Towards More Efficient and Natural Proof Systems

Alessio Guglielmi

University of Bath

Joint work with
Paola Bruscoli, Tom Gundersen, Michel Parigot and Lutz Straßburger

23 May 2012

This talk is available at `http://cs.bath.ac.uk/ag/t/TMENPS.pdf`
Deep inference web site: `http://alessio.guglielmi.name/res/cos/`

Outline

Problem: Getting rid of bureaucracy in proofs

Cut Elimination by Experiments: Gentzen's structure is too rigid

Open Deduction (Deep Inference): locality (atomicity + linearity)

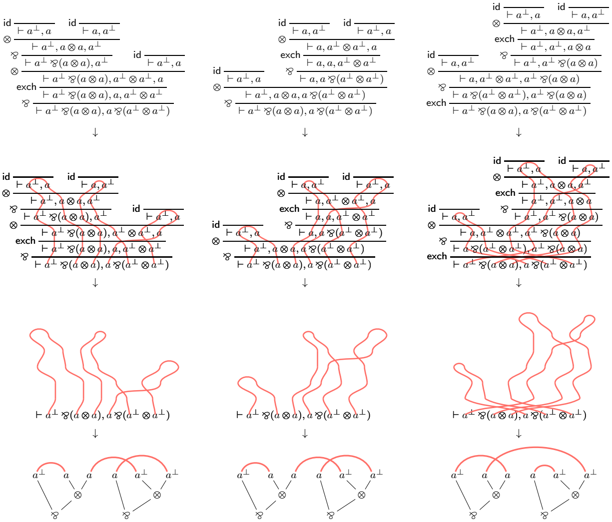
Deep Inference and Proof Complexity: proofs are small, so it is OK

Atomic Flows: locality brings geometry

Normalisation With Atomic Flows: geometry is enough to normalise

The Future, Incorporating Substitution: more geometry, more efficiency, more naturality

Problem: getting rid of bureaucracy in proofs



Picture taken from [Straßburger, 2006]

- From ‘different’ Gentzen sequent proofs we get proof nets (Girard),

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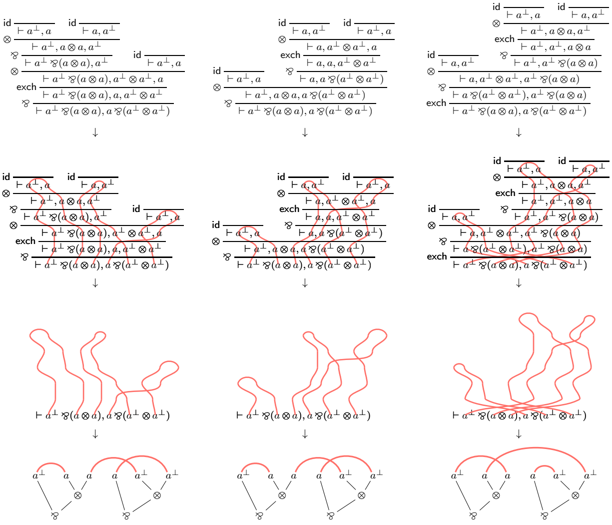

Picture taken from [Straßburger, 2006]

- From 'different' Gentzen sequent proofs we get proof nets (Girard),
- but they are too small: for propositional logic, they probably do not form a proof system.

Proof Systems

- Proof system = algorithm checking proofs in polytime.

Proof Systems

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- Theorem (Cook and Reckhow):

$\exists$ *super proof system*

iff

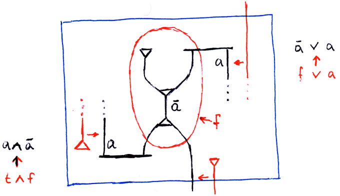
NP = co-NP

where

super = with polysize proofs over the proved tautology

Cut Elimination by 'Experiments'

Experiment over a proof:



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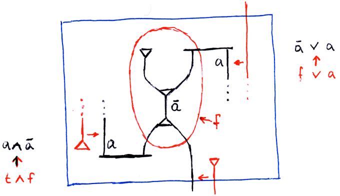

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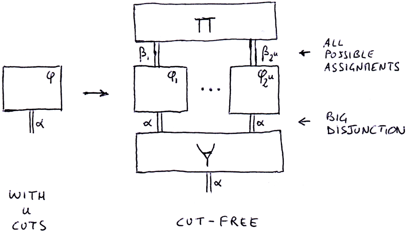

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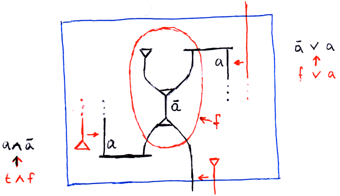

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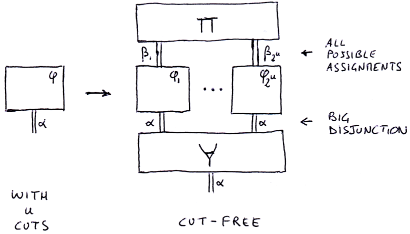

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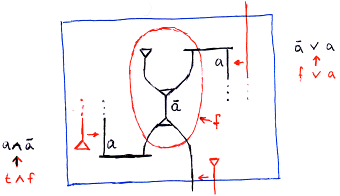

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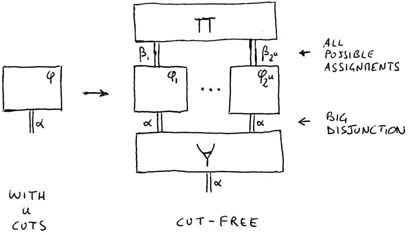

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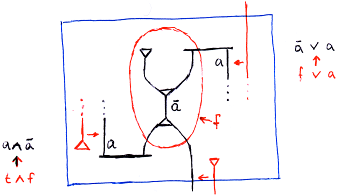

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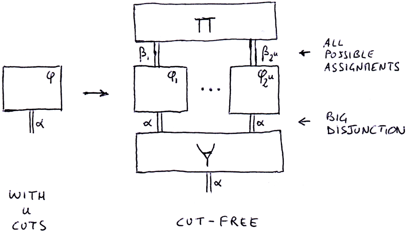

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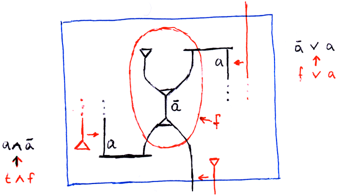

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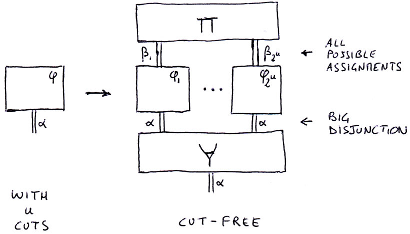

- Simple, exponential cut elimination;
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The secret of success is in the proof composition mechanism.

(Proof) System SKS [Brünnler and Tiu, 2001]

- Atomic rules:

$$\mathsf{ai}{\downarrow}\,\frac{\mathsf{t}}{a \vee \bar{a}} \qquad \mathsf{aw}{\downarrow}\,\frac{\mathsf{f}}{a} \qquad \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$

identity *weakening* *contraction*

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\mathsf{f}} \qquad \mathsf{aw}{\uparrow}\,\frac{a}{\mathsf{t}} \qquad \mathsf{ac}{\uparrow}\,\frac{a}{a \wedge a}$$

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- Linear rules:

$$
\mathsf{s}\,\frac{A \wedge [B \vee C]}{(A \wedge B) \vee C} \qquad \mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}
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- Plus an '$=$' linear rule (associativity, commutativity, units).

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$$\mathsf{ai}\!\uparrow\frac{a \wedge \bar{a}}{\mathsf{f}}\ \textit{cut} \qquad \mathsf{aw}\!\uparrow\frac{a}{\mathsf{t}}\ \textit{coweakening} \qquad \mathsf{ac}\!\uparrow\frac{a}{a \wedge a}\ \textit{cocontraction}$$

- Linear rules:

$$\mathsf{s}\frac{A \wedge [B \vee C]}{(A \wedge B) \vee C}\ \textit{switch} \qquad \mathsf{m}\frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}\ \textit{medial}$$

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switch — *medial*

- Plus an '$=$' linear rule (associativity, commutativity, units).
- Negation on atoms only.
- Cut is atomic.
- SKS is complete for propositional logic.

Examples in Open Deduction (Deep Inference)

- $$\left(\mathsf{m}\,\dfrac{\dfrac{a}{a \wedge a} \vee \dfrac{b}{b \wedge b}}{[a \vee b] \wedge [a \vee b]}\right) \wedge \dfrac{a}{a \wedge a}$$

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Proofs are composed by the same operators as formulae.

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Top-down symmetry: so inference steps can be made atomic (the medial rule, m, is impossible in Gentzen).

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(In [Guglielmi et al., 2010a].)

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E.g., atomic cocontraction:

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In Gentzen:

- no locality for (co)contraction (counterexample in [Brünnler, 2004]),
- no local reduction of cut into atomic form.

Slogans

Deep inference $=$ locality $(+$ symmetry$)$.

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Deep inference = locality (+ symmetry).

Locality = linearity + atomicity.

Geometry = syntax independence (elimination of bureaucracy).

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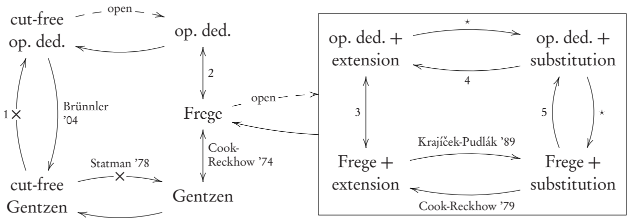
Locality $\rightarrow$ geometry $\rightarrow$ semantics of proofs.

Deep Inference and Proof Complexity



$\longrightarrow$ = 'polynomially simulates'.

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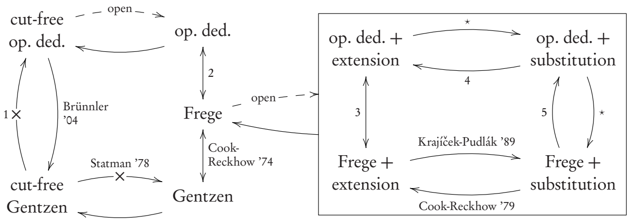

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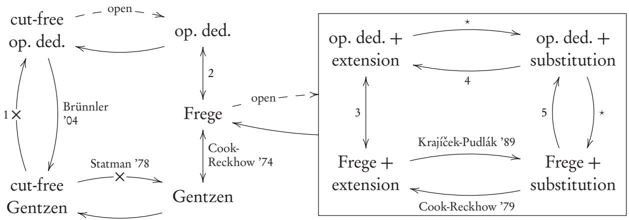

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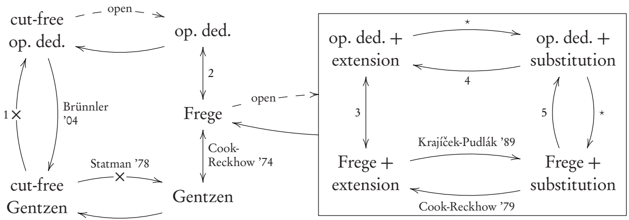

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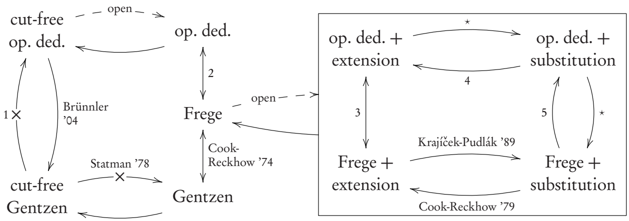

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Open deduction has as small proofs as the best formalisms
and
it has a normalisation theory
and
its cut-free proof systems are more powerful than Gentzen ones
and
cut elimination is quasipolynomial (instead of exponential).
(See [Jeřábek, 2009, Bruscoli and Guglielmi, 2009, Bruscoli et al., 2010]).

Atomic Flows

$$
{\scriptstyle \mathsf{s}}\,\dfrac{{\scriptstyle \mathsf{m}}\,\dfrac{\dfrac{\mathsf{t}}{a \vee \bar{a}}}{[a \vee \mathsf{t}] \wedge [\mathsf{t} \vee \bar{a}]}}{\left[{\scriptstyle \mathsf{s}}\,\dfrac{[a \vee \mathsf{t}] \wedge \bar{a}}{\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee \mathsf{t}} \vee \mathsf{t} \right]}
\quad = \dfrac{\left({\scriptstyle \mathsf{s}}\,\dfrac{a \wedge \left[\bar{a} \vee \dfrac{\mathsf{t}}{\bar{a} \vee a} \right]}{\dfrac{a \wedge \dfrac{\bar{a} \vee \bar{a}}{\bar{a}}}{\mathsf{f}} \vee \dfrac{a}{a \wedge a}} \wedge \bar{a} \right)}{a \wedge \dfrac{a \wedge \bar{a}}{\mathsf{f}}}
\qquad
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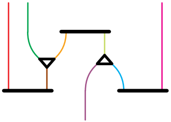
$$


Below proofs, their (atomic) flows are shown:

Atomic Flows

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=\frac{\left(\mathsf{s}\dfrac{a\wedge\left[\bar a\vee\dfrac{\mathsf{t}}{\bar a\vee a}\right]}{\dfrac{a\wedge\dfrac{\bar a\vee\bar a}{\bar a}}{\mathsf{f}}\vee\dfrac{a}{a\wedge a}}\wedge\bar a\right)}{a\wedge\dfrac{a\wedge\bar a}{\mathsf{f}}}
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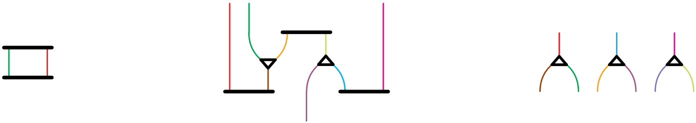
\mathsf{m}\frac{\dfrac{a}{a\wedge a}\vee\dfrac{b}{b\wedge b}}{[a\vee b]\wedge[a\vee b]}\wedge\frac{a}{a\wedge a}$$



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- only structural information is retained in flows;

Atomic Flows

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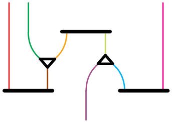

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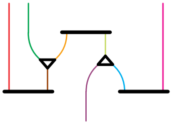
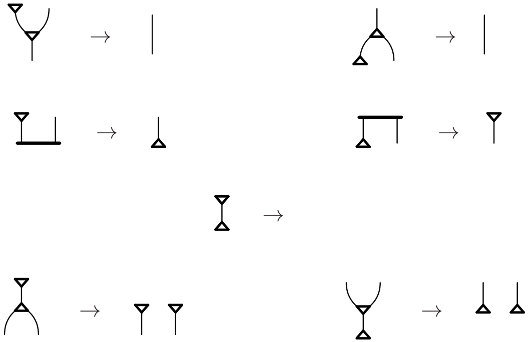


Below proofs, their (atomic) flows are shown:

- only structural information is retained in flows;
- logical information is lost;
- flow size is polynomially related to derivation size.

Flow Reductions: (Co)Weakening (1)



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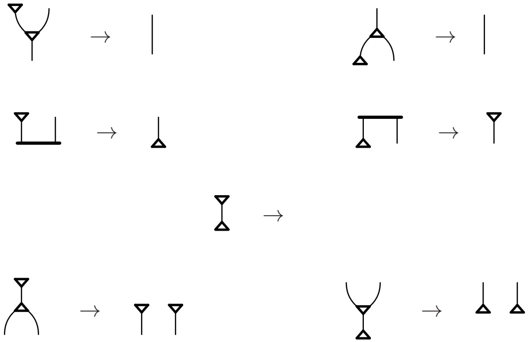
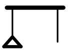


Each flow reduction corresponds to a correct proof reduction.

Flow Reductions: (Co)Weakening (2)

E.g.,  $\to$  specifies that

$$
\begin{array}{c}
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\alpha
\end{array}
\quad \text{becomes} \quad
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\end{array}
$$

Flow Reductions: (Co)Weakening (2)

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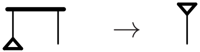
\Pi'' \Big\Vert \\
\xi \left[\mathsf{t} \vee \dfrac{\mathsf{f}}{\bar{a}} \right] \\
\Phi\{a^{\epsilon}/\mathsf{t}\} \Big\Vert \\
\zeta\{\mathsf{t}\} \\
\Psi \Big\Vert \\
\alpha
\end{array}
$$

We can operate on flow reductions instead than on derivations:

- much easier,

Flow Reductions: (Co)Weakening (2)

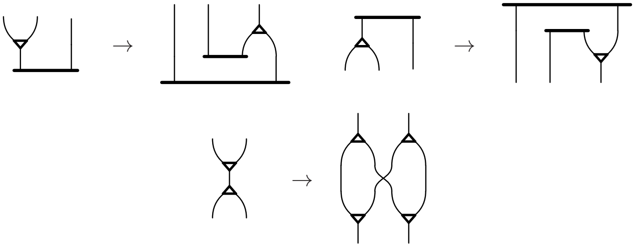
E.g.,  specifies that

$$
\begin{array}{c}
\Pi'' \Big\Vert \\
\xi \left\{ \dfrac{\mathsf{t}}{a^{\epsilon} \vee \bar{a}} \right\} \\
\Phi \Big\Vert \\
\zeta \left\{ \dfrac{a^{\epsilon}}{\mathsf{t}} \right\} \\
\Psi \Big\Vert \\
\alpha
\end{array}
\qquad \text{becomes} \qquad
\begin{array}{c}
\Pi'' \Big\Vert \\
\xi \left[\mathsf{t} \vee \dfrac{\mathsf{f}}{\bar{a}} \right] \\
\Phi\{a^{\epsilon}/\mathsf{t}\} \Big\Vert \\
\zeta\{\mathsf{t}\} \\
\Psi \Big\Vert \\
\alpha
\end{array}
$$

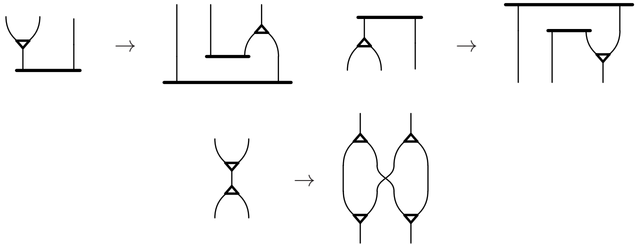
We can operate on flow reductions instead than on derivations:

- much easier,
- we get natural, syntax-independent induction measures.

Flow Reductions: (Co)Contraction

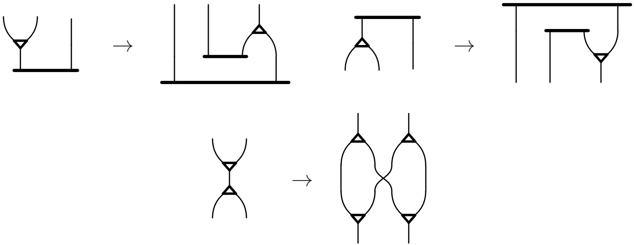


Flow Reductions: (Co)Contraction



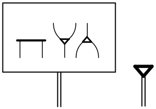
- These reductions conserve the number and length of paths.

Flow Reductions: (Co)Contraction

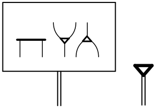


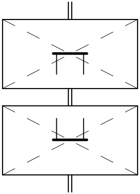
- These reductions conserve the number and length of paths.
- Open problem: does cocontraction yield exponential compression?

Generalising the Cut-Free Form

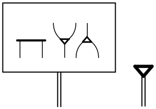
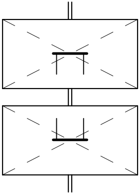
- Normalised proof: 

Generalising the Cut-Free Form

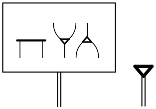
- Normalised proof: 

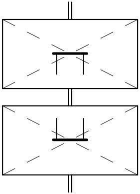
- Normalised derivation: 

Generalising the Cut-Free Form

- Normalised proof: 
- Normalised derivation: 
- The symmetric form is called streamlined.

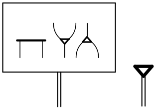
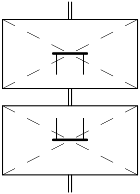
Generalising the Cut-Free Form

- Normalised proof: 

- Normalised derivation: 

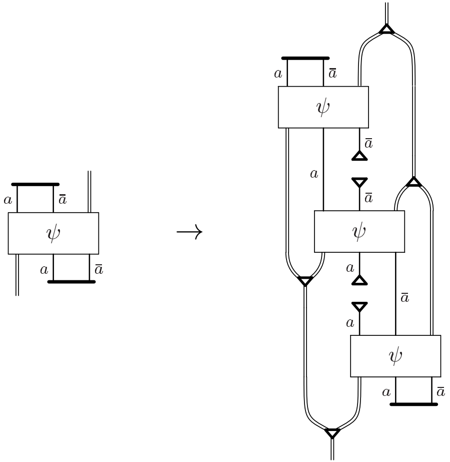
- The symmetric form is called streamlined.
- Cut elimination is a corollary of streamlining.

Generalising the Cut-Free Form

- Normalised proof: 
- Normalised derivation: 
- The symmetric form is called streamlined.
- Cut elimination is a corollary of streamlining.
- We just need to break the paths between identities and cuts, and (co)weakenings do the rest.

How Do We Break Paths?

With the path breaker [Guglielmi et al., 2010b]:



Even if there is a path between identity and cut on the left, there is none on the right.

We Can Do This on Derivations, of Course

$$
\dfrac{\dfrac{A}{[a \vee \bar{a}] \wedge A}}{\begin{array}{c} \Psi \Big\| \\ B \vee (a \wedge \bar{a}) \end{array}} \Bigg/ B
\quad \longrightarrow \quad
\begin{array}{c}
A \\
\Big\| \{\mathsf{c}\uparrow, \mathsf{ai}\downarrow, =\} \\
(([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\
{\scriptstyle (\Psi \wedge A) \wedge A} \Big\| \\
([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\
{\scriptstyle \Phi_a \wedge A} \Big\| \\
[B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\
{\scriptstyle [B \vee \Psi] \wedge A} \Big\| \\
B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\
{\scriptstyle B \vee \Phi_{\bar{a}}} \Big\| \\
B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\
{\scriptstyle B \vee [B \vee \Psi]} \Big\| \\
B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\
\Big\| \{\mathsf{c}\downarrow, \mathsf{ai}\uparrow, =\} \\
B
\end{array}
$$

We Can Do This on Derivations, of Course

$$
\begin{array}{c}
\dfrac{A}{[a \vee \bar{a}] \wedge A} \\
\Psi \Big\| \\
\dfrac{B \vee (a \wedge \bar{a})}{B}
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
A \\
\Big\| \{\mathsf{c}\uparrow, \mathsf{ai}\downarrow, =\} \\
(([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\
(\Psi \wedge A) \wedge A \Big\| \\
([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\
\Phi_a \wedge A \Big\| \\
[B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\
[B \vee \Psi] \wedge A \Big\| \\
B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\
B \vee \Phi_{\bar{a}} \Big\| \\
B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\
B \vee [B \vee \Psi] \Big\| \\
B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\
\Big\| \{\mathsf{c}\downarrow, \mathsf{ai}\uparrow, =\} \\
B
\end{array}
$$

- We can compose this as many times as there are paths between identities and cut.

We Can Do This on Derivations, of Course

$$
\begin{array}{c}
A \\ \hline
[a \vee \bar{a}] \wedge A \\
{\scriptstyle \Psi} \Big\| \\
B \vee (a \wedge \bar{a}) \\ \hline
B
\end{array}
\qquad \longrightarrow \qquad
\begin{array}{c}
A \\
\Big\| {\scriptstyle \{\mathsf{c}\uparrow, \mathsf{ai}\downarrow, =\}} \\
(([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\
{\scriptstyle (\Psi \wedge A) \wedge A} \Big\| \\
([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\
{\scriptstyle \Phi_a \wedge A} \Big\| \\
[B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\
{\scriptstyle [B \vee \Psi] \wedge A} \Big\| \\
B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\
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B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\
{\scriptstyle B \vee [B \vee \Psi]} \Big\| \\
B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\
\Big\| {\scriptstyle \{\mathsf{c}\downarrow, \mathsf{ai}\uparrow, =\}} \\
B
\end{array}
$$

- We can compose this as many times as there are paths between identities and cut.
- We obtain a family of normalisers that only depends on n.

We Can Do This on Derivations, of Course

$$
\begin{array}{c}
A \\ \hline
[a \vee \bar{a}] \wedge A \\
\Psi \Vert \\
B \vee (a \wedge \bar{a}) \\ \hline
B
\end{array}
\qquad \longrightarrow \qquad
\begin{array}{c}
A \\
\Vert \{\mathsf{c}\uparrow, \mathsf{ai}\downarrow, =\} \\
(([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\
{\scriptstyle (\Psi \wedge A) \wedge A} \Vert \\
([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\
{\scriptstyle \Phi_a \wedge A} \Vert \\
[B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\
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B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\
{\scriptstyle B \vee \Phi_{\bar{a}}} \Vert \\
B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\
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B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\
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\end{array}
$$

- We can compose this as many times as there are paths between identities and cut.
- We obtain a family of normalisers that only depends on n.
- The construction is exponential.

We Can Do This on Derivations, of Course

$$
\begin{array}{c}
A \\ \hline
[a \vee \bar{a}] \wedge A \\
{\scriptstyle \Psi}\Big\| \\
B \vee (a \wedge \bar{a}) \\ \hline
B
\end{array}
\qquad \longrightarrow \qquad
\begin{array}{c}
A \\
\Big\| {\scriptstyle \{\mathsf{c}\uparrow, \mathsf{ai}\downarrow, =\}} \\
(([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\
{\scriptstyle (\Psi \wedge A) \wedge A}\Big\| \\
([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\
{\scriptstyle \Phi_a \wedge A}\Big\| \\
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B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\
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B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\
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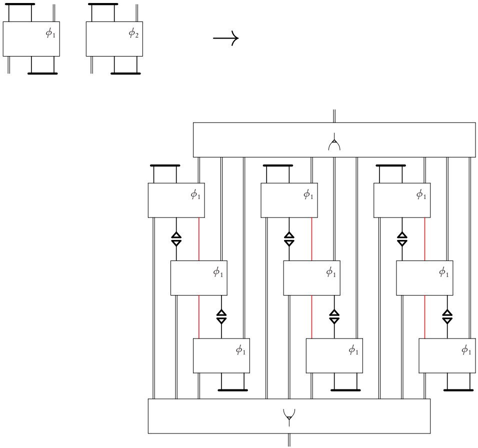
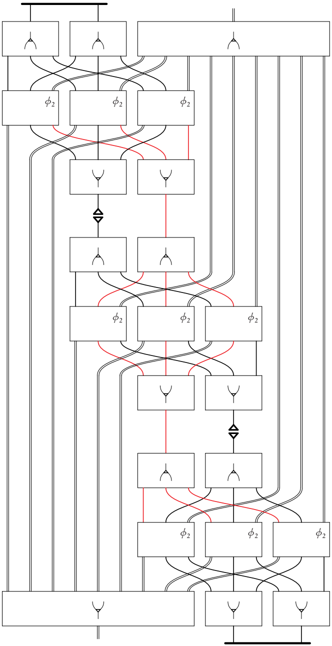
B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\
\Big\| {\scriptstyle \{\mathsf{c}\downarrow, \mathsf{ai}\uparrow, =\}} \\
B
\end{array}
$$

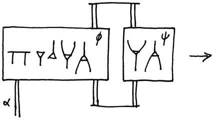
- We can compose this as many times as there are paths between identities and cut.
- We obtain a family of normalisers that only depends on *n*.
- The construction is exponential.
- Finding something like this is unthinkable without flows.

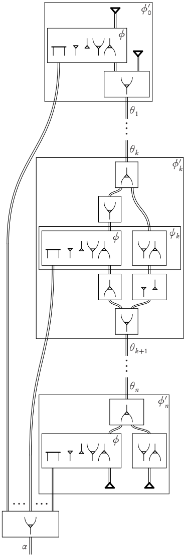
Example for $n = 2$





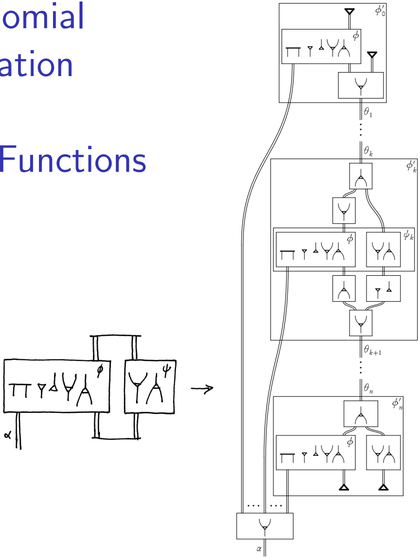
Quasipolynomial Cut Elimination by Threshold Functions





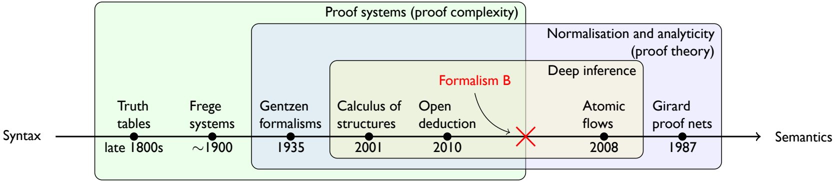
- Only $n + 1$ copies of the proof are stitched together.

Quasipolynomial Cut Elimination by Threshold Functions

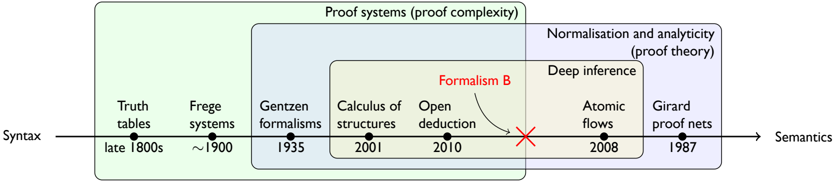


- Only $n + 1$ copies of the proof are stitched together.
- Note local cocontraction ($=$ better sharing, not available in Gentzen).

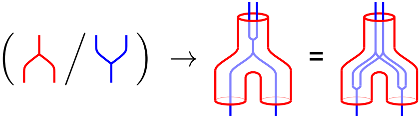
Formalism B: Incorporating Substitution



Formalism B: Incorporating Substitution



Achieving the power of Frege + extension (possibly optimal proof system) by incorporating substitution, guided by flow geometry:



Work is in progress.

Conclusion

- Deep inference and atomic flows reach geometry by exploiting locality;

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This talk is available at `http://cs.bath.ac.uk/ag/t/TMENPS.pdf`

Deep inference web site: `http://alessio.guglielmi.name/res/cos/`

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