bar{R}\bar{J}_2 \wedge \bar{R}\bar{J}_2, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\wedge_L \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \bar{R}\bar{J}_2 \wedge \bar{R}\bar{J}_2, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 \frac{}{\exists_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \bar{R}\bar{J}_2 \wedge \bar{R}\bar{J}_2 \wedge R(\bar{J}_2^{\bar{J}_2})^{\bar{J}_2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 \frac{}{\exists_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \bar{R}\bar{J}_2 \wedge \bar{R}\bar{J}_2, \exists y. (\bar{R}\bar{J}_2 \wedge \bar{R}y \wedge R(\bar{J}_2^{\bar{J}_2})^{\bar{J}_2})} \\
 \frac{}{\exists_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \bar{R}\bar{J}_2 \wedge \bar{R}\bar{J}_2, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(\bar{J}_2^{\bar{J}_2})^{\bar{J}_2})} \\
 \frac{\exists_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \bar{R}\bar{J}_2 \wedge \bar{R}\bar{J}_2 \wedge R(\bar{J}_2^{\bar{J}_2})^{\bar{J}_2, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}{\exists_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 \frac{\exists_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \exists y. (\bar{R}\bar{J}_2 \wedge \bar{R}y \wedge R(\bar{J}_2^{\bar{J}_2})^{\bar{J}_2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y))}{\exists_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 \frac{\exists_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\exists_R \quad \bar{R}\bar{J}_2, R(J_2^{\bar{J}_2})^{\bar{J}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}
 \end{array}$$

Infinite choice of witnesses: no real subformula property; no real analyticity.

GETTING REAL ANALYTICITY

$$\begin{array}{c}
 \frac{\neg_R \frac{R\sqrt{2}^{\bar{v}_2} \vdash R\sqrt{2}^{\bar{v}_2}}{\vdash R\sqrt{2}, \bar{R}\sqrt{2}^{\bar{v}_2}}}{w_L} \\
 \frac{w_L \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash R\sqrt{2}^{\bar{v}_2}, \bar{R}\sqrt{2}^{\bar{v}_2}}{\vdash R\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash R\sqrt{2}^{\bar{v}_2}, \bar{R}\sqrt{2}^{\bar{v}_2}}}{w_R} \\
 \frac{w_R \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\vdash R\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash R\sqrt{2}^{\bar{v}_2}, \bar{R}\sqrt{2}^{\bar{v}_2}}}{w_L} \\
 \frac{w_L \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash R\sqrt{2}^{\bar{v}_2}, \bar{R}\sqrt{2}^{\bar{v}_2} \wedge \bar{R}\sqrt{2}}{\vdash R\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash R\sqrt{2}^{\bar{v}_2}, \bar{R}\sqrt{2}^{\bar{v}_2} \wedge \bar{R}\sqrt{2} \wedge R(\bar{J}_2^{\bar{v}_2})^{\bar{v}_2}}}{w_R} \\
 \frac{w_R \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash R\sqrt{2}^{\bar{v}_2}, \bar{R}\sqrt{2}^{\bar{v}_2} \wedge \bar{R}\sqrt{2} \wedge R(\bar{J}_2^{\bar{v}_2})^{\bar{v}_2}}{\exists_R \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \bar{R}\sqrt{2} \wedge \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}{w_L}}}{w_R} \\
 \frac{w_L \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \bar{R}\sqrt{2} \wedge \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}{\vdash R\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \bar{R}\sqrt{2} \wedge \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}}{\exists_R} \\
 \frac{\exists_R \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \bar{R}\sqrt{2} \wedge \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}{\exists_R \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}{w_R}}}{\exists_R} \\
 \frac{\exists_R \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}{\exists_R \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}{w_L}}}{\exists_R} \\
 \end{array}$$

$\neg_R \frac{R\sqrt{2}^{\bar{v}_2} \vdash R\sqrt{2}^{\bar{v}_2}}{\vdash R\sqrt{2}, \bar{R}\sqrt{2}^{\bar{v}_2}}$ $w_R \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\vdash R\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash R\sqrt{2}^{\bar{v}_2}, \bar{R}\sqrt{2}^{\bar{v}_2}}$
 $\wedge_R \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\vdash \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}$ $\wedge_R \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash R\sqrt{2}^{\bar{v}_2}, \bar{R}\sqrt{2}^{\bar{v}_2}}{\vdash \bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash R\sqrt{2}^{\bar{v}_2}, \bar{R}\sqrt{2}^{\bar{v}_2} \wedge \bar{R}\sqrt{2}}$
 $\wedge_R \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R(\bar{J}_2^{\bar{v}_2})^{\bar{v}_2}}{\vdash \bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R(\bar{J}_2^{\bar{v}_2})^{\bar{v}_2}}$
 $\exists_R \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \bar{R}\sqrt{2} \wedge \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}{\vdash R\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \bar{R}\sqrt{2} \wedge \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}$
 $\exists_R \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \bar{R}\sqrt{2} \wedge \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}{\exists_R \frac{\bar{R}\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}{\vdash R\sqrt{2}, R(J_2^{\bar{v}_2})^{\bar{v}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R(x^y))}}$

GETTING REAL ANALYTICITY

$$\begin{array}{c}
 \wedge_R \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}} \\
 \wedge_R \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{y2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 \wedge_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{y2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{y2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}
 \end{array}$$

$$\begin{array}{c}
 \neg_R \frac{R\sqrt{2}^{y2} \vdash R\sqrt{2}^{y2}}{\vdash R\sqrt{2}^{y2}, \bar{R}\sqrt{2}^{y2}} \quad w_R \frac{\bar{R}y \vdash \bar{R}y}{\bar{R}y \vdash R\sqrt{2}^{y2}, \bar{R}y} \\
 2w_L \frac{}{\vdash R\sqrt{2}^{y2}, \bar{R}\sqrt{2}^{y2}} \quad w_L \frac{}{\bar{R}y \vdash R\sqrt{2}^{y2}, \bar{R}y} \\
 \wedge_R \frac{\bar{R}y, R(\sqrt{2})^y \vdash R\sqrt{2}^{y2}, \bar{R}\sqrt{2}^{y2}}{\bar{R}y, R(\sqrt{2})^y \vdash R\sqrt{2}^{y2}, \bar{R}y} \quad \wedge_R \frac{\bar{R}y, R(\sqrt{2})^y \vdash R\sqrt{2}^{y2}, \bar{R}\sqrt{2}^{y2} \wedge \bar{R}y}{\bar{R}y, R(\sqrt{2})^y \vdash R\sqrt{2}^{y2}, \bar{R}\sqrt{2}^{y2} \wedge \bar{R}y \wedge R(\sqrt{2})^y} \\
 \wedge_R \frac{\bar{R}y, R(\sqrt{2})^y \vdash R\sqrt{2}^{y2}, \bar{R}\sqrt{2}^{y2} \wedge \bar{R}y \wedge R(\sqrt{2})^y}{\bar{R}y, R(\sqrt{2})^y \vdash R\sqrt{2}^{y2}, \exists y. (\bar{R}\sqrt{2}^{y2} \wedge \bar{R}y \wedge R(\sqrt{2})^y)} \quad \{y \leftarrow \sqrt{2}\} \\
 \exists_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{y2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{y2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{y2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}
 \end{array}$$

$$\begin{array}{c}
 \exists_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{y2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{y2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{y2} \vdash \exists y. (\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 \exists_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{y2} \vdash \exists y. (\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{y2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 c_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{y2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{y2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}
 \end{array}$$

↑

indicated substitution
applies inside the box

GETTING REAL ANALYTICITY

\wedge_R	$\frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}$
\wedge_L	$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)$
\wedge_L	$\bar{R}\sqrt{2}, R(\bar{R}\sqrt{2})^{y2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)$

\neg_a	$\frac{R\sqrt{2}^{i^2} \vdash R\sqrt{2}^{i^2}}{\vdash R\sqrt{2}^{i^2}, \bar{R}\sqrt{2}^{i^2}}$	w_R	$\frac{\bar{R}y \vdash \bar{R}y}{\bar{R}y \vdash R\sqrt{2}^{i^2}, \bar{R}y}$
$\exists w_L$	$\bar{R}y, R(J_2^{i^2})^y \vdash R\sqrt{2}^{i^2}, \bar{R}\sqrt{2}^{i^2}$	w_L	$\bar{R}y, R(J_2^{i^2})^y \vdash R\sqrt{2}^{i^2}, \bar{R}y$
\wedge_R	$\bar{R}y, R(J_2^{i^2})^y \vdash R\sqrt{2}^{i^2}, \bar{R}\sqrt{2}^{i^2} \wedge \bar{R}y$		w_R
\wedge_R	$\bar{R}y, R(J_2^{i^2})^y \vdash R\sqrt{2}^{i^2}, \bar{R}\sqrt{2}^{i^2} \wedge \bar{R}y$		$R(\sqrt{2}^{i^2})^y \vdash R(\sqrt{2}^{i^2})^y$
\exists_R	$\bar{R}y, R(J_2^{i^2})^y \vdash R\sqrt{2}^{i^2}, \bar{R}\sqrt{2}^{i^2} \wedge \bar{R}y \wedge R(J_2^{i^2})^y$		$R(\sqrt{2}^{i^2})^y \vdash R\sqrt{2}^{i^2}, R(J_2^{i^2})^y$
\exists_R	$\bar{R}y, R(J_2^{i^2})^y \vdash R\sqrt{2}^{i^2}, \exists y. (\bar{R}\sqrt{2}^{i^2} \wedge \bar{R}y \wedge R(J_2^{i^2})^y)$		$\bar{R}y, R(J_2^{i^2})^y \vdash R\sqrt{2}^{i^2}, R(J_2^{i^2})^y$
\exists_L	$\bar{R}\sqrt{2}, R(J_2^{i^2})^{i^2} \vdash R\sqrt{2}^{i^2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)$		$\{y \leftarrow \sqrt{2}\}$

\exists_R	$\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^y, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)$
\exists_R	$\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \exists y. (\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)$
\exists_R	$\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)$
\leftarrow_R	$\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)$

GETTING REAL ANALYTICITY

$$\begin{array}{c}
 \wedge_R \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}} \\
 \wedge_R \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)} \\
 \wedge_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^j, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}
 \end{array}$$

$$\begin{array}{c}
 \neg_R \frac{}{R x \vdash R x} \quad w_R \frac{}{\bar{R}y \vdash \bar{R}y} \\
 \vdash_R \frac{}{\vdash R x, \bar{R}x} \quad w_L \frac{}{\bar{R}y \vdash R x, \bar{R}y} \\
 \wedge_R \frac{\bar{R}y, R x^y \vdash R x, \bar{R}x \quad \bar{R}y, R x^y \vdash R x, \bar{R}y}{\bar{R}y, R x^y \vdash R x, \bar{R}x \wedge \bar{R}y} \\
 \exists_R \frac{\bar{R}y, R x^y \vdash R x, \bar{R}x \wedge \bar{R}y \wedge R x^y}{\bar{R}y, R x^y \vdash R x, \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)} \quad \{y \leftarrow \sqrt{2}\} \\
 \exists_R \frac{\bar{R}y, R x^y \vdash R x, \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash R x, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)} \quad \{x \leftarrow \sqrt{2}\}
 \end{array}$$

$$\begin{array}{c}
 \exists_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^j, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \exists y. (\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^j), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)} \\
 \exists_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \exists y. (\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^j), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)} \\
 c_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}
 \end{array}$$

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$$\begin{array}{c}
 \wedge_R \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \\
 \wedge_R \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \\
 \hline
 \wedge_R \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y) \\
 \wedge_R \quad \bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)
 \end{array}$$

$\neg_R \frac{R x \vdash R x}{\vdash R x, \bar{R}x}$ $\wedge_L \frac{}{\vdash R x, \bar{R}x}$ $\wedge_R \frac{}{\vdash \bar{R}y, R x^y \vdash R x, \bar{R}x}$	$w_R \frac{\bar{R}y \vdash \bar{R}y}{\bar{R}y \vdash R x, \bar{R}y}$ $w_L \frac{}{\bar{R}y, R x^y \vdash R x, \bar{R}y}$ $\wedge_R \frac{}{\vdash \bar{R}y, R x^y \vdash R x, \bar{R}x \wedge \bar{R}y}$	$w_R \frac{R x^y \vdash R x^y}{R x^y \vdash R x, R x^y}$ $w_L \frac{}{R x^y \vdash R x, R x^y}$ $\wedge_R \frac{}{\vdash \bar{R}y, R x^y \vdash R x, R x^y}$
		$\exists_R \frac{}{\vdash \bar{R}y, R x^y \vdash R x, \bar{R}x \wedge \bar{R}y \wedge R x^y}$ $\exists_R \frac{}{\vdash \bar{R}y, R x^y \vdash R x, \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$
		$\exists_R \frac{}{\vdash \bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash R x, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$
		$\{y \leftarrow \sqrt{2}\}$
	$\exists_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R(\sqrt{2})^{j_2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \exists y. (\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R(\sqrt{2})^{j_2}), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$	
		$\exists_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \exists y. (\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R(\sqrt{2})^{j_2}), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$
		$\{x \leftarrow \sqrt{2}\}$
	$\exists_R \frac{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\sqrt{2})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$	

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$$\wedge_R \frac{\bar{R}z \vdash \bar{R}z \quad \bar{R}z \vdash \bar{R}z}{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z}$$

$$\frac{w_R \quad w_L}{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$$

$$\exists_R \frac{\bar{R}z, R(\sqrt{z})^{j_2} \vdash \bar{R}z \wedge \bar{R}z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}z, R(\sqrt{z})^z \vdash \exists y. (\bar{R}z \wedge \bar{R}y \wedge R \sqrt{z}^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$$

$$c_R \frac{\bar{R}\sqrt{z}, R(\sqrt{z})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{z}, R(\sqrt{z})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$$

$\neg_R \frac{Rx \vdash Rx}{\vdash Rx, \bar{R}x}$	$w_R \frac{\bar{R}y \vdash \bar{R}y}{\bar{R}y \vdash Rx, \bar{R}y}$	$w_R \frac{Rx^y \vdash Rx^y}{Rx^y \vdash Rx, Rx^y}$
$\wedge_R \frac{\vdash Rx, \bar{R}x \quad \bar{R}y \vdash Rx, \bar{R}y}{\bar{R}y, Rx^y \vdash Rx, \bar{R}x}$	$w_L \frac{\bar{R}y, Rx^y \vdash Rx, \bar{R}y}{\bar{R}y, Rx^y \vdash Rx, Rx^y}$	$w_L \frac{\bar{R}y, Rx^y \vdash Rx, Rx^y}{\bar{R}y, Rx^y \vdash Rx, Rx^y}$
$\wedge_R \frac{\bar{R}y, Rx^y \vdash Rx, \bar{R}x \quad \bar{R}x \wedge \bar{R}y}{\bar{R}y, Rx^y \vdash Rx, \bar{R}x \wedge \bar{R}y}$	$\exists_R \frac{\bar{R}y, Rx^y \vdash Rx, \bar{R}x \wedge \bar{R}y \wedge Rx^y}{\bar{R}y, Rx^y \vdash Rx, \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$	$\{y \leftarrow z\}$
$\exists_R \frac{\bar{R}z, R(\sqrt{z})^z \vdash \bar{R}z \wedge \bar{R}z \wedge R\sqrt{z}^z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}z, R(\sqrt{z})^z \vdash \exists y. (\bar{R}z \wedge \bar{R}y \wedge R \sqrt{z}^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$	$\{z \leftarrow \sqrt{z}\}$	

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	$\frac{R_x \vdash R_x}{\neg_R}$	$\frac{\bar{R}_y \vdash \bar{R}_y}{w_R}$	$\frac{R_x^y \vdash R_x^y}{w_R}$
	$\frac{\vdash R_x, \bar{R}_x}{z w_L}$	$\frac{\bar{R}_y \vdash R_x, \bar{R}_y}{w_L}$	$\frac{R_x^y \vdash R_x, R_x^y}{w_L}$
	$\frac{\neg_R \bar{R}_y, R_x^y \vdash R_x, \bar{R}_x}{\wedge_R \bar{R}_y, R_x^y \vdash R_x, \bar{R}_x}$	$\frac{\bar{R}_y, R_x^y \vdash R_x, \bar{R}_y}{\wedge_R \bar{R}_y, R_x^y \vdash R_x, \bar{R}_x \wedge \bar{R}_y}$	$\frac{\bar{R}_y, R_x^y \vdash R_x, R_x^y}{\bar{R}_y, R_x^y \vdash R_x, \bar{R}_x \wedge \bar{R}_y \wedge R_x^y}$
	$\frac{\neg_R \bar{R}_z \vdash \bar{R}_z}{w_R}$	$\frac{\bar{R}_z \vdash \bar{R}_z \wedge \bar{R}_z}{w_L}$	$\frac{\bar{R}_z, R(\sqrt{z})^{j_2} \vdash \bar{R}_z \wedge \bar{R}_z, \exists x. \exists y. (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}{\exists_R \bar{R}_z, R(\sqrt{z})^z \vdash \bar{R}_z \wedge \bar{R}_z \wedge R\sqrt{z}^z, \exists x. \exists y. (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}$
	$\frac{\bar{R}_z \vdash \bar{R}_z \wedge \bar{R}_z, \exists x. \exists y. (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}{w_L}$	$\frac{\bar{R}_y, R_x^y \vdash R_x, \bar{R}_x \wedge \bar{R}_y \wedge R_x^y}{\exists_R \bar{R}_y, R_x^y \vdash R_x, \exists y. (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}$	$\{y \leftarrow z\}$
	$\frac{\bar{R}_z, R(\sqrt{z})^z \vdash \bar{R}_z \wedge \bar{R}_z \wedge R\sqrt{z}^z, \exists x. \exists y. (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}{\exists_R \bar{R}_z, R(\sqrt{z})^z \vdash \exists y. (\bar{R}_z \wedge \bar{R}_y \wedge R\sqrt{z}^y), \exists x. \exists y. (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}$	$\{z \leftarrow \sqrt{z}\}$	
\exists_R	$\frac{\bar{R}\sqrt{z}, R(\sqrt{z})^{j_2} \vdash \exists x. \exists y. (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y), \exists x. \exists y. (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}{\bar{R}\sqrt{z}, R(\sqrt{z})^{j_2} \vdash \exists x. \exists y. (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}$		
	$\frac{\bar{R}\sqrt{z}, R(\sqrt{z})^{j_2} \vdash \exists x. \exists y. (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}{\exists_R \bar{R}\sqrt{z}, R(\sqrt{z})^{j_2} \vdash \exists x. \exists y. (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}$		

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$$\wedge_R \frac{\bar{R}z \vdash \bar{R}z \quad \bar{R}z \vdash \bar{R}z}{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z}$$

$$\frac{\wedge_R \frac{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}z, R(\bar{R}z)^{j_2} \vdash \bar{R}z \wedge \bar{R}z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}}{\wedge_L \frac{}{\bar{R}z, R(w^z)^z \vdash \bar{R}z \wedge \bar{R}z \wedge R w^z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}}$$

$\neg_R \frac{R x \vdash R x}{\vdash R x, \bar{R}x}$ $\wedge_R \frac{\vdash R x, \bar{R}x}{\bar{R}y, R x^y \vdash R x, \bar{R}x}$ $\wedge_R \frac{\bar{R}y, R x^y \vdash R x, \bar{R}x}{\bar{R}y, R x^y \vdash R x, \bar{R}x \wedge \bar{R}y}$ $\exists_R \frac{\bar{R}y, R x^y \vdash R x, \bar{R}x \wedge \bar{R}y \wedge R x^y}{\bar{R}y, R x^y \vdash R x, \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$ $\exists_R \frac{\bar{R}y, R x^y \vdash R x, \bar{R}x \wedge \bar{R}y \wedge R x^y}{\bar{R}z, R x^z \vdash R x, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$ $\exists_R \frac{\bar{R}z, R(x^z)^z \vdash \bar{R}z \wedge \bar{R}z \wedge R w^z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}z, R(w^z)^z \vdash \exists y. (\bar{R}z \wedge \bar{R}y \wedge R w^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$ $\exists_R \frac{\bar{R}z, R(w^z)^z \vdash \exists y. (\bar{R}z \wedge \bar{R}y \wedge R w^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}w, R(w^w)^w \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$ $\exists_R \frac{\bar{R}w, R(w^w)^w \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\bar{R}z)^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$	$w_R \frac{\bar{R}y \vdash \bar{R}y}{\vdash R x^y \vdash R x, \bar{R}y}$ $w_L \frac{\bar{R}y, R x^y \vdash R x, \bar{R}y}{\bar{R}y, R x^y \vdash R x, R x^y}$ $w_L \frac{\bar{R}y, R x^y \vdash R x, R x^y}{\bar{R}y, R x^y \vdash R x, \{y \leftarrow z\}}$ $\{y \leftarrow z\}$ $\{x \leftarrow w^z\}$ $\{z \leftarrow w\}$ $\{w \leftarrow \sqrt{2}\}$
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c_R

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$$\wedge_R \frac{\bar{R}z \vdash \bar{R}z \quad \bar{R}z \vdash \bar{R}z}{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z}$$

$$w_R \frac{}{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}$$

$$w_L \frac{}{\bar{R}z, R(\bar{z}^{\bar{z}})^{\bar{z}} \vdash \bar{R}z \wedge \bar{R}z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}$$

$$\wedge_R \frac{\bar{R}z, R(w^z)^z \vdash \bar{R}z \wedge \bar{R}z \wedge Rw^z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}{\bar{R}z, R(w^z)^z \vdash \exists y. (\bar{R}z \wedge \bar{R}y \wedge Rw^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}$$

$$\exists_R \frac{\bar{R}w, R(w^w)^w \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}{\bar{R}\sqrt{2}, R(\bar{z}^{\bar{z}})^{\bar{z}} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}$$

$$\exists_R \frac{}{\bar{R}\sqrt{2}, R(\bar{z}^{\bar{z}})^{\bar{z}} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}$$

We can remove weakenings (for clarity).

$\neg_R \frac{Rx \vdash Rx}{\vdash Rx, \bar{R}x}$ $\wedge_R \frac{w_L \frac{\bar{R}y \vdash \bar{R}y}{\bar{R}y \vdash Rx, \bar{R}y}}{\bar{R}y, Rx^y \vdash Rx, \bar{R}x}$ $\wedge_R \frac{w_L \frac{\bar{R}y, Rx^y \vdash Rx, \bar{R}x}{\bar{R}y, Rx^y \vdash Rx, \bar{R}y}}{\bar{R}y, Rx^y \vdash Rx, \bar{R}x \wedge \bar{R}y}$ $\exists_R \frac{w_L \frac{\bar{R}y, Rx^y \vdash Rx, \bar{R}x}{\bar{R}y, Rx^y \vdash Rx, \bar{R}y \wedge Rx^y}}{\bar{R}y, Rx^y \vdash Rx, \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$ $\exists_R \frac{}{\bar{R}z, Rx^z \vdash Rx, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$	$w_R \frac{Ry \vdash Ry}{\vdash Ry, \bar{R}y}$ $w_L \frac{\bar{R}y \vdash Rx, \bar{R}y}{\bar{R}y, Rx^y \vdash Rx, \bar{R}y}$ $w_L \frac{\bar{R}y, Rx^y \vdash Rx, \bar{R}y}{\bar{R}y, Rx^y \vdash Rx, Rx^y}$ $\{y \leftarrow z\}$ $\{x \leftarrow w^z\}$
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$\exists_R \frac{\bar{R}z, R(w^z)^z \vdash \bar{R}z \wedge \bar{R}z \wedge Rw^z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}{\bar{R}z, R(w^z)^z \vdash \exists y. (\bar{R}z \wedge \bar{R}y \wedge Rw^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}$ $\exists_R \frac{}{\bar{R}w, R(w^w)^w \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}$ $\{z \leftarrow w\}$

$\exists_R \frac{\bar{R}\sqrt{2}, R(\bar{z}^{\bar{z}})^{\bar{z}} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}{\bar{R}\sqrt{2}, R(\bar{z}^{\bar{z}})^{\bar{z}} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rxy)}$ $\{w \leftarrow \sqrt{2}\}$
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$$\wedge_R \frac{\bar{R}z \vdash \bar{R}z \quad \bar{R}z \vdash \bar{R}z}{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z}$$

$$w_R \frac{}{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$$

$$w_L \frac{}{\bar{R}z, R(\bar{z}^{j_2})^{j_2} \vdash \bar{R}z \wedge \bar{R}z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$$

$$\wedge_R \frac{\bar{R}z, R(w^z)^z \vdash \bar{R}z \wedge \bar{R}z \wedge R w^z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}z, R(w^z)^z \vdash \exists y. (\bar{R}z \wedge \bar{R}y \wedge R w^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$$

$$\exists_R \frac{\bar{R}w, R(w^w)^w \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\bar{z}^{j_2})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}$$

We have a **real subformula property**
because terms that go up either disappear or eventually come down.

$$\begin{array}{c}
 \frac{R x \vdash R x}{\neg_R \frac{}{\vdash R x, \bar{R}x}} \quad \frac{\bar{R}y \vdash \bar{R}y}{w_R \frac{}{\bar{R}y \vdash R x, \bar{R}y}} \\
 \frac{\vdash R x, \bar{R}x}{w_L \frac{}{\bar{R}y, R x^y \vdash R x, \bar{R}x}} \quad \frac{\bar{R}y \vdash R x, \bar{R}y}{w_L \frac{}{R x^y \vdash R x, R x^y}} \\
 \wedge_R \frac{\bar{R}y, R x^y \vdash R x, \bar{R}x}{\exists_R \frac{\bar{R}y, R x^y \vdash R x, \bar{R}x \wedge \bar{R}y}{\exists_R \frac{\bar{R}y, R x^y \vdash R x, \bar{R}x \wedge \bar{R}y \wedge R x^y}{\exists_R \frac{\bar{R}y, R x^y \vdash R x, \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}z, R x^z \vdash R x, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}}}} \quad \{y \leftarrow z\} \\
 \wedge_R \frac{\bar{R}z, R(\bar{z}^{j_2})^{j_2} \vdash \bar{R}z \wedge \bar{R}z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\exists_R \frac{\bar{R}z, R(w^z)^z \vdash \bar{R}z \wedge \bar{R}z \wedge R w^z, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\exists_R \frac{\bar{R}z, R(w^z)^z \vdash \exists y. (\bar{R}z \wedge \bar{R}y \wedge R w^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\exists_R \frac{\bar{R}w, R(w^w)^w \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}{\bar{R}\sqrt{2}, R(\bar{z}^{j_2})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)}}}} \quad \{z \leftarrow w\} \\
 \end{array}$$

$$c_R \quad \bar{R}\sqrt{2}, R(\bar{z}^{j_2})^{j_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge R x^y)$$

DEEP INFERENCE = freely compose proofs via any given connectives — and substitution (w.i.p.)

DEEP INFERENCE

freely compose proofs via any given connectives — and substitution (w.i.p.)

$$i \frac{}{t} \\ R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}$$

$$2s \frac{i \frac{}{t} \\ Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y)}{Rx \wedge \bar{R}y \wedge Rx^y} \wedge Rx^y \\ Rx \vee Ry \vee 2s \frac{}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

$$2s \frac{\bar{R}x \wedge i \frac{}{t} \\ Ry \vee \bar{R}x^y \vee (\bar{R}y \wedge Rx^y)}{\bar{R}x \wedge \bar{R}y \wedge Rx^y} \\ Ry \vee \bar{R}x^y \vee 2s \frac{}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

$$2c \frac{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vee c \frac{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \vee \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

The two possibilities are clearly represented — nice semantics of proofs.

REAL ANALYTICITY IN DEEP INFERENCE

freely compose proofs via any given connectives — and substitution (w.i.p.)

$$i \frac{t}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}}$$

=

$$2s \frac{i \frac{t}{R_x \vee R_y \vee (\bar{R}_x \wedge \bar{R}_y)} \wedge R_x^y}{\bar{R}_x \wedge \bar{R}_y \wedge R_x^y}$$

$$R_x \vee R_y \vee \exists \exists . (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)$$

$\{x \leftarrow \sqrt{2}\} \vee$

$$2s \frac{\bar{R}_x \wedge i \frac{t}{R_y \vee \bar{R}_x^y \vee (\bar{R}_y \wedge R_x^y)}}{\bar{R}_x \wedge \bar{R}_y \wedge R_x^y}$$

$$R_y \vee \bar{R}_x^y \vee \exists \exists . (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)$$

$\{x \leftarrow \sqrt{2}\} \quad \{y \leftarrow \sqrt{2}\}$

=

$$2c \frac{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vee c \frac{\exists \exists . \exists y . (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y) \vee \exists \exists . \exists y . (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}{\exists \exists . \exists y . (\bar{R}_x \wedge \bar{R}_y \wedge R_x^y)}$$

Real subformula property.

NONELEMENTARY SPEED-UP

Corollary of Aguilera-Baaz*: Deep inference has a non-elementary speed-up over cut-free Gentzen proofs of the predicate calculus.

Moral reason:

	$\vdash Pz,$	\overline{Pz}
w	$\vdash Pz,$	\overline{Pz}, Py
v	$\vdash \overline{Px}, Pz,$	\overline{Pz}, Py
v	$\vdash \overline{Px}, Pz,$	$\overline{Pz} \vee Py$
v	$\vdash \overline{Px} \vee Pz,$	$\overline{Pz} \vee Py$
v	$\vdash \overline{Px} \vee Pz,$	$\forall y.(\overline{Pz} \vee Py)$
<u>\exists</u>	$\vdash \overline{Px} \vee Pz,$	$\exists x. \forall y. (\overline{Px} \vee Py)$
<u>\forall</u>	$\vdash \forall y. (\overline{Px} \vee Py), \exists x. \forall y. (\overline{Px} \vee Py)$	
<u>\exists</u>	$\vdash \exists x. \forall y. (\overline{Px} \vee Py), \exists x. \forall y. (\overline{Px} \vee Py)$	
<u>\bot</u>	$\vdash \exists x. \forall y. (\overline{Px} \vee Py)$	

Take proofs of the drikker formula.

Gentzen bureaucracy
demands contractions
(and nonanalytic \exists rules) ...

... deep inference does not.

$$\begin{aligned} & \boxed{t} \\ = & \exists x. \overline{Px} \vee \forall y. Py \\ = & \exists x. (\overline{Px} \vee \forall y. Py) \\ = & \exists x. \forall y. (\overline{Px} \vee Py) \end{aligned}$$

* J.P. Aguilera and M. Bazz. Unsound Inferences Make Proofs Shorter. 2016, t.b.c. on JSL.