

TOWARDS SUBSTITUTION VIA LINEARITY AND PROJECTION

Alessio Guglielmi

joint work with Chris Barrett and Victoria Barrett

Bath 23/3/21

Talk available from AG's home page and at <https://people.bath.ac.uk/ag248/t/TSLP.pdf>
All about deep inference at <http://alessio.guglielmi.name/res/cas>

TOWARDS SUBSTITUTION VIA LINEARITY AND PROJECTION

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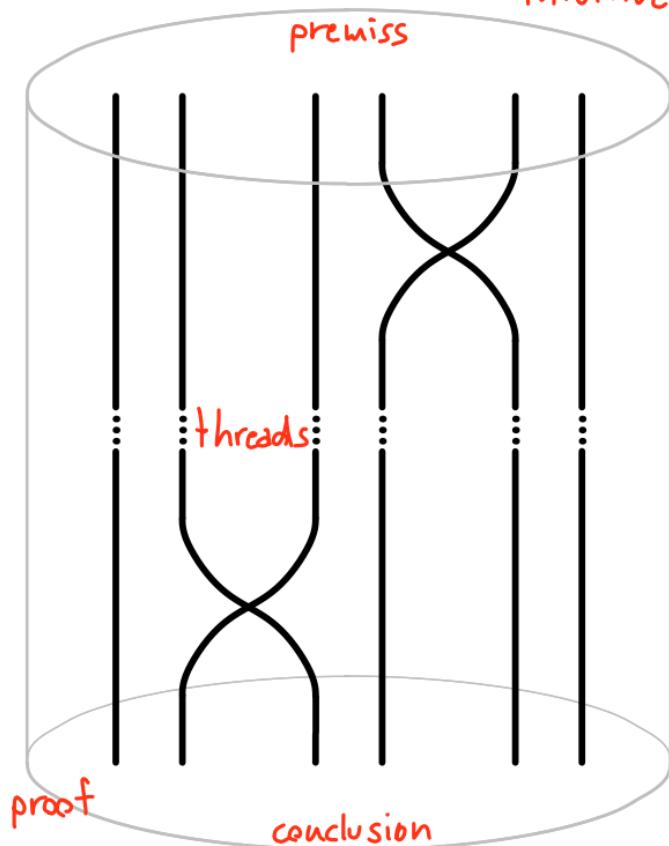
Survey talk with significant (and honest) simplifications

Talk available from AG's home page and at <https://people.bath.ac.uk/ag248/t/TSLP.pdf>
All about deep inference at <http://alessio.guglielmi.name/res/cas>

STATE OF THE ART AND PROBLEM

'Topological model'

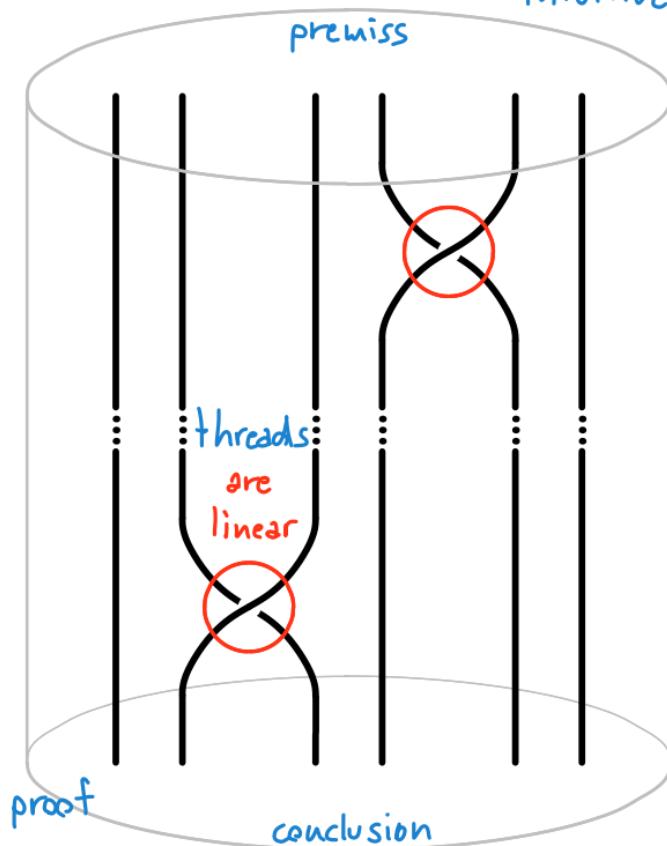
informal
intuitive



STATE OF THE ART AND PROBLEM

'Topological model'

informal
intuitive



STATE OF THE ART AND PROBLEM

Subatomic logic*

formal, unfriendly

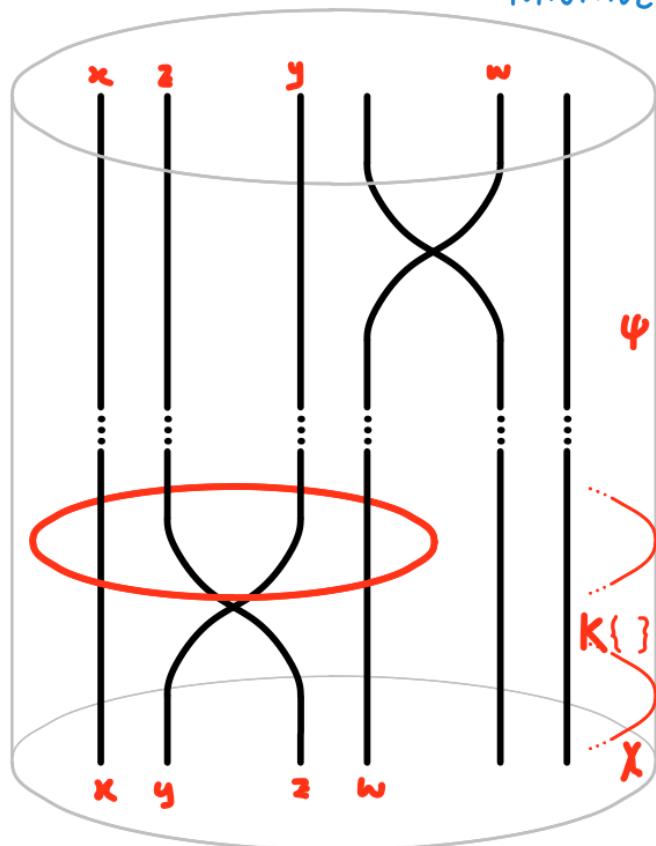
$$K \{ (x \ z \alpha y \ w) \}$$

X

*Aler Tubella, Guglielmi
ACM ToCL 2018

'Topological model'

informal
intuitive



STATE OF THE ART AND PROBLEM

Subatomic logic*

formal, unfriendly

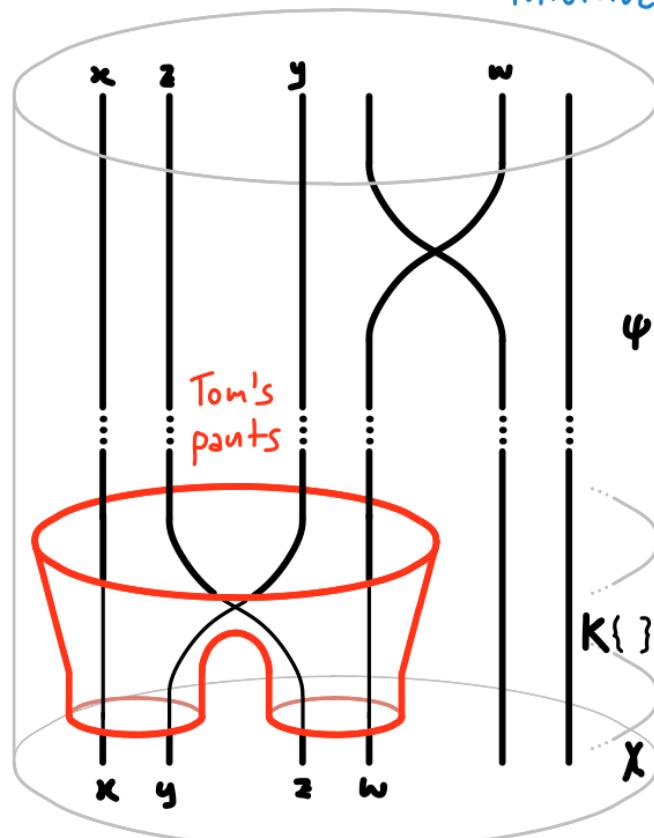
$$\frac{\psi}{K \left\{ \begin{array}{l} (x \sqsupseteq z \alpha y \sqsupseteq w) \\ (x \alpha y) \quad (z \alpha w) \end{array} \right\}}$$

X

*Aler Tubella, Guglielmi
ACM ToCL 2018

'Topological model'

informal
intuitive



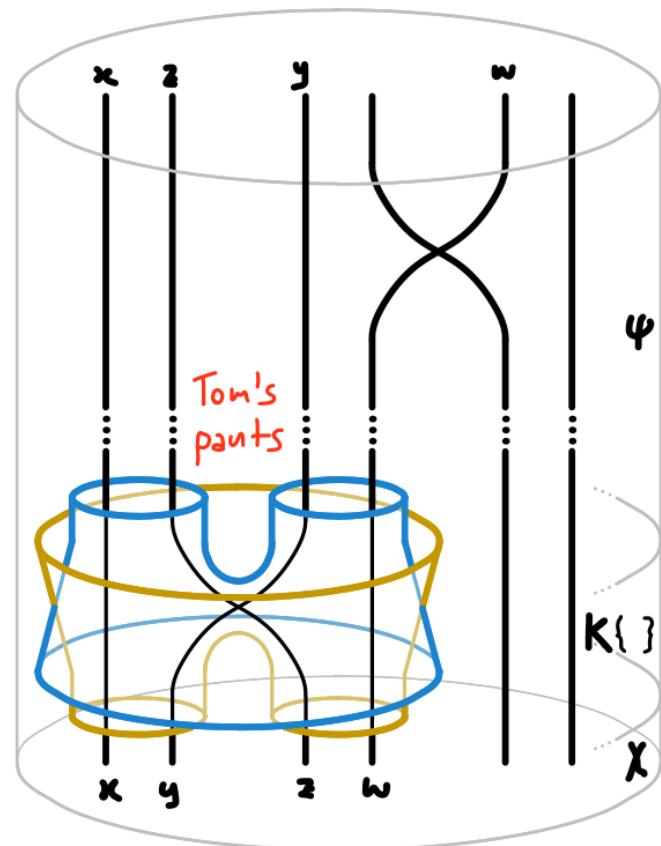
STATE OF THE ART AND PROBLEM

Subatomic logic

$$\frac{\psi}{K \left\{ \frac{(x \beta z) \alpha (y \beta w)}{(x \alpha y) \beta (z \alpha w)} \right\}}$$

χ

'Topological model'



STATE OF THE ART AND PROBLEM

Open deduction*

medial

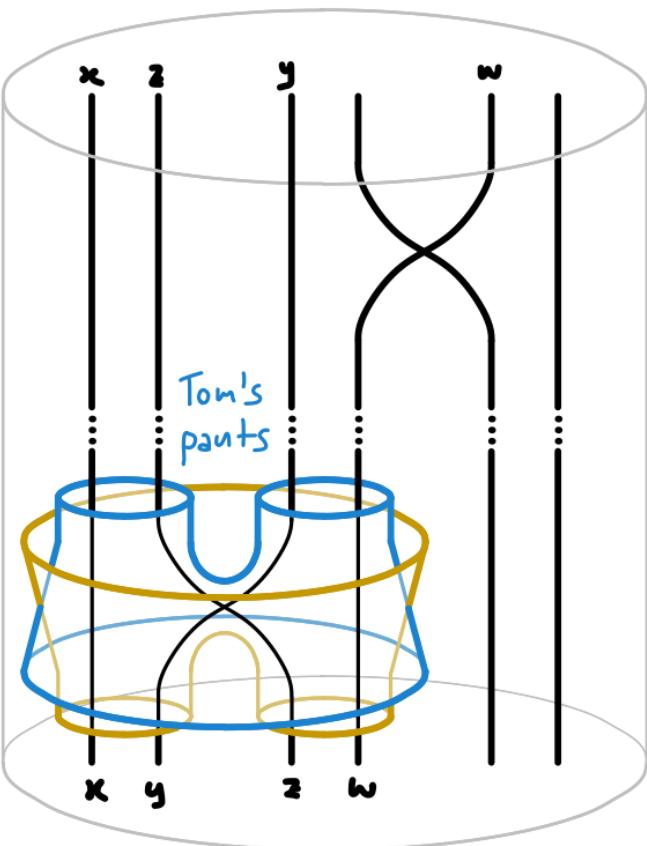
interpretation

$$K \frac{(x \wedge z) \vee (y \wedge w)}{(x \vee y) \wedge (z \vee w)}$$

X

Subatomic logic

'Topological model'



*Guglielmi, Gundersen, Parigot, RTA 2010

STATE OF THE ART AND PROBLEM

Open deduction*

contraction

$$\frac{\psi}{K \left\{ \frac{a \vee a}{a} \right\}}$$

X

Subatomic logic

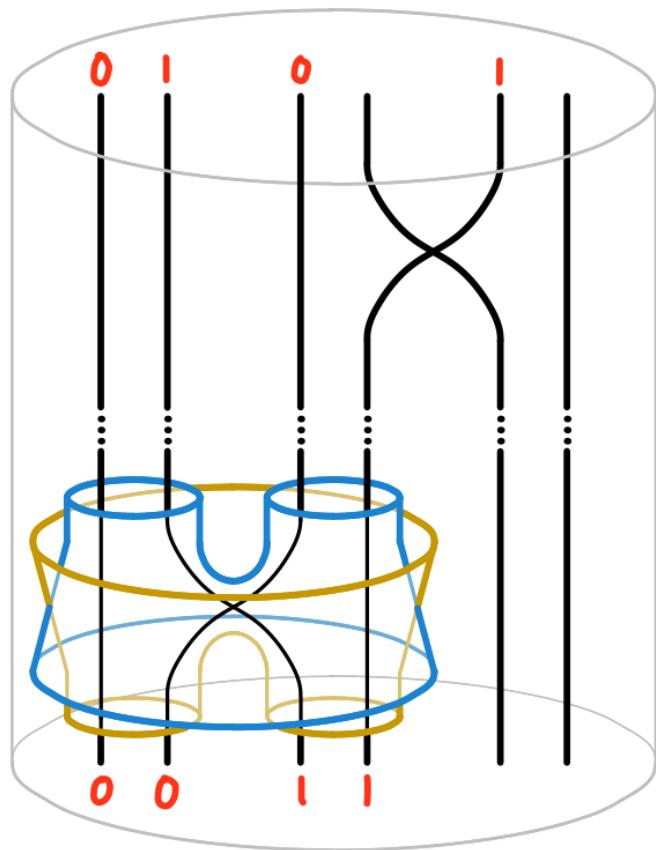
interpretation

$$\frac{\psi}{K \left\{ \frac{(0a1) \vee (0a1)}{(0vo)a(lvi)} \right\}}$$

X

*Guglielmi, Gundersen, Parigot, RTA 2010

'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows*



track atoms

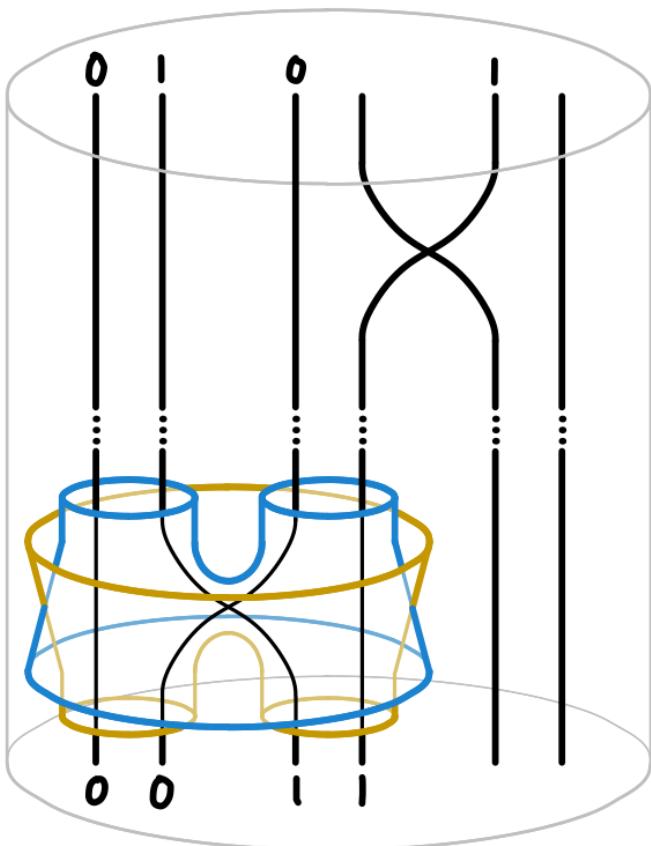
$$\text{K} \left\{ \frac{\psi}{\alpha} \middle| \frac{\alpha \vee \alpha}{\alpha} \right\} \chi$$

contraction

Open deduction

Subatomic logic

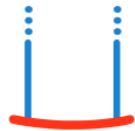
'Topological model'



*Guglielmi, Gundersen, Straßburger, LICS 2010

STATE OF THE ART AND PROBLEM

Atomic flows *



track atoms

$$K \left\{ \frac{a \wedge \bar{a}}{0} \right\}$$

X

cut

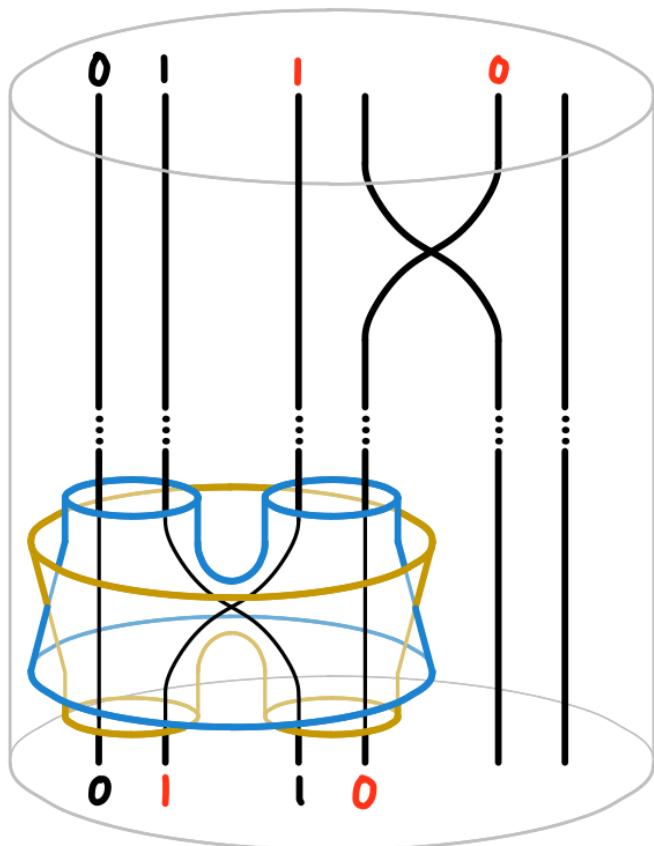
Open deduction

Subatomic logic

$$K \left\{ \frac{(0a1) \wedge (1a0)}{(0 \wedge 1)a(1 \wedge 0)} \right\}$$

X

'Topological model'



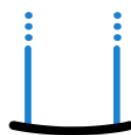
*Guglielmi, Gundersen,
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STATE OF THE ART AND PROBLEM

Atomic flows

Open deduction

Subatomic logic



track atoms

cut

$$K \left\{ \frac{a \wedge \bar{a}}{0} \right\}$$

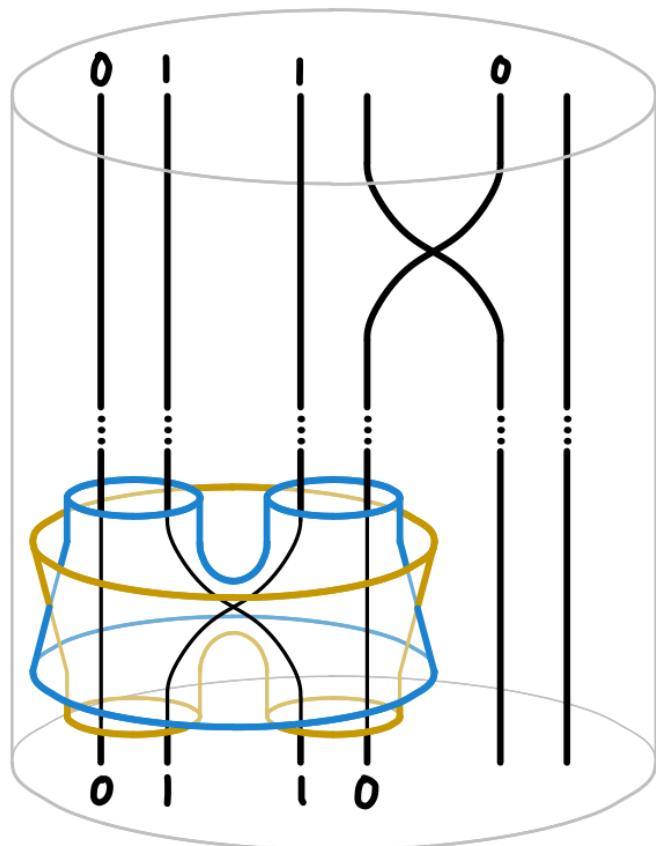
X

negation is not
in the formalism

$$K \left\{ \frac{(0 \wedge 1) \wedge 1 \wedge 0}{(0 \wedge 1) \wedge (1 \wedge 0)} \right\}$$

X

'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

Open deduction

Subatomic logic



track atoms

$$K \left\{ \frac{a \wedge \bar{a}}{0} \right\}$$

X

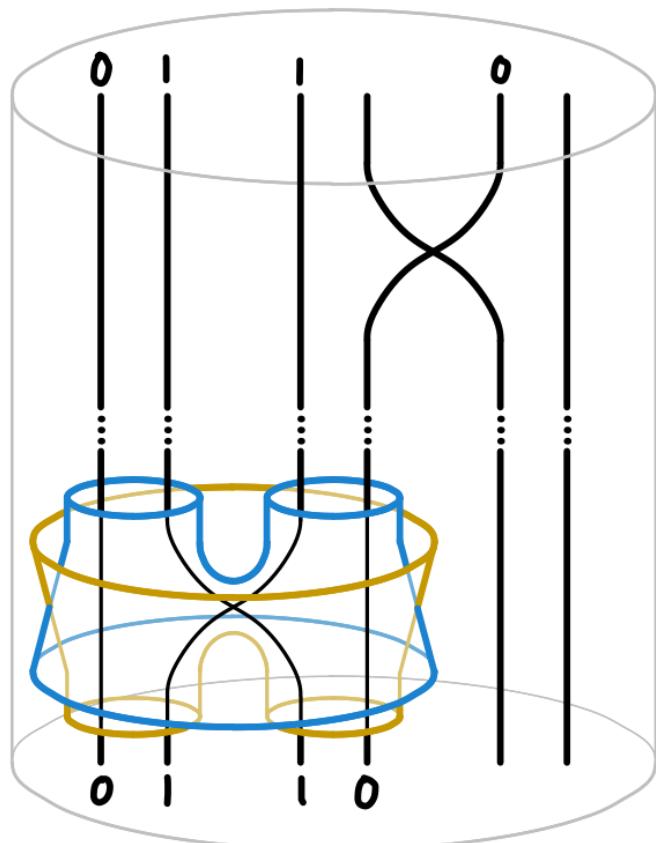
cut

is in
negation
the interpretation

$$K \left\{ \frac{(0 \wedge 1) \wedge (1 \wedge 0)}{(0 \wedge 1) \wedge (1 \wedge 0)} \right\}$$

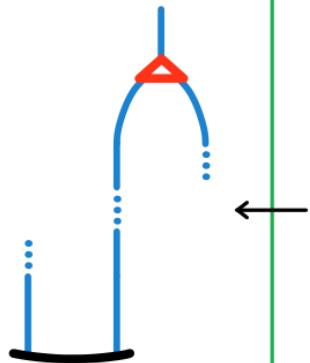
X

'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows



Open deduction

$$H \left\{ \frac{\varphi}{\bar{a} \wedge \bar{a}} \right\}$$

$$K \left\{ \frac{\psi}{a \wedge \bar{a}} \right\}$$

X

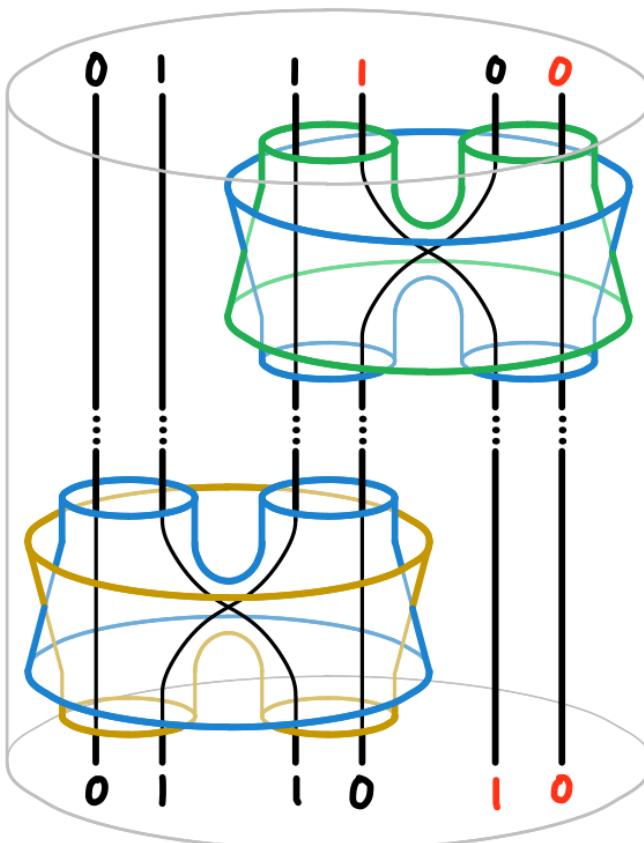
Subatomic logic

$$H \left\{ \frac{\varphi}{(I \wedge I) a (O \wedge O)} \right\}$$

$$K \left\{ \frac{\psi}{(O a I) \wedge (I a O)} \right\}$$

X

'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows



Open deduction

$$\frac{\varphi}{H \left\{ \frac{I}{a \vee \bar{a}} \right\}}$$

$$\frac{\psi}{K \left\{ \frac{a \wedge \bar{a}}{O} \right\}}$$

$$X$$

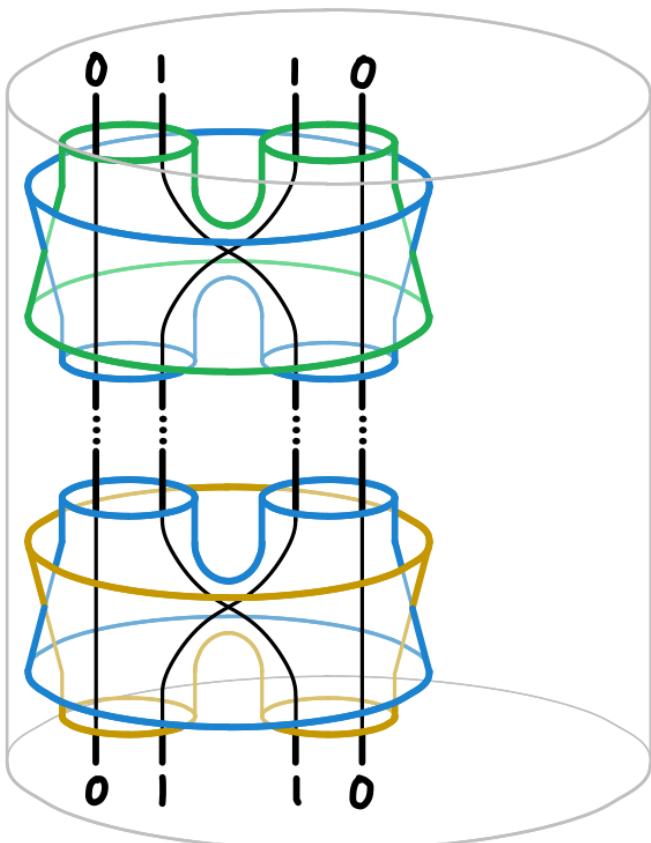
Subatomic logic

$$\frac{\varphi}{H \left\{ \frac{(O \vee I) a (I \vee O)}{(O a I) \vee (I a O)} \right\}}$$

$$\frac{\psi}{K \left\{ \frac{(O a I) \wedge (I a O)}{(O \wedge I) a (I \wedge O)} \right\}}$$

$$X$$

'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

cycle



Open deduction

$$\frac{\varphi}{H \left\{ \frac{1}{a \vee \bar{a}} \right\}}$$



$$\frac{\psi}{K \left\{ \frac{a \wedge \bar{a}}{0} \right\}}$$

X

Subatomic logic

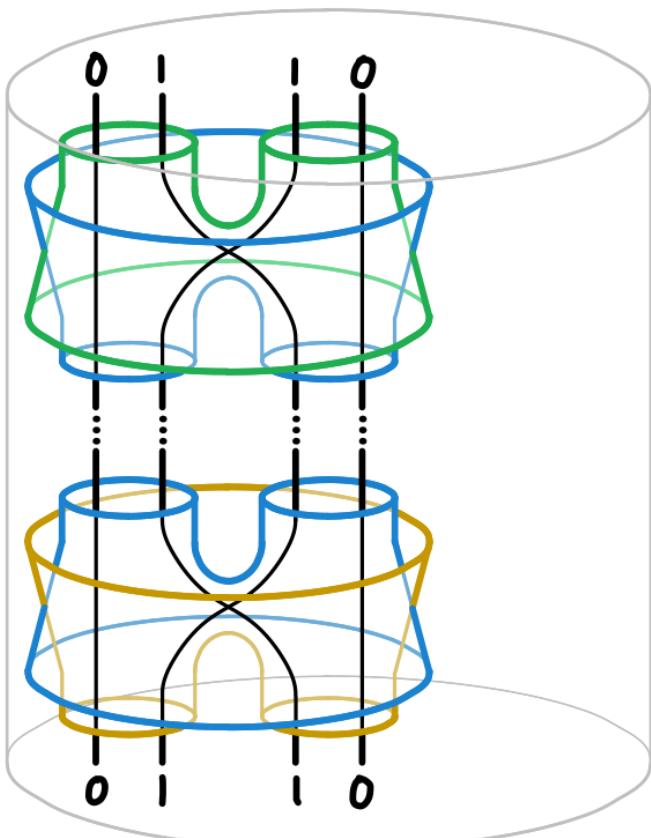
$$\frac{\varphi}{H \left\{ \frac{(0 \vee 1) a (1 \vee 0)}{(0 a 1) \vee (1 a 0)} \right\}}$$



$$\frac{\psi}{K \left\{ \frac{(0 a 1) \wedge (1 a 0)}{(0 \wedge 1) a (1 \wedge 0)} \right\}}$$

X

'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

cycle



Open deduction

- more general proof theory than Gentzen's
- has significant speed-ups
- better normalisation theory

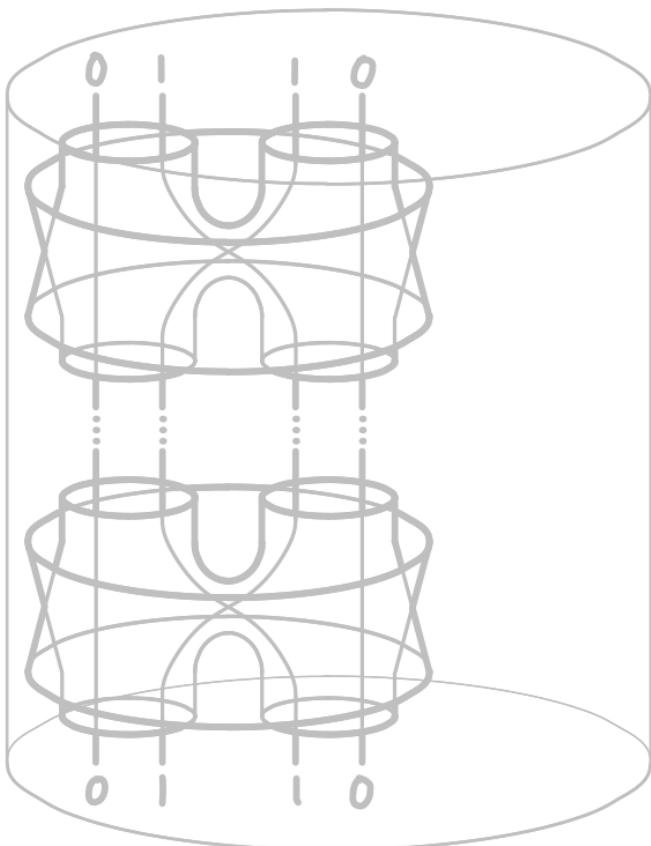
Subatomic logic

$$\varphi \\ \dots \\ H \left\{ \frac{(OVI)a(IVO)}{(OaI)V(IaO)} \right\} \\ \dots$$

$$\psi \\ \dots \\ K \left\{ \frac{(DaI)\Lambda(IaO)}{(O\Lambda I)a(I\Lambda O)} \right\} \\ \dots$$

X

'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

cycle



Open deduction

- more general proof theory than Gentzen's
- has significant speed-ups
- better normalisation theory

Subatomic logic

- makes proof systems extremely regular - all inferences are like this:

$$\frac{(x\beta z)\alpha(y\beta w)}{(x\alpha y)\beta(z\alpha w)}$$

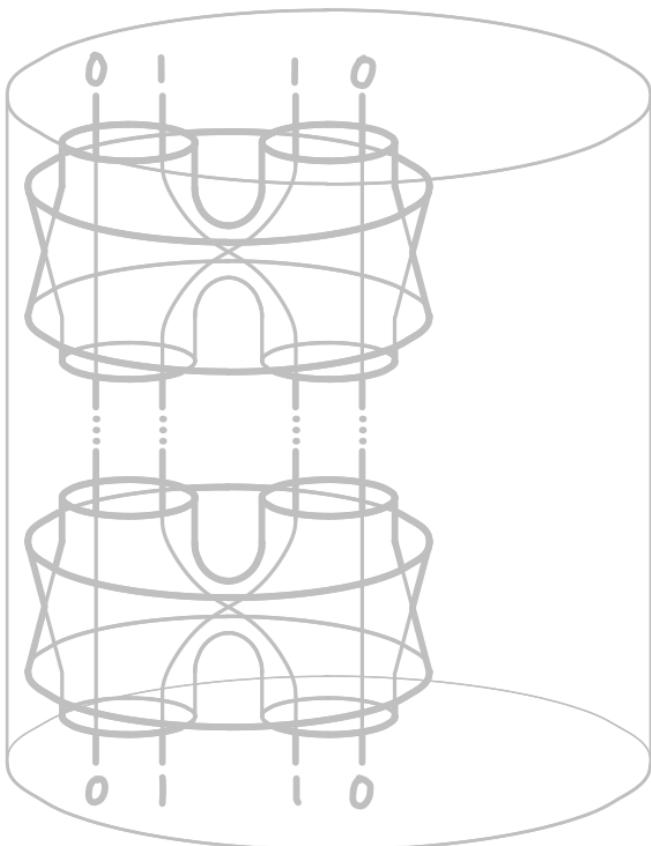
$$(x\alpha y)\beta(z\alpha w)$$

except* for unit equations, e.g. $A \wedge I = A$

- even better normalisation theory

*in this talk we ignore saturation

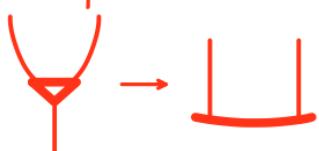
'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

- yield topological invariants that provide induction measures for normalisation
- yield normalisation for proof substitution



Open deduction

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- better normalisation theory

Subatomic logic

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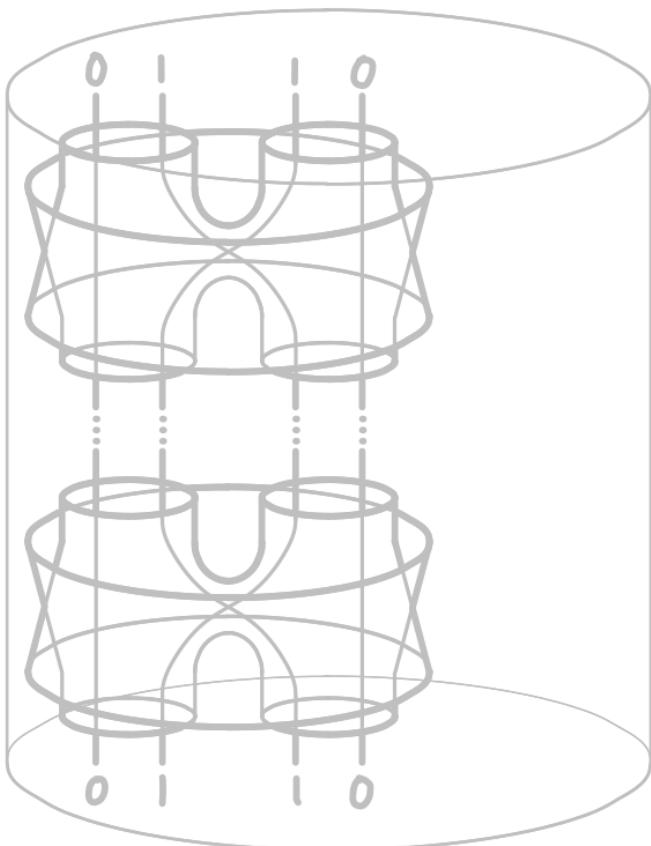
$$\frac{(x\beta z)\alpha(y\beta w)}{(x\alpha y)\beta(z\alpha w)}$$

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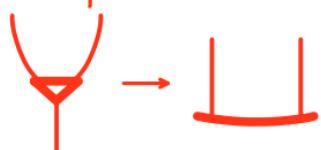
'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

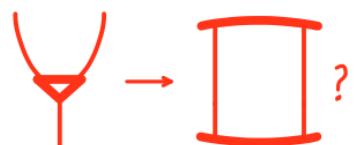
- yield topological invariants that provide induction measures for normalisation
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Open deduction

- more general proof theory than Gentzen's
- has significant speed-ups
- better normalisation theory

• how do we realize



Subatomic logic

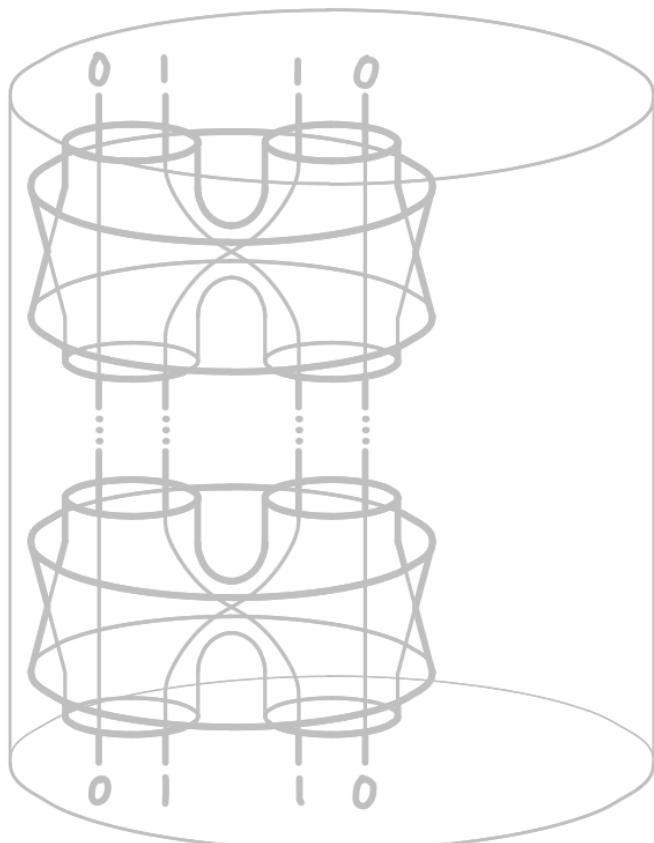
- makes proof systems extremely regular - all inferences are like this:

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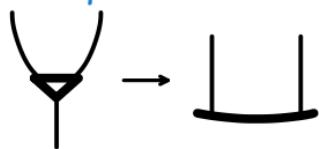
'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

- yield topological invariants that provide induction measures for normalisation
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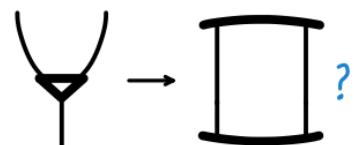
Open deduction

Subatomic logic

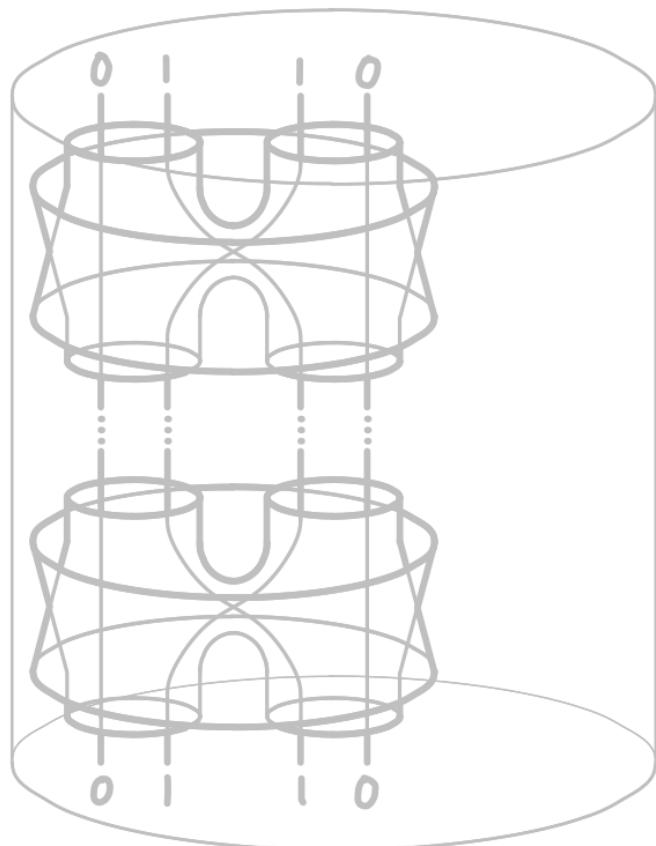
Substitution of proofs into atoms

$$[\varphi \rightarrow a][\varphi \rightarrow b] \cdots \chi$$

• how do we realize



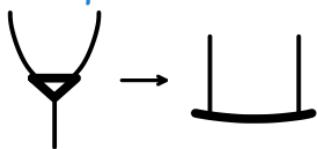
'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

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Open deduction

Subatomic logic

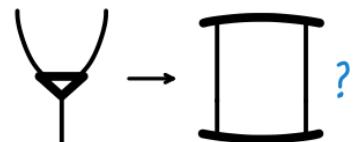
Substitution of proofs into atoms

open deduction +

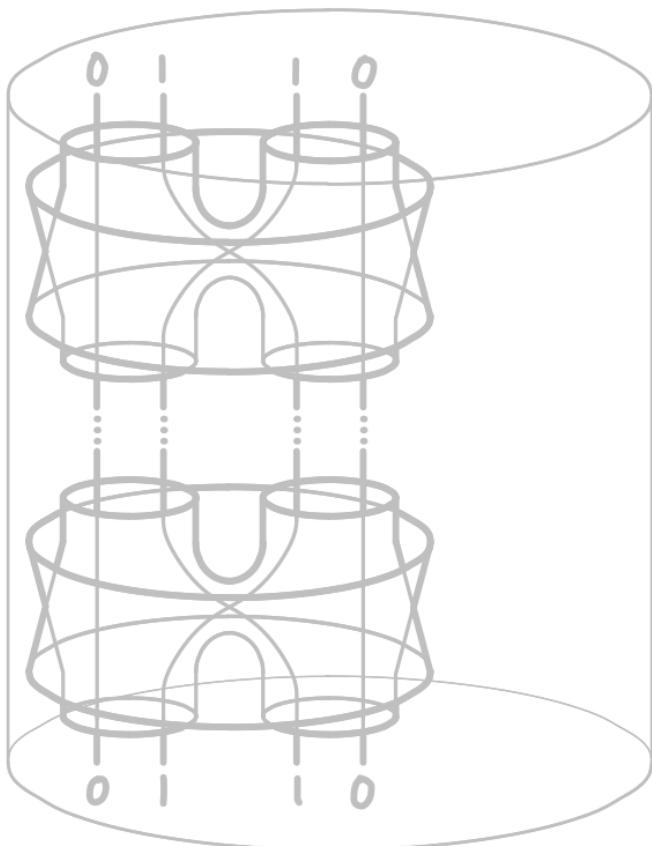
$[\varphi \rightarrow a][\varphi \rightarrow b] \dots \chi$

indicated substitutions

• how do we realize



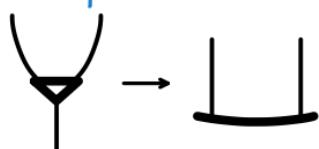
'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

- yield topological invariants that provide induction measures for normalisation



- yield normalisation for proof substitution

Open deduction

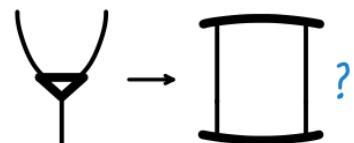
Subatomic logic

Substitution of proofs into atoms

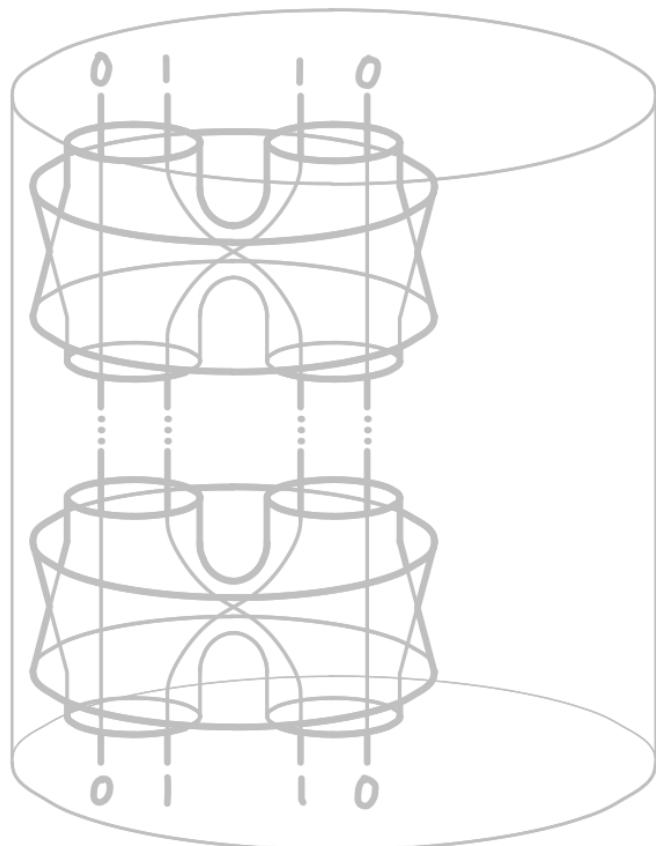
$$[\varphi \rightarrow a][\varphi \rightarrow b] \cdots \chi$$

- exponential compression
- unique factorisation of proofs

• how do we realize



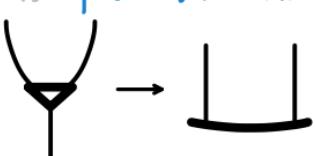
'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

- yield topological invariants that provide induction measures for normalisation



Open deduction

Subatomic logic

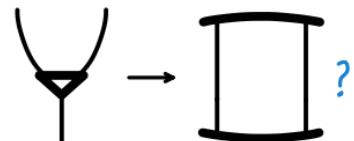
Substitution of proofs into atoms

$$[\varphi \rightarrow a][\varphi \rightarrow b] \cdots \chi$$

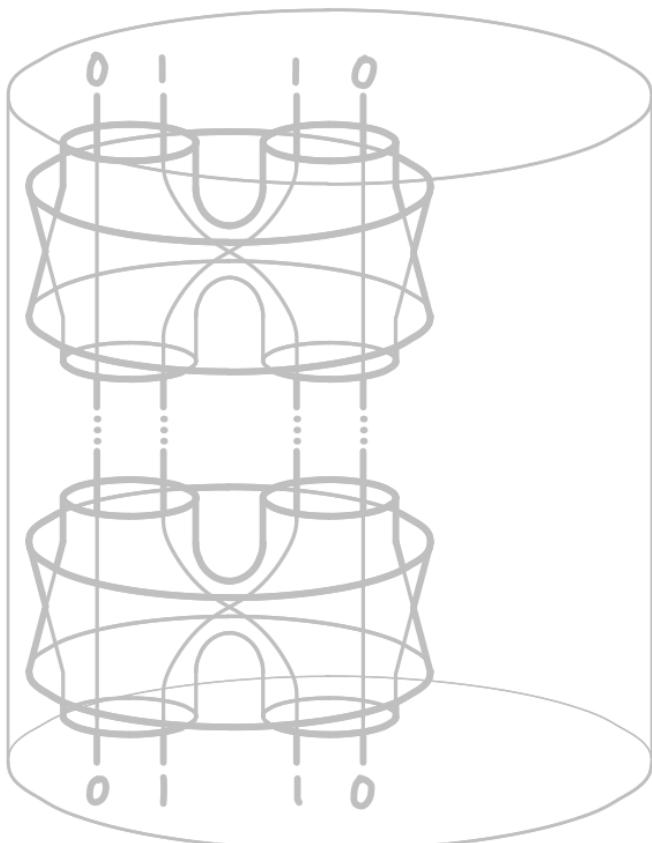
- exponential compression
- unique factorisation of proofs

How? Atomic flows guide us.

- how do we realize



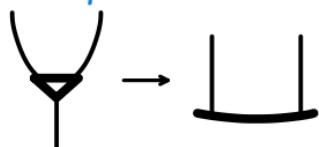
'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

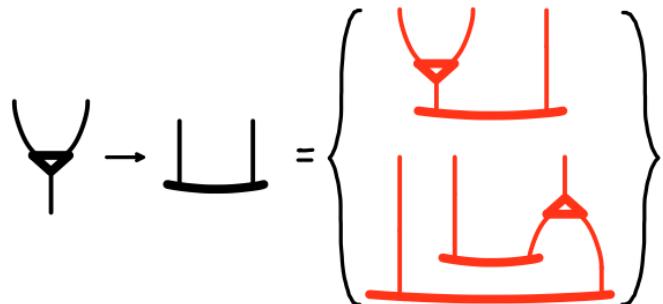
- yield topological invariants that provide induction measures for normalisation
- yield normalisation for proof substitution



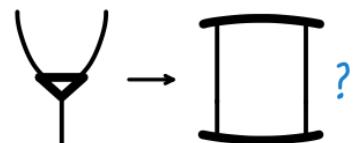
Open deduction

Subatomic logic

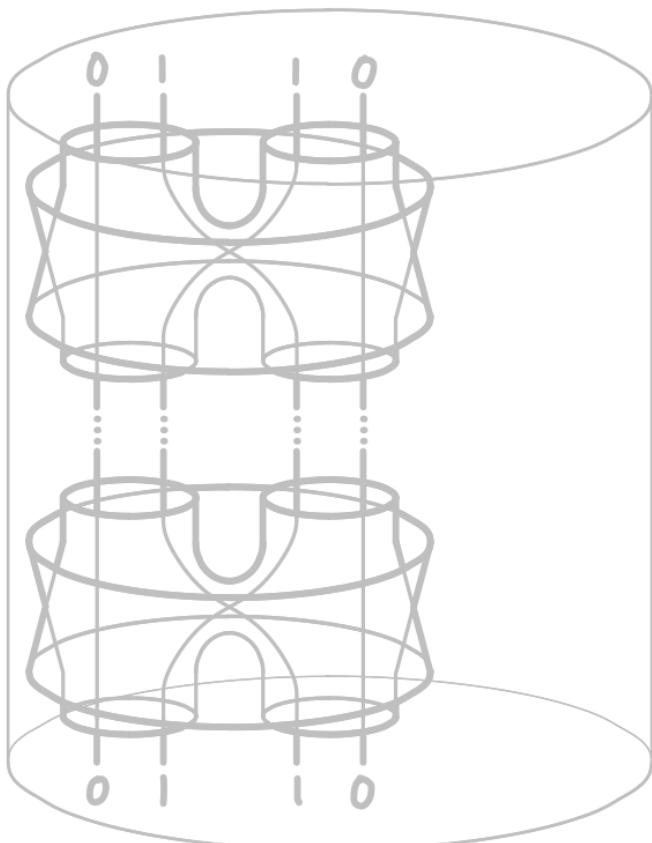
Substitution of proofs into atoms



• how do we realize



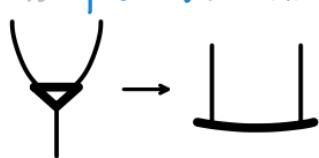
'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

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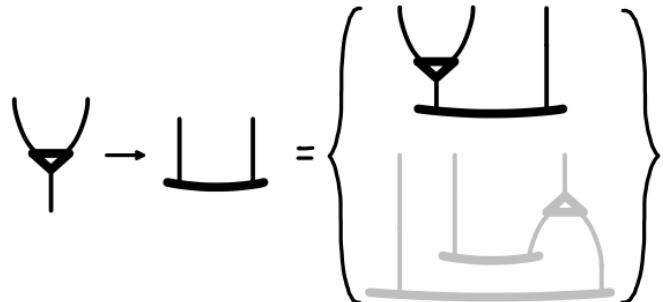


Open deduction

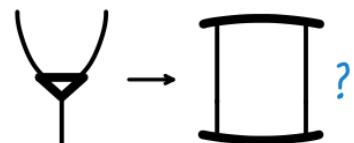
Subatomic logic

Substitution of proofs into atoms

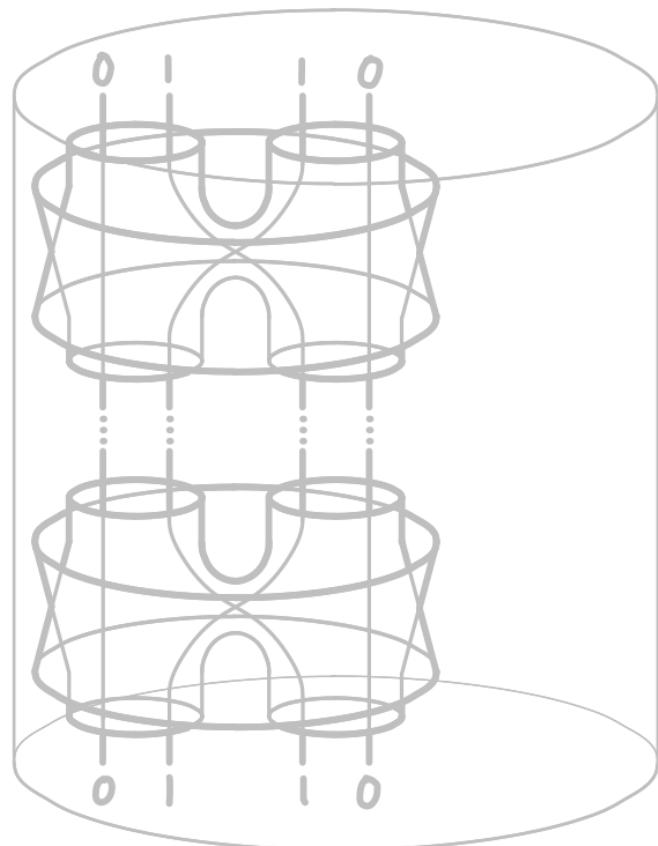
straightforward



• how do we realize



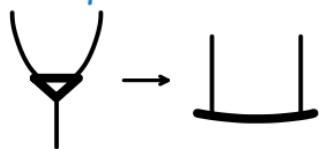
'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

- yield topological invariants that provide induction measures for normalisation
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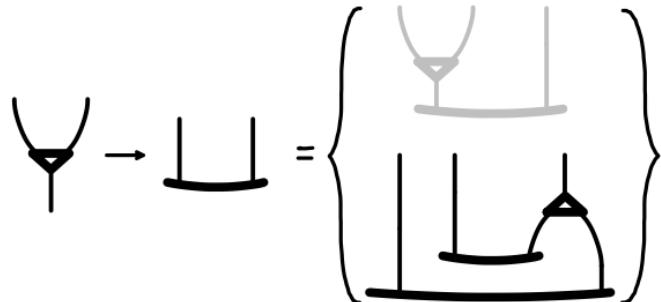


Open deduction

Subatomic logic

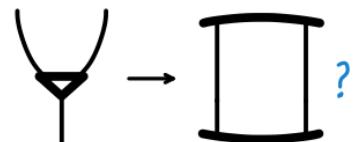
Substitution of proofs into atoms

straightforward

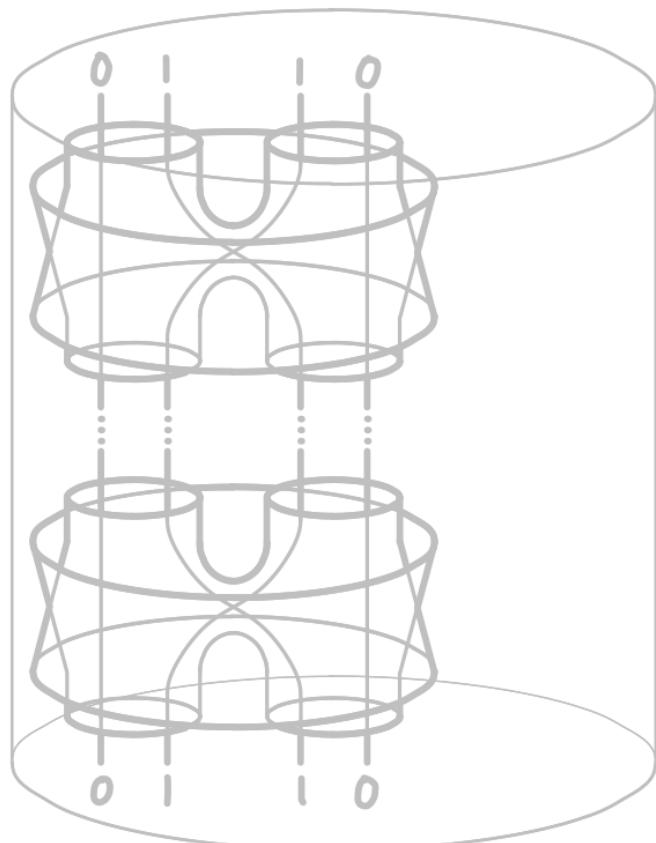


not straightforward but OK - Tom's trick

• how do we realize



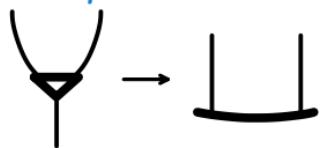
'Topological model'



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Open deduction

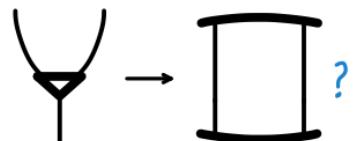
Subatomic logic

Substitution of proofs into atoms

$$\text{Diagram illustrating proof substitution: } \text{Left: } \text{A single vertical line with a central node.} \rightarrow \text{A horizontal bar connecting two vertical lines.} = \left\{ \begin{array}{c} \text{Top: A vertical line with a central node above a horizontal bar.} \\ \text{Bottom: A vertical line with a central node below a horizontal bar.} \end{array} \right.$$

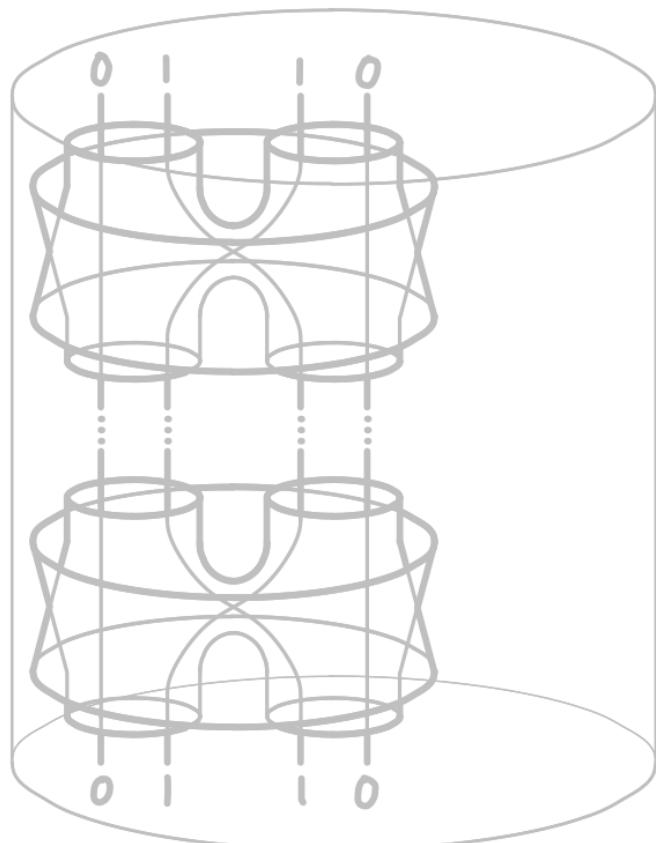
What is the negation of a proof?

- how do we realize



- no negation

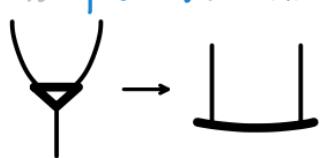
'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

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Open deduction

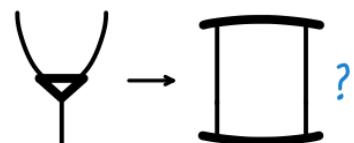
Subatomic logic

Substitution of proofs into atoms

$$\text{Diagram showing a transformation rule: } \text{Complex Shape} \rightarrow \text{Simpler Shape} = \left\{ \text{Red Substitution Variants} \right\}$$

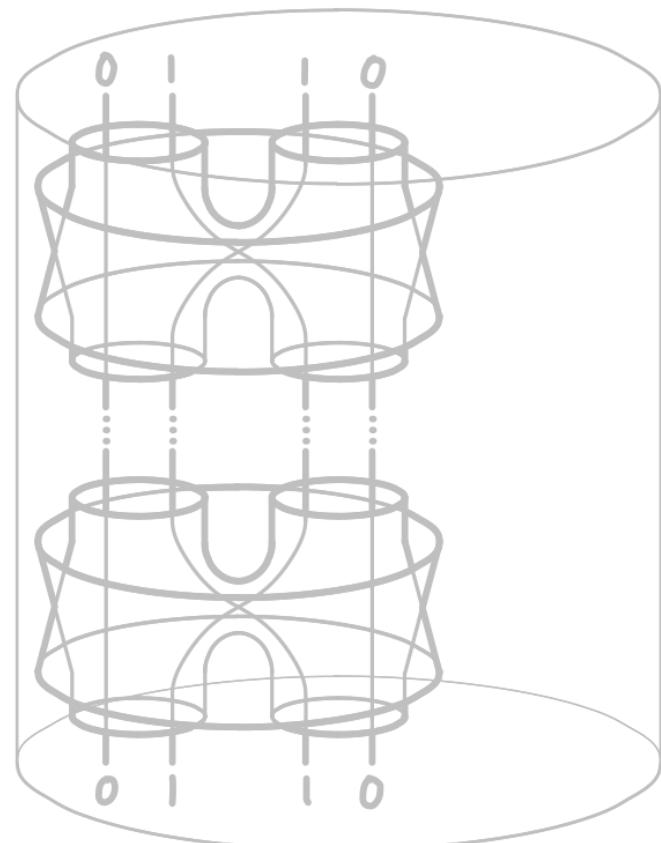
A diagram showing a transformation rule. On the left, a complex black line with a triangle is transformed by an arrow into a simpler black line. To the right of the arrow is an equals sign followed by a set brace containing two red diagrams. The first red diagram shows a complex branching structure with a triangle at the bottom. The second red diagram shows a more complex branching structure with two triangles at the bottom.

• how do we realize



• no negation

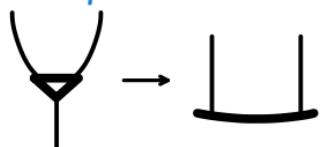
'Topological model'



STATE OF THE ART AND PROBLEM

Atomic flows

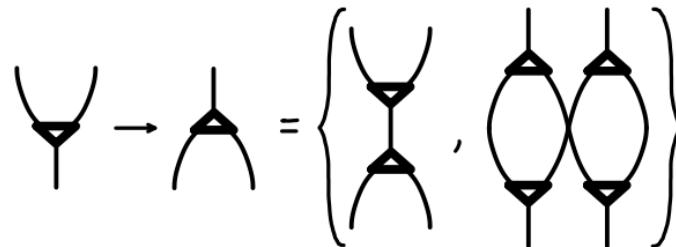
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Open deduction

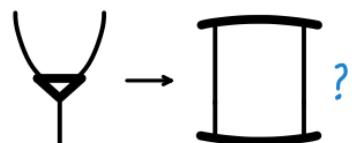
Subatomic logic

Substitution of proofs into atoms



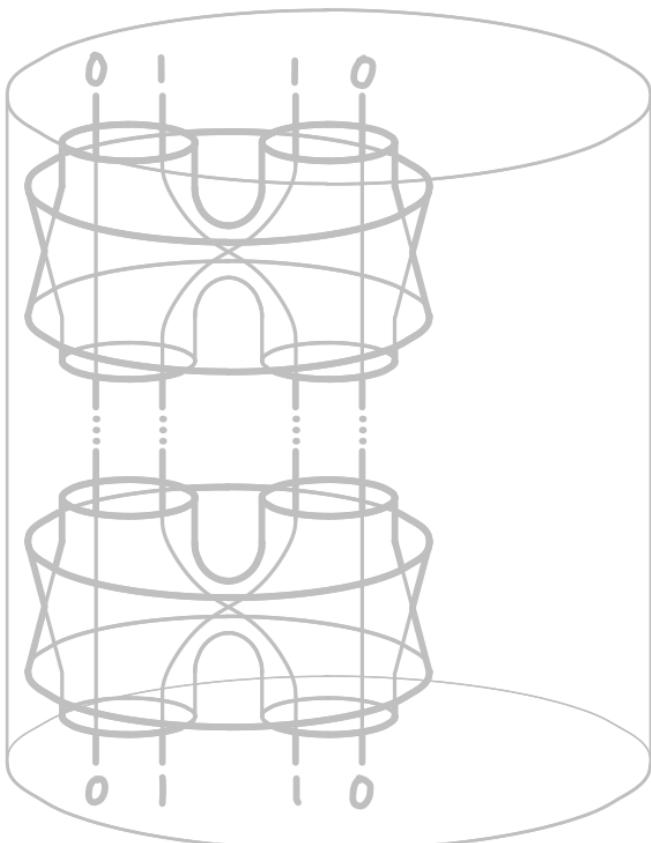
logically straightforward
not topologically equivalent

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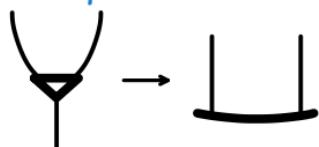
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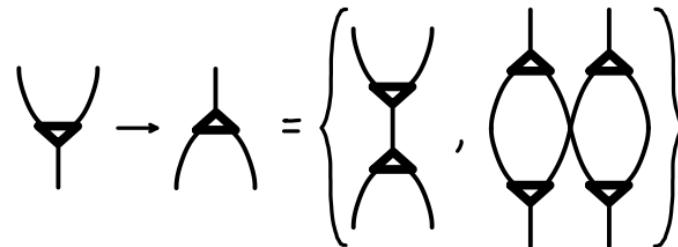
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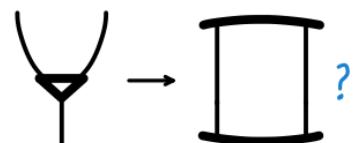
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Is there an underlying linear model?

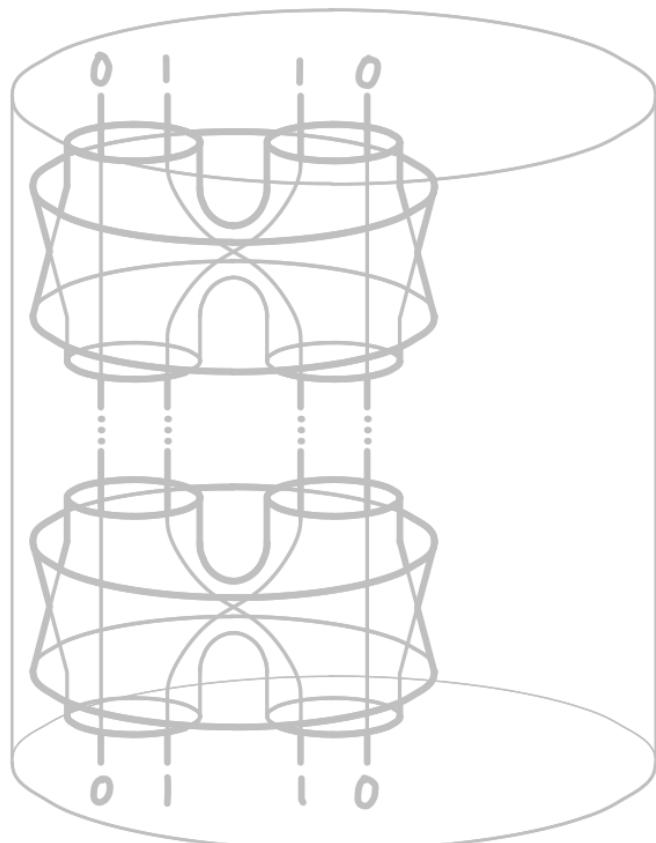
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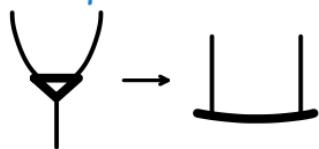
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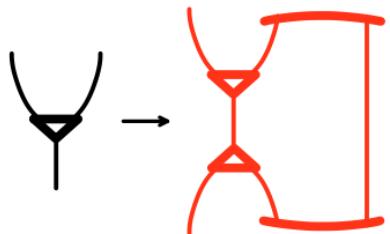
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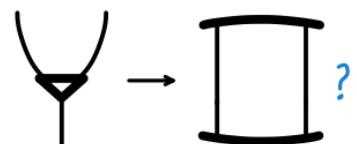
Subatomic logic

Substitution of proofs into cycles



neither logically nor topologically straightforward

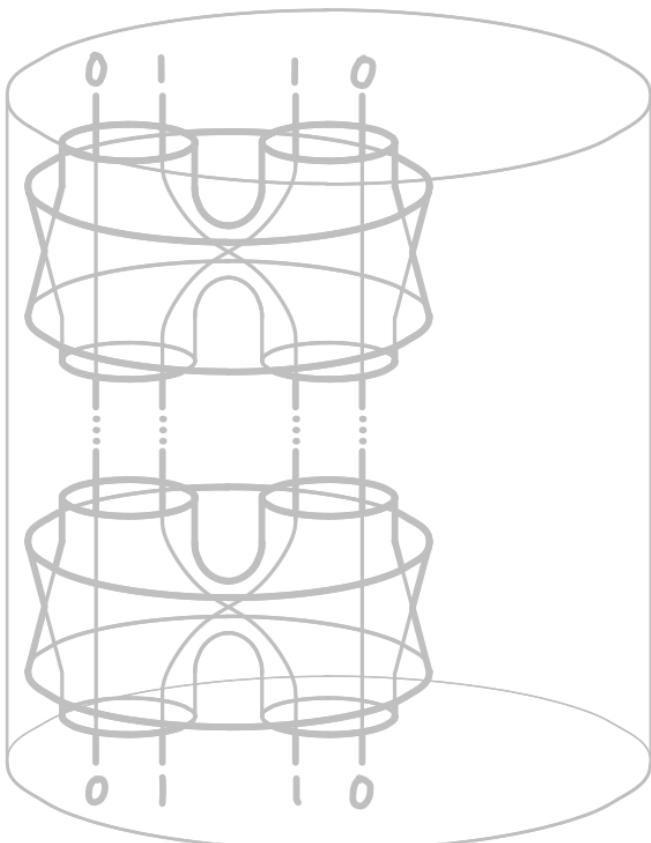
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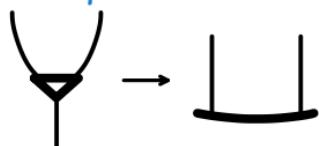
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STATE OF THE ART AND PROBLEM

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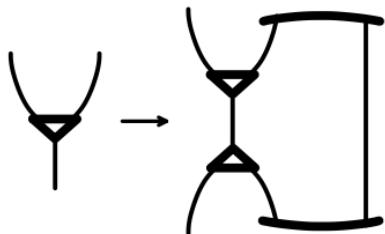
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Subatomic logic

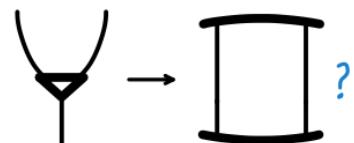
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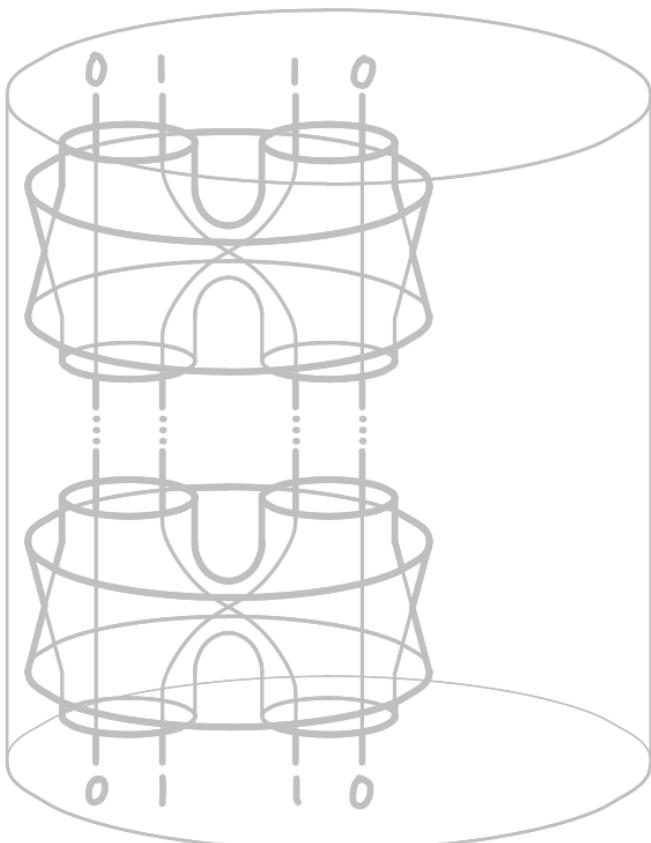
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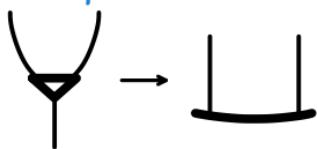
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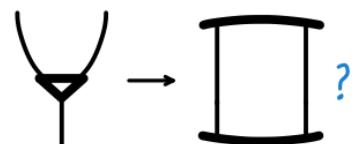
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Open deduction

- more general proof theory than Gentzen's
- has significant speed-ups
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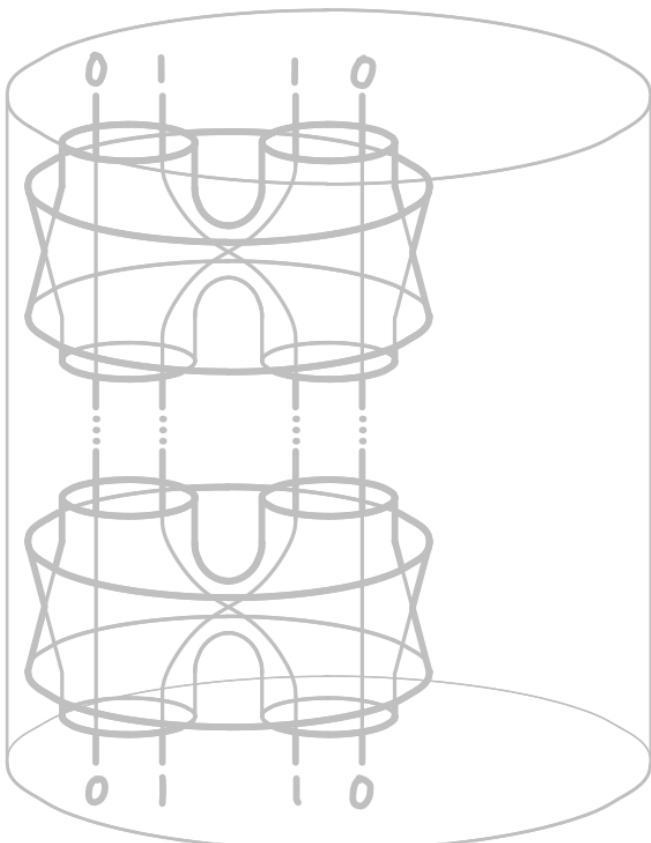
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$$\frac{(x\beta z)\alpha(y\beta w)}{(x\alpha y)\beta(z\alpha w)}$$

except for unit equations, e.g. $A \wedge I = A$

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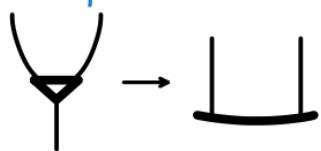
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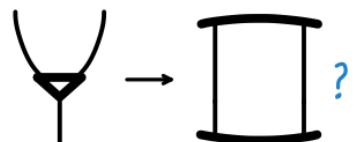
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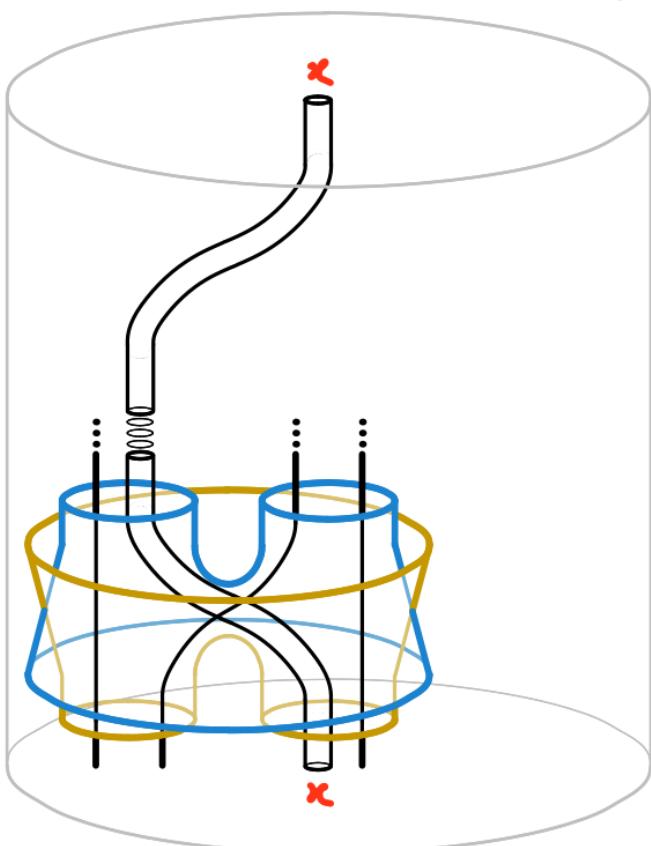
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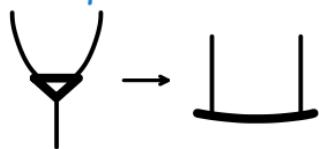
yes



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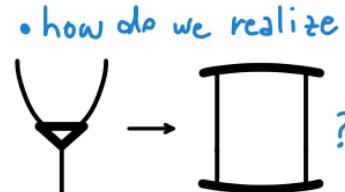
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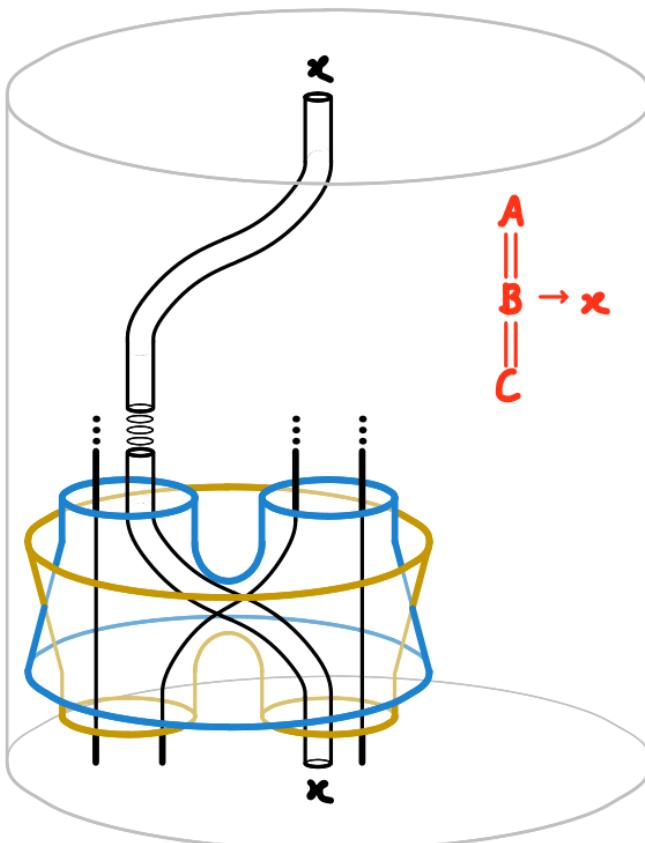
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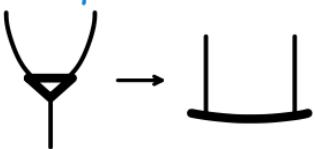
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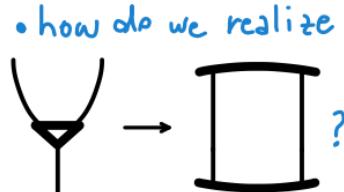
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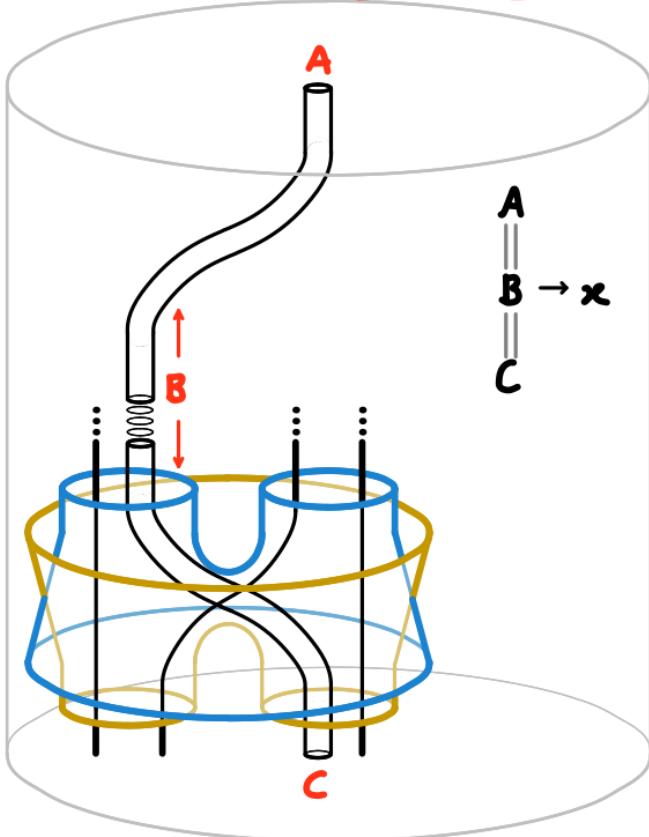
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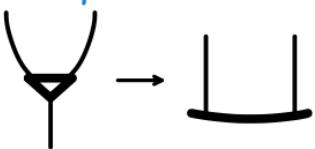
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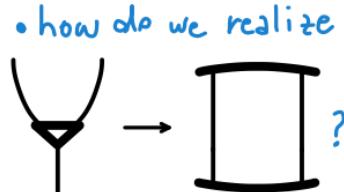
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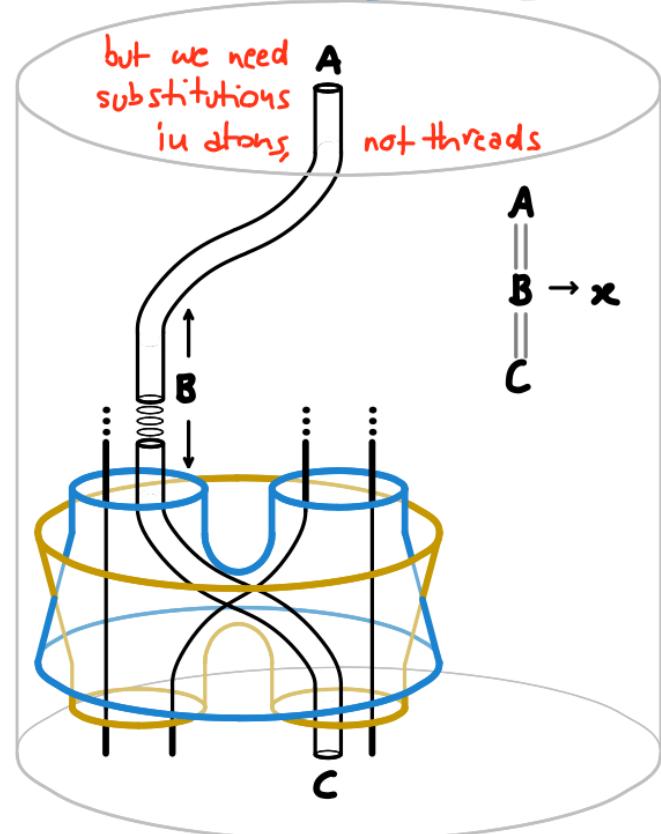
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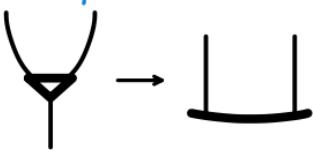
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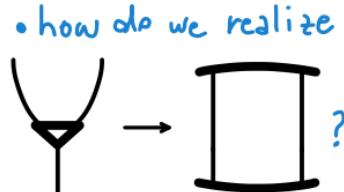
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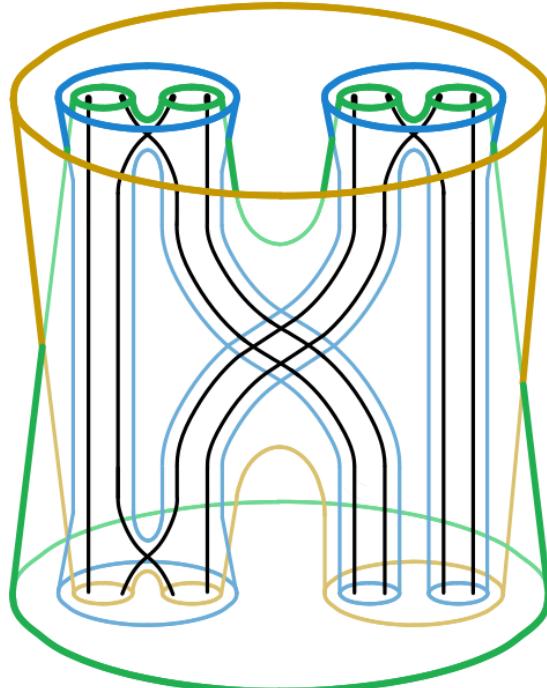
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'Topological model' projections

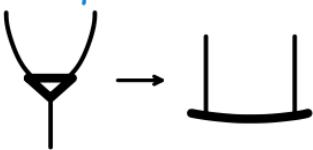
- remove the blue connected component and its right threads
- get the same shape



STATE OF THE ART AND PROBLEM

Atomic flows

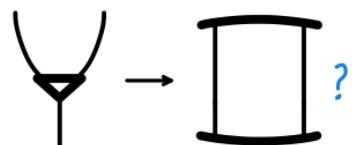
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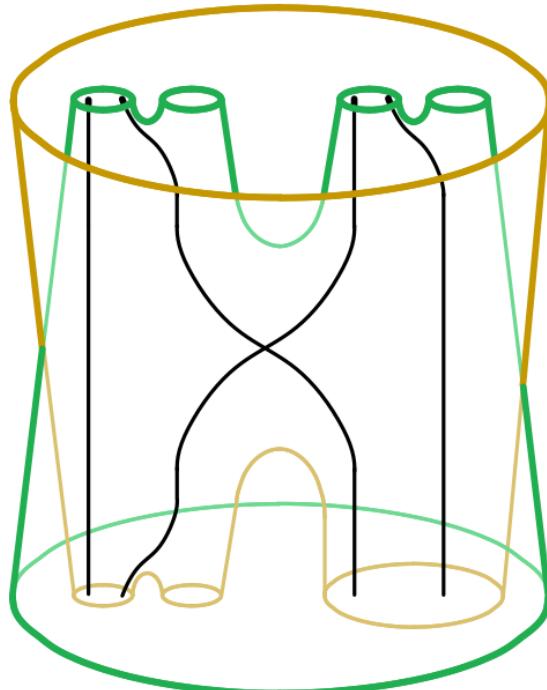
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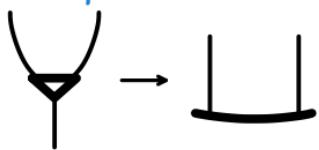
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STATE OF THE ART AND ~~PROBLEM~~ HOPE

Atomic flows

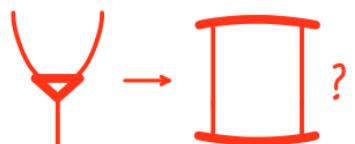
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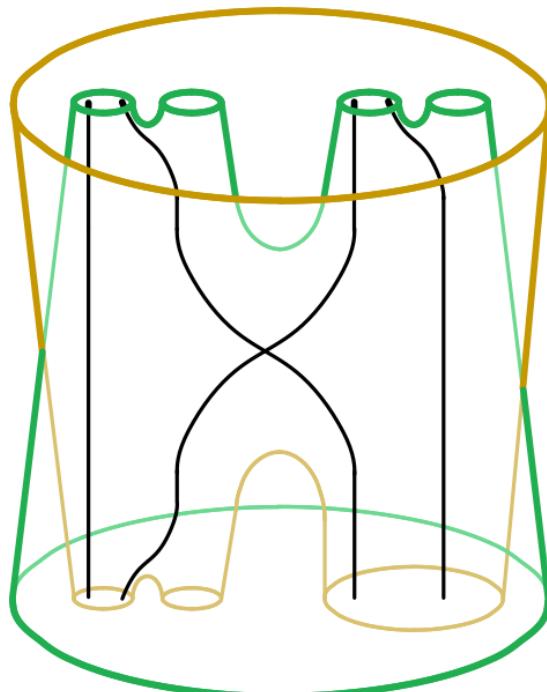
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'Topological model'

$(\text{proof} \rightarrow \text{atom}) = (\text{proofs} \rightarrow \text{threads})$
+ projection



Two RESULTS

1 With Victoria Barrett, in preparation

Unit equations are admissible for subatomic logic, i.e., subatomic logic is totally linear.

2 With Chris Barrett, submitted:

Projections prove cut elimination and other normalisation results.

TWO RESULTS

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Unit equations are admissible for subatomic logic, i.e., subatomic logic is totally linear.

Bonus: all the proof-theoretic constructions needed only use substitutions.

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LINEARITY

φ'
 $K(P)$
 ψ'
 $H(Q)$
 χ'

=

$$\begin{array}{c}
 \varphi \\
 \dots \\
 K\left\{\frac{P}{P \otimes I}\right\} \\
 \dots \\
 [I \Rightarrow y]\psi \\
 \dots \\
 H\left\{\frac{Q \otimes I}{Q}\right\} \\
 \dots \\
 \chi
 \end{array}$$

Subatomic

$$\begin{array}{c}
 \varphi \\
 \dots \\
 K\left\{\frac{P}{P \amalg D(y)}\right\} \\
 \dots \\
 \Psi \\
 \dots \\
 H\left\{\frac{Q' \cap \hat{C}(y)}{Q'}\right\} \\
 \dots \\
 \chi
 \end{array}$$

L

$$\rho \begin{bmatrix} T \Rightarrow K(\cdot) \\ T \Rightarrow Q \end{bmatrix}$$

$$\begin{array}{c}
 \varphi \amalg \hat{K}(D(y)) \\
 K(P) \\
 \downarrow \\
 K(P \amalg D(y)) \\
 \sigma \dots \\
 \Psi \\
 \dots \\
 (\sigma H) \left\{ \begin{array}{c} Q \amalg [\hat{Q} \Rightarrow \hat{C}] \hat{C} \\ \Downarrow \\ [\hat{C} \Rightarrow \hat{Q}] \hat{Q} \end{array} \right\} \\
 \dots \\
 [(Q_j \cap \hat{C}) \Rightarrow Q_j]_j Q \\
 \dots \\
 [(Q_j \cap \hat{C}) \Rightarrow Q_j]_j \sigma \chi
 \end{array}$$

where $\sigma = [(\underline{K}_i \{\cdot\}) \amalg \hat{K}_i \{\cdot\}]_i$;
 $\hat{C} = \sigma \hat{C}(y)$
 $Q = \sigma Q'$

Standard

LINEARITY

Lemma Given any pure formulae B, C and D and any α , there exist derivations

$$[(C^i \alpha D^j) \Rightarrow x_i]_{\underline{B}} \check{B}$$

$$\text{and } [C^i \Rightarrow x_i]_{\underline{B}} \check{B} \alpha [D^j \Rightarrow x_i]_{\underline{B}} \check{B}$$

$$[C^i \Rightarrow x_i]_{\underline{\hat{B}}} \hat{B} \prec [D^j \Rightarrow x_i]_{\underline{\hat{B}}} \hat{B}$$

$$[(C^i \alpha D^j) \Rightarrow x_i]_{\underline{\hat{B}}} \hat{B}$$

where the C^i 's and D^j 's are renamings of C and D .

Proof By induction on B . For the derivation at the left, if $\check{B} \equiv E \gamma F$:

$$\frac{\begin{array}{c} [(C^h \alpha D^k) \Rightarrow x_h]_{\underline{E}} E \\ \parallel \star \hat{x} \\ [C^h \Rightarrow x_h]_{\underline{E}} E \alpha [D^k \Rightarrow x_h]_{\underline{E}} E \end{array} \quad \begin{array}{c} [(C^h \alpha D^k) \Rightarrow x_k]_{\underline{F}} F \\ \parallel \star \hat{x} \\ [C^k \Rightarrow x_k]_{\underline{F}} F \alpha [D^k \Rightarrow x_k]_{\underline{F}} F \end{array}}{\alpha \gamma \quad [(C^i \Rightarrow x_i)]_{E \gamma F} (E \gamma F) \alpha [D^i \Rightarrow x_i]_{E \gamma F} (E \gamma F)}$$

Note $\star \hat{x}$ can be replaced by $\hat{x} A$ and $\star \hat{x}$ can be replaced by $\check{x} A$.

Proposition Given any pure formulae A and B , there exist derivations

$$[A^i \Rightarrow x_i]_{\underline{\check{B}}} \check{B}$$

$$\text{and } [\check{B}^j \Rightarrow y_j]_{\underline{A}} A$$

$$[\check{B}^j \Rightarrow y_j]_{\underline{A}} A$$

$$[A^i \Rightarrow x_i]_{\underline{\hat{B}}} \hat{B}$$

where the A^i 's (resp. \check{B}^j 's, \hat{B}^j 's) are renamings of A (resp. \check{B} , \hat{B}) with no threads in common and $U_i A^i = U_j \check{B}^j = U_j \hat{B}^j$.

Proof By induction on A . For the derivation at the left, if $A \equiv C \alpha D$:

$$[(C^i \alpha D^j) \Rightarrow x_i]_{\underline{\check{B}}} \check{B}$$

$$[(C^i \Rightarrow x_i)]_{\underline{\check{B}}} \check{B}$$

$$[(D^j \Rightarrow x_i)]_{\underline{\check{B}}} \check{B}$$

$$[(\check{B}^k \Rightarrow y_n)]_{\underline{C}} C$$

$$[(\check{B}^k \Rightarrow y_n)]_{\underline{D}} D$$

TWO RESULTS

2

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Projections prove cut elimination and other normalisation results.

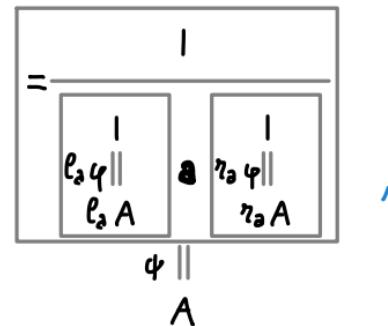
Bonus: we get a good proof system for decision trees and boolean connectives and short analytic proofs for Statman tautologies.

PROJECTIONS

The previous proposition gives analytic proofs for the semantic equivalence $A \leftrightarrow \ell_a A \wedge r_a A$. It is all we need for cut elimination:

Theorem Given a derivation $\frac{}{\parallel}$ we can build a cut-free derivation $\frac{}{\parallel}$.

Proof If φ is the given derivation and a an atom appearing in a cut instance, build



where ψ is given by the previous proposition; the derivation so produced is free of cuts in a . Repeat for all the atoms appearing in cuts in φ .

PROJECTIONS

Proposition For any A of size u there exists a cut-free derivation $\frac{e_a A \text{ and } r_d A}{A}$ of height $O(u^2)$ and width $O(u)$.

Proof We build the derivation as follows, by induction on A :

- if $e_a A \equiv r_d A \equiv A$ then take $\frac{A \text{ and } A}{A}$; $e_a B \text{ and } r_d C$
- otherwise, if $A \equiv B \text{ or } C$, take $\boxed{e_a B \text{ and } r_d B} \quad \boxed{e_a C \text{ and } r_d C}$, where φ is made of two DT-weakening constructions;

- otherwise, if $A \equiv B \alpha C$, where $\alpha \neq \text{and}$, take

$$\frac{\alpha \downarrow (e_a B \text{ and } e_a C) \text{ and } (r_d B \text{ and } r_d C)}{\boxed{e_a B \text{ and } r_d B} \quad \boxed{e_a C \text{ and } r_d C}}.$$

Remark When $\frac{A}{e_a A \text{ and } r_d A}$ is not a cut, we can similarly build a cut-free $\frac{A}{e_a A \text{ and } r_d A}$.