

TWO DESIGN QUESTIONS

Alessio Guglielmi

Bath

Proof, Computation, Complexity – Djursholm, 15–19 July 2019

Talk available from my home page and at <http://cs.bath.ac.uk/ag/t/TDQ.pdf>

All about deep inference at <http://alessio.guglielmi.name/res/cos>

17/7/19

MOTIVATION: SPEED-UPS FOR ANALYTIC PROOFS

Corollary of [Aguilera-Baaz, JSL, 2019]

Deep inference has a non-elementary speed-up over cut-free Gentzen proofs of the predicate calculus.

MOTIVATION: SPEED-UPS FOR ANALYTIC PROOFS

Corollary of [Aguilera-Baaz, JSL, 2019]

Deep inference has a non-elementary speed-up over cut-free Gentzen proofs of the predicate calculus.

Gentzen bureaucracy
demands contractions.



$\vdash P_z,$	$\overline{P_z}$
$\vdash P_z,$	$\overline{P_z}, P_y$
$\vdash \overline{P_x}, P_z,$	$\overline{P_z}, P_y$
$\vdash \overline{P_x}, P_z,$	$\overline{P_z} \vee P_y$
$\vdash \overline{P_x} \vee P_z,$	$\overline{P_z} \vee P_y$
$\vdash \overline{P_x} \vee P_z,$	$\forall y. (\overline{P_z} \vee P_y)$
$\vdash \overline{P_x} \vee P_z,$	$\exists x. \forall y. (\overline{P_x} \vee P_y)$
$\vdash \forall y. (\overline{P_x} \vee P_y), \exists x. \forall y. (\overline{P_x} \vee P_y)$	
$\vdash \exists x. \forall y. (\overline{P_x} \vee P_y), \exists x. \forall y. (\overline{P_x} \vee P_y)$	
	$\vdash \exists x. \forall y. (\overline{P_x} \vee P_y)$

MOTIVATION: SPEED-UPS FOR ANALYTIC PROOFS

Corollary of [Aguilera-Baaz, JSL, 2019]

Deep inference has a non-elementary speed-up over cut-free Gentzen proofs of the predicate calculus.

$$\frac{t}{\exists x. \overline{P}_x \vee \forall y. P_y}$$

$$\frac{\overline{P}_x \vee \forall y. P_y}{\exists x. \forall y. (\overline{P}_x \vee P_y)}$$

Deep inference does not.

Gentzen bureaucracy demands contractions.

$$\frac{\vdash P_z, \quad \overline{P}_z}{\vdash P_z, \quad \overline{P}_z, P_y}$$

$$\frac{}{\vdash \overline{P}_x, P_z, \quad \overline{P}_z, P_y}$$

$$\frac{}{\vdash \overline{P}_x, P_z, \quad \overline{P}_z \vee P_y}$$

$$\frac{}{\vdash \overline{P}_x \vee P_z, \quad \overline{P}_z \vee P_y}$$

$$\frac{}{\vdash \overline{P}_x \vee P_z, \quad \forall y. (\overline{P}_z \vee P_y)}$$

$$\frac{}{\vdash \overline{P}_x \vee P_z, \quad \exists x. \forall y. (\overline{P}_x \vee P_y)}$$

$$\frac{}{\vdash \forall y. (\overline{P}_x \vee P_y), \quad \exists x. \forall y. (\overline{P}_x \vee P_y)}$$

$$\frac{}{\vdash \exists x. \forall y. (\overline{P}_x \vee P_y), \quad \exists x. \forall y. (\overline{P}_x \vee P_y)}$$

$$\vdash \exists x. \forall y. (\overline{P}_x \vee P_y)$$

MOTIVATION: SPEED-UPS FOR ANALYTIC PROOFS

Corollary of [Aguilera-Baaz, JSL, 2019]

Deep inference has a non-elementary speed-up over cut-free Gentzen proofs of the predicate calculus.

$$\frac{t}{\exists x. \overline{P}x \vee \forall y. Py}$$

$$\frac{\overline{P}x \vee \forall y. Py}{\exists x. \forall y. (\overline{P}x \vee Py)}$$

Success is due to
quantifier shifts.



Is there a good reason to do
without quantifier shifts?

$$\frac{\vdash Pz, \quad \overline{P}z}{\vdash Pz, \quad \overline{P}z, Py}$$

$$\frac{\vdash \overline{Px}, Pz, \quad \overline{P}z}{\vdash \overline{Px}, Pz, \quad \overline{P}z, Py}$$

$$\frac{\vdash \overline{Px}, Pz, \quad \overline{P}z \vee Py}{\vdash \overline{Px} \vee Pz, \quad \overline{P}z \vee Py}$$

$$\frac{\vdash \overline{Px} \vee Pz, \quad \overline{P}z \vee Py}{\vdash \overline{Px} \vee Pz, \quad \forall y. (\overline{P}z \vee Py)}$$

$$\frac{\vdash \overline{Px} \vee Pz, \quad \exists x. \forall y. (\overline{Px} \vee Py)}{\vdash \forall y. (\overline{Px} \vee Py), \quad \exists x. \forall y. (\overline{Px} \vee Py)}$$

$$\frac{\vdash \exists x. \forall y. (\overline{Px} \vee Py), \quad \exists x. \forall y. (\overline{Px} \vee Py)}{\vdash \exists x. \forall y. (\overline{Px} \vee Py)}$$

MOTIVATION: SPEED-UPS FOR ANALYTIC PROOFS

Corollary of [Aguilera-Baaz, JSL, 2019]

Deep inference has a non-elementary speed-up over cut-free Gentzen proofs of the predicate calculus.

$$\frac{t}{\exists x. \overline{P}x \vee \forall y. Py}$$
$$\frac{}{\exists x. \boxed{\overline{P}x \vee \forall y. Py}}$$
$$\frac{}{\exists x. \boxed{\forall y. (\overline{P}x \vee Py)}}$$

Success is due to quantifier shifts.



Is there a good reason to do without quantifier shifts?

Theorems

The speed-up for analytic propositional proofs is exponential.

[Bruscoli-Cuglielmi, ACM ToCL, 2009]

MOTIVATION: SPEED-UPS FOR ANALYTIC PROOFS

Corollary of [Aguilera-Baaz, JSL, 2019]

Deep inference has a non-elementary speed-up over cut-free Gentzen proofs of the predicate calculus.

$$\frac{t}{\exists x. \overline{P}x \vee \forall y. Py}$$
$$\frac{}{\exists x. \boxed{\overline{P}x \vee \forall y. Py}}$$
$$\frac{}{\exists x. \forall y. (\overline{P}x \vee Py)}$$

Success is due to quantifier shifts.



Is there a good reason to do without quantifier shifts?

Theorems

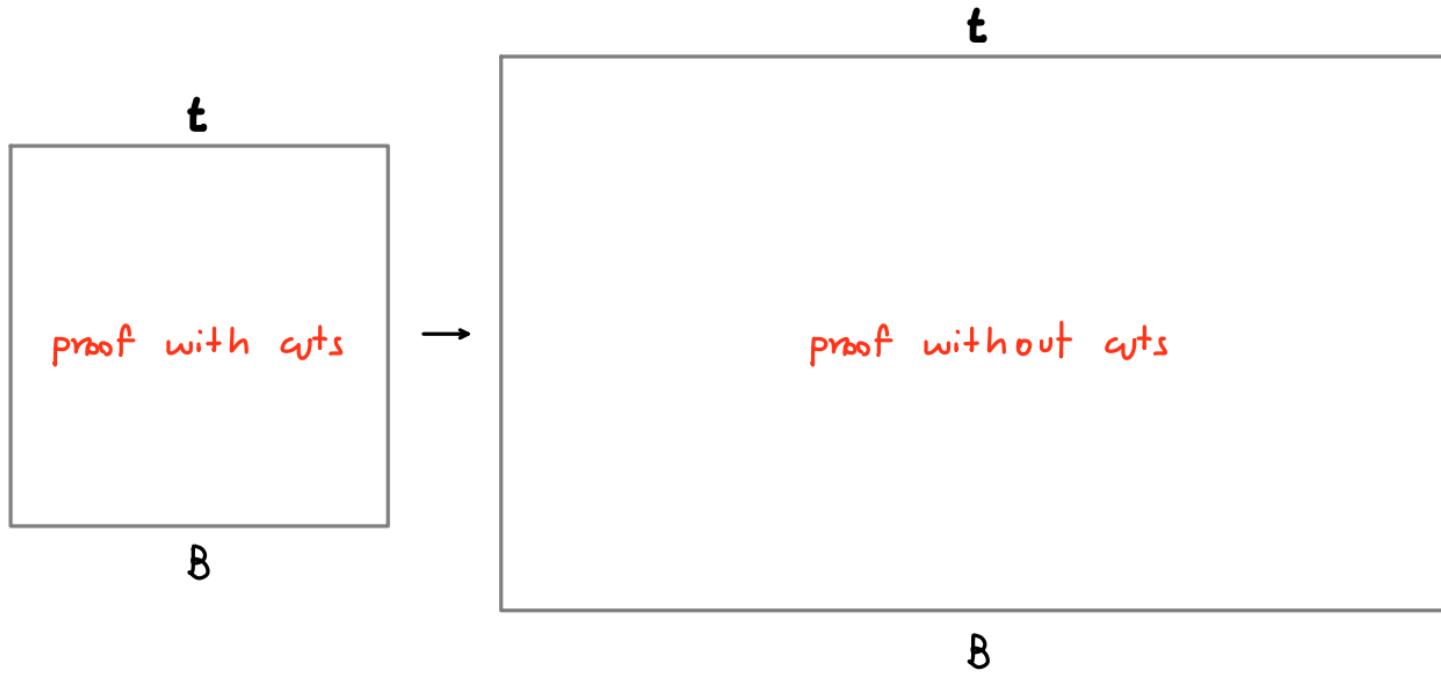
The speed-up for analytic propositional proofs is exponential.

[Bruscoli-Caviglioni, ACM ToCL, 2009]

Cut-elimination for propositional classical logic is quasipolynomial.
[Jeřábek, JLC, 2009]

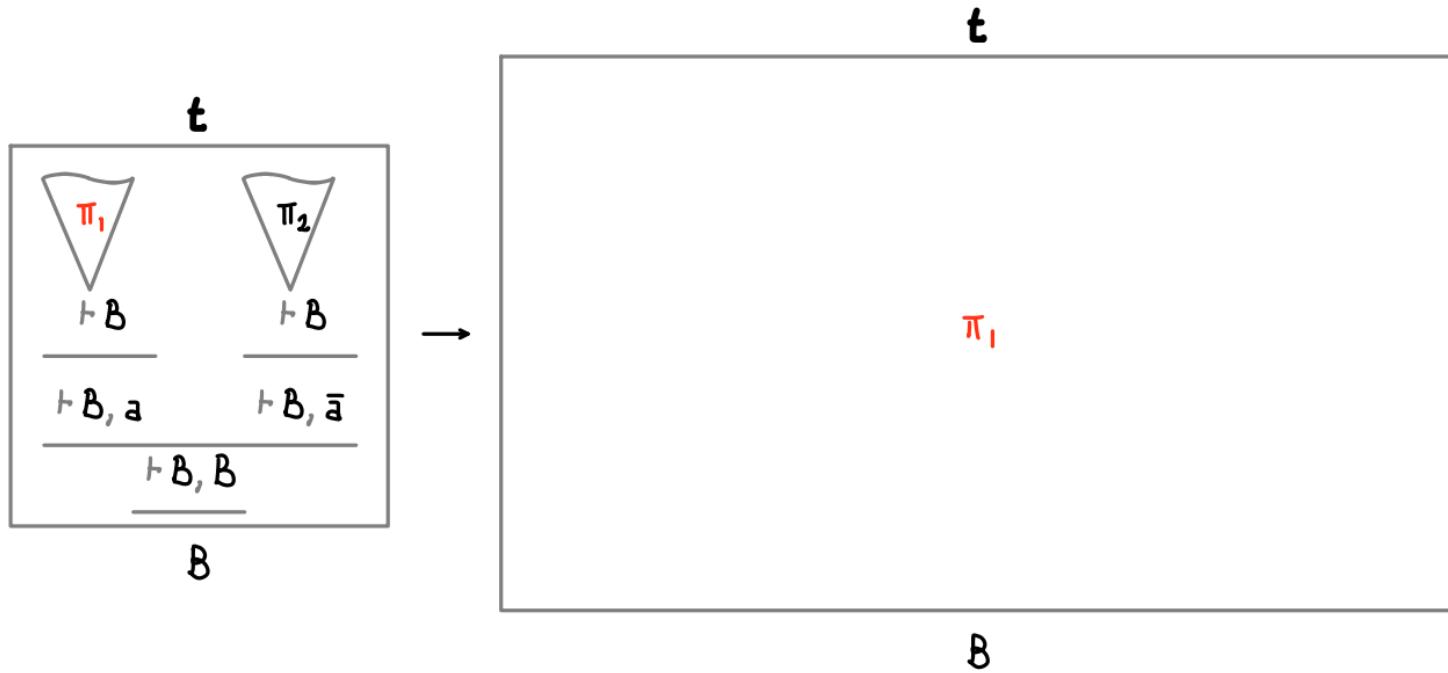
MOTIVATION: CANONICAL FORMS FOR PROOF SEMANTICS

Cut elimination for propositional classical logic:



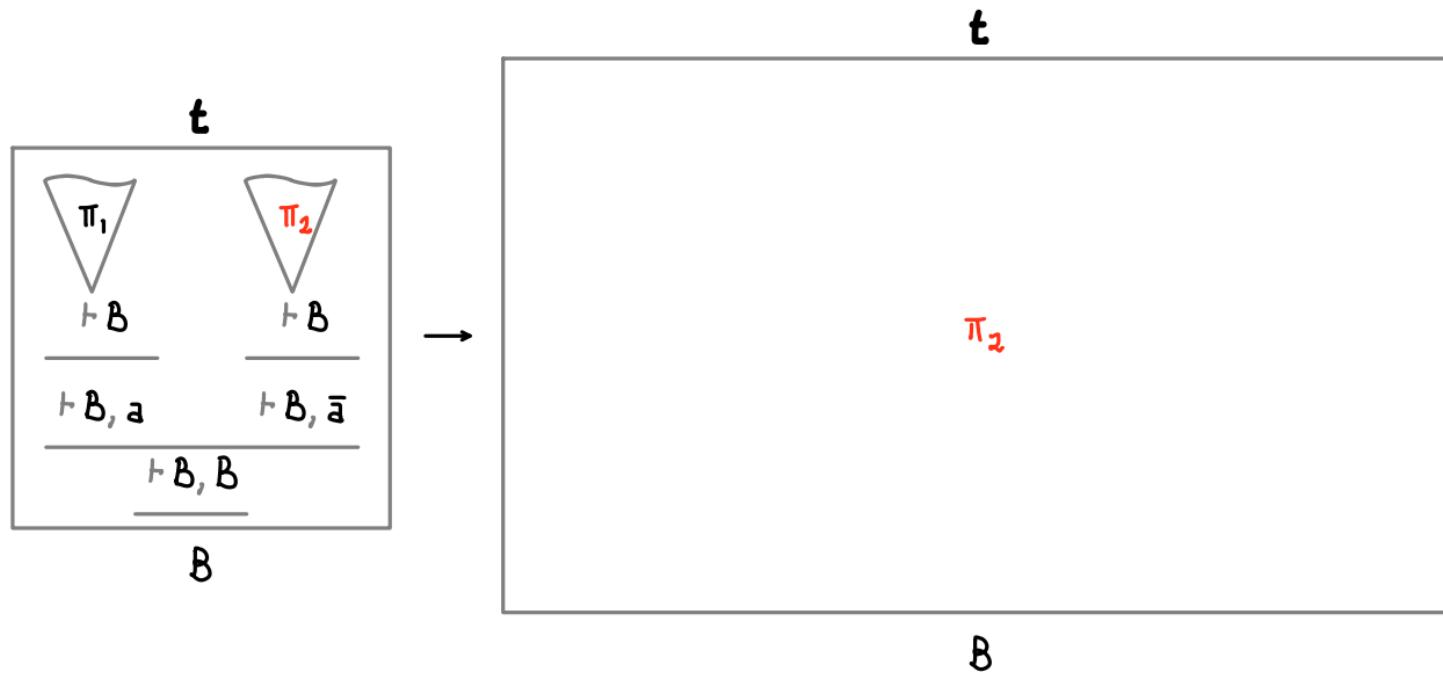
MOTIVATION: CANONICAL FORMS FOR PROOF SEMANTICS

Cut elimination for propositional classical logic, in Gentzen:



MOTIVATION: CANONICAL FORMS FOR PROOF SEMANTICS

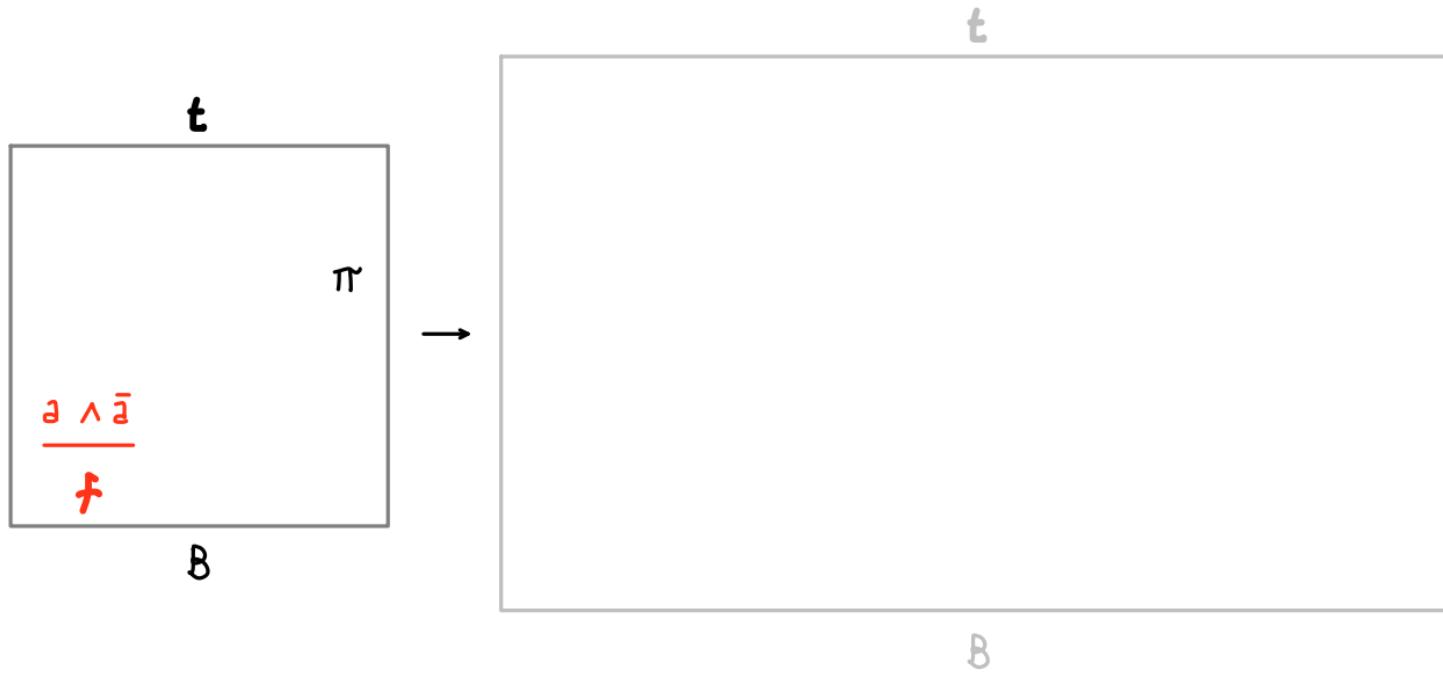
Cut elimination for propositional classical logic, in Gentzen:



No canonical form, therefore bad semantics.

MOTIVATION: CANONICAL FORMS FOR PROOF SEMANTICS

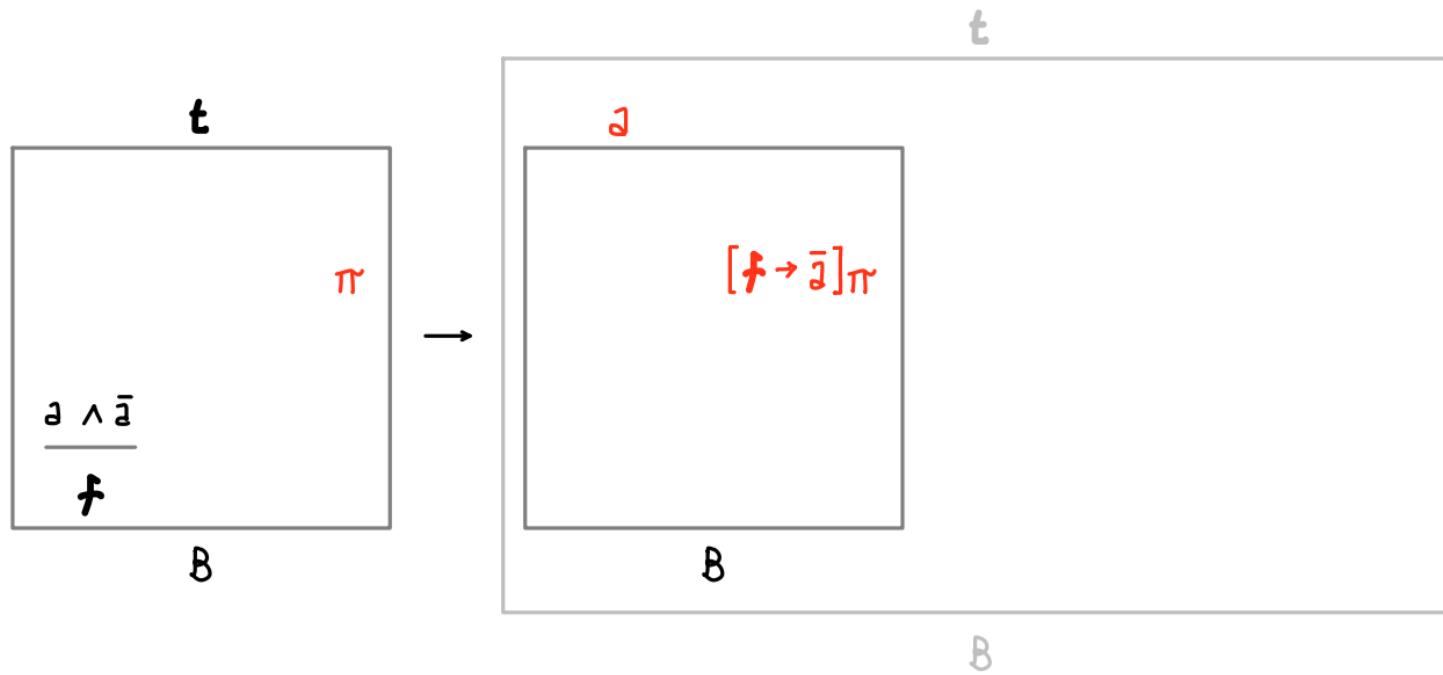
Cut elimination for propositional classical logic, in deep inference^{*}:



* Proofs can be composed by any connective.

MOTIVATION: CANONICAL FORMS FOR PROOF SEMANTICS

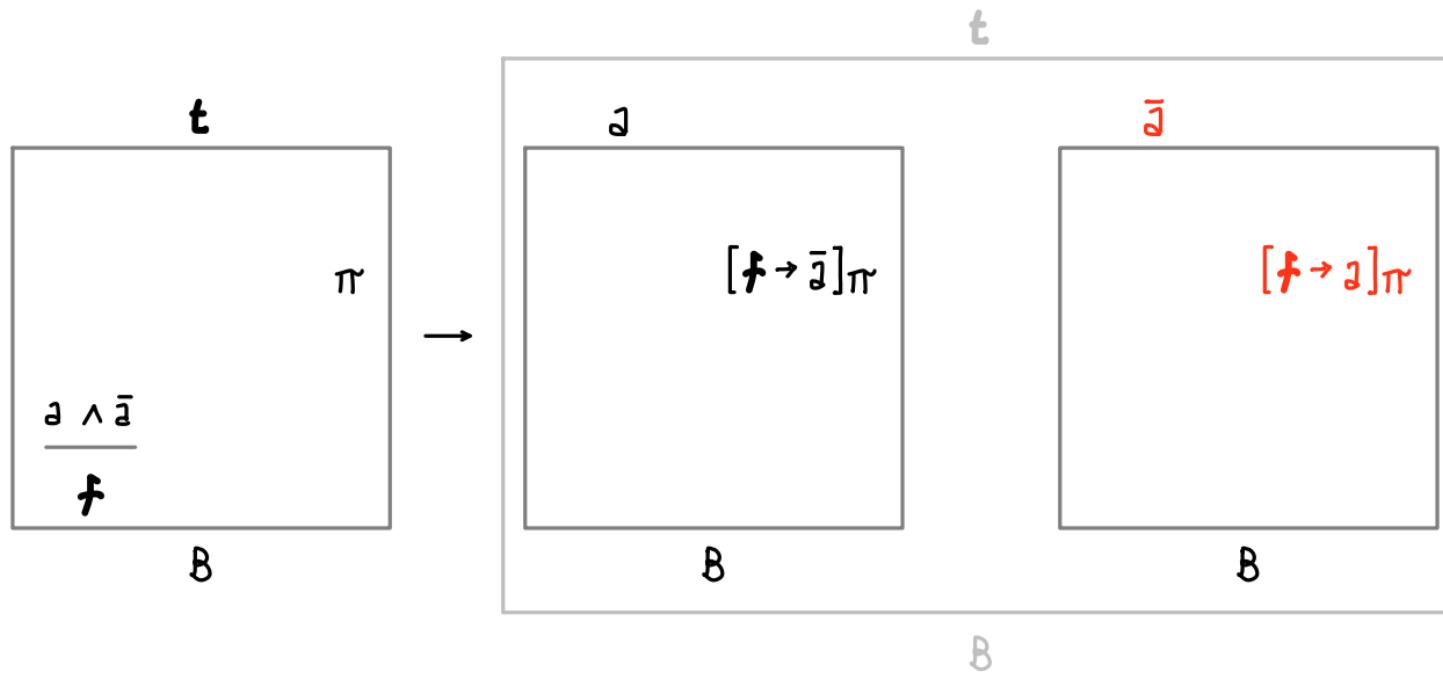
Cut elimination for propositional classical logic, in deep inference:



No a s in cuts.

MOTIVATION: CANONICAL FORMS FOR PROOF SEMANTICS

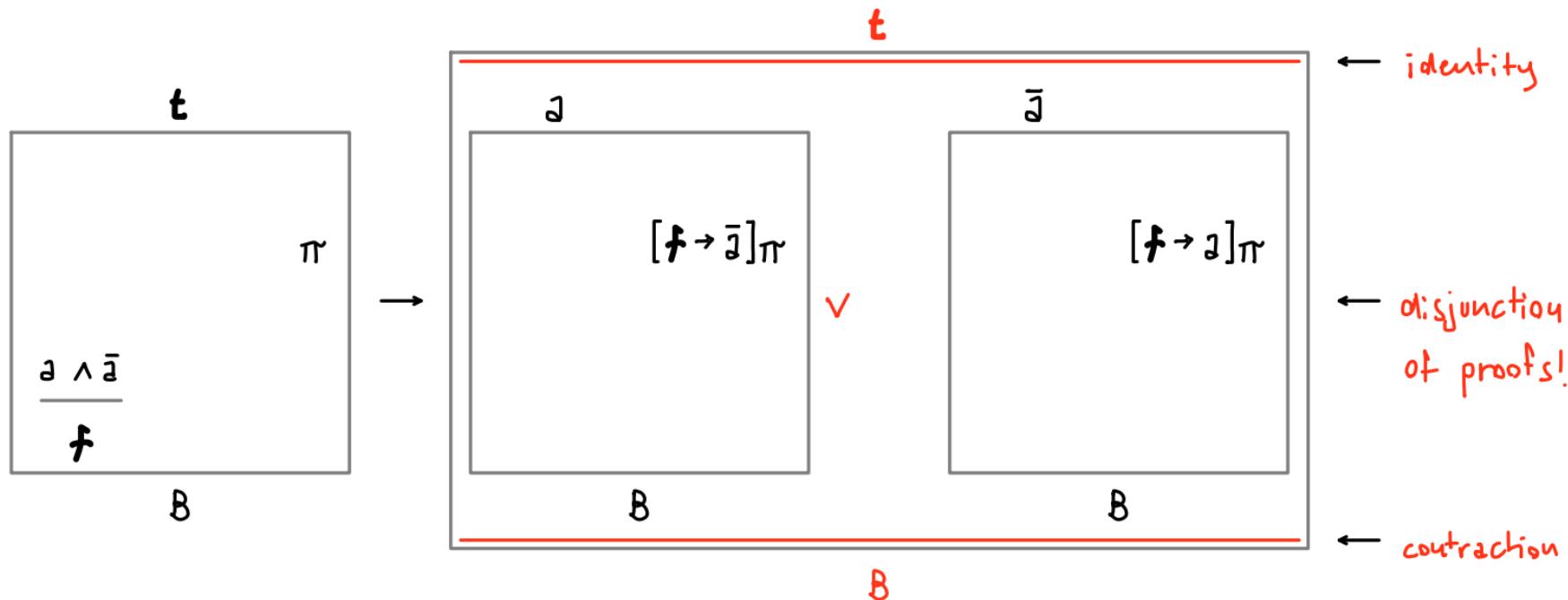
Cut elimination for propositional classical logic, in deep inference:



No a s in cuts in both proofs.

MOTIVATION: CANONICAL FORMS FOR PROOF SEMANTICS

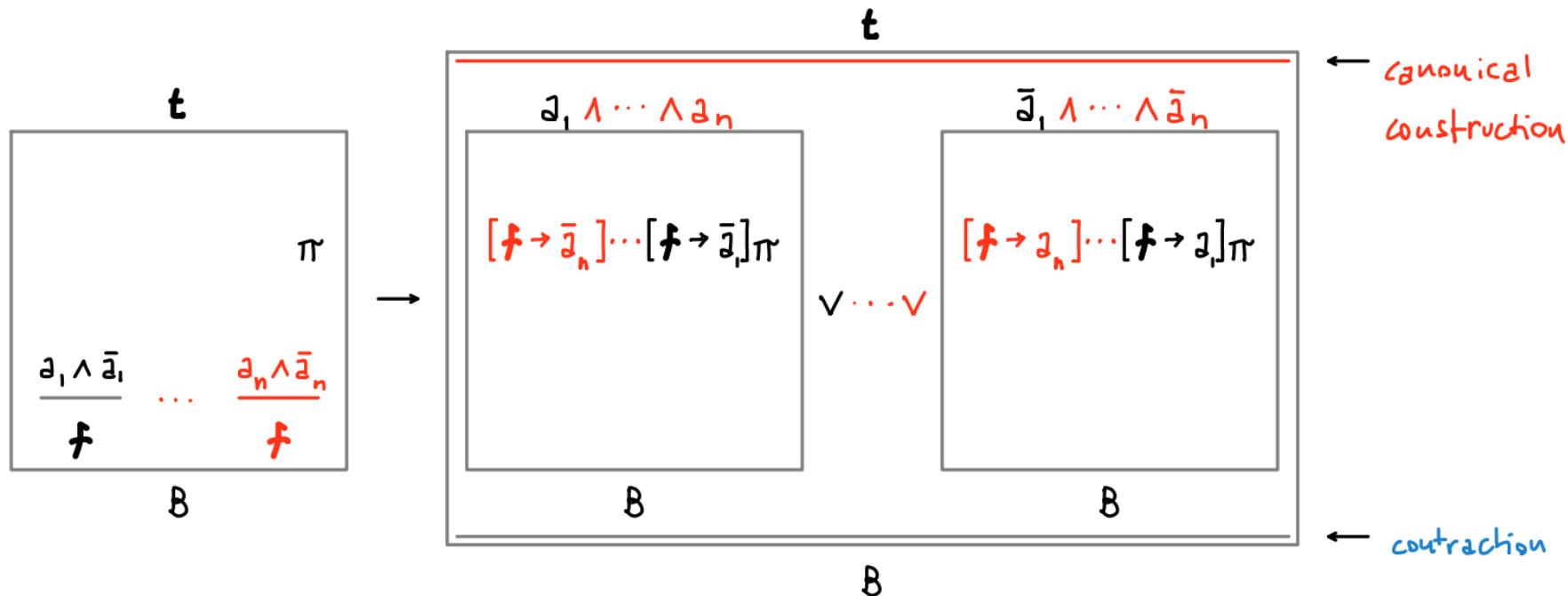
Cut elimination for propositional classical logic, in deep inference:



No \exists s in cuts.

MOTIVATION: CANONICAL FORMS FOR PROOF SEMANTICS

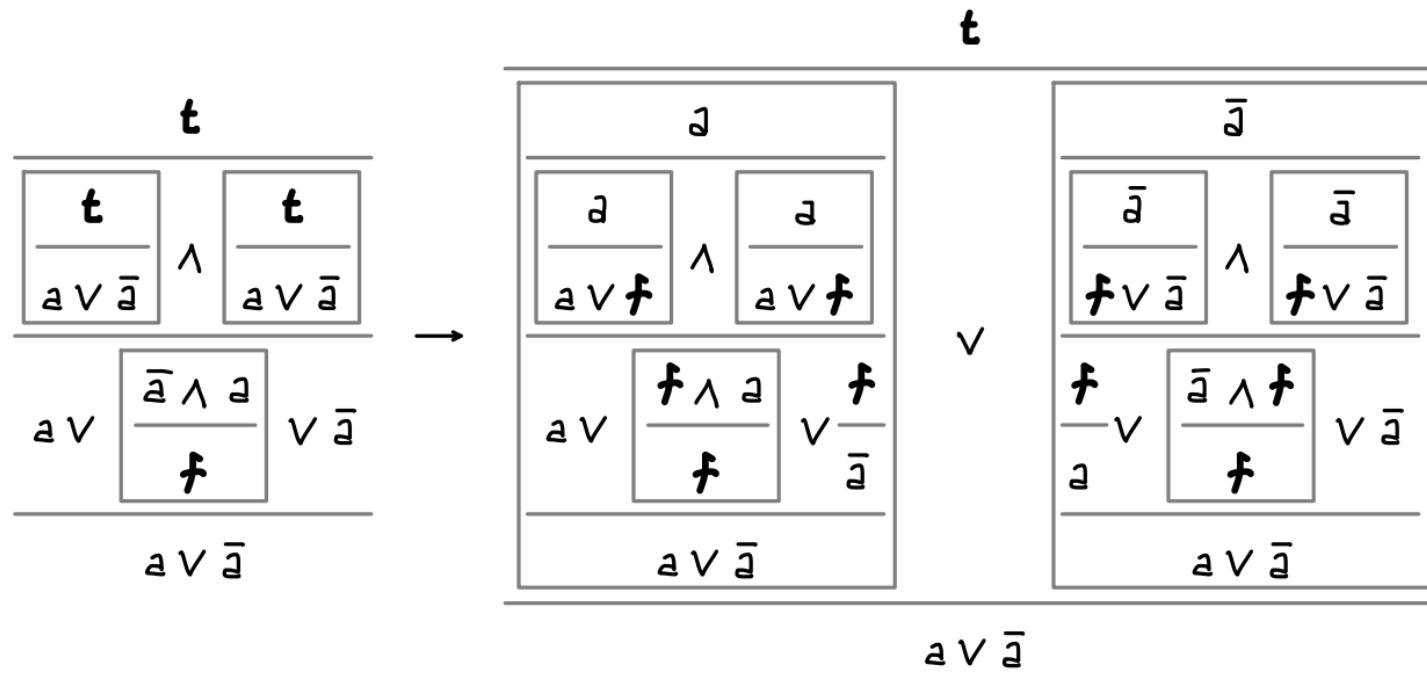
Cut elimination for propositional classical logic, in deep inference:



No cuts. Canonical modulo associativity and commutativity.

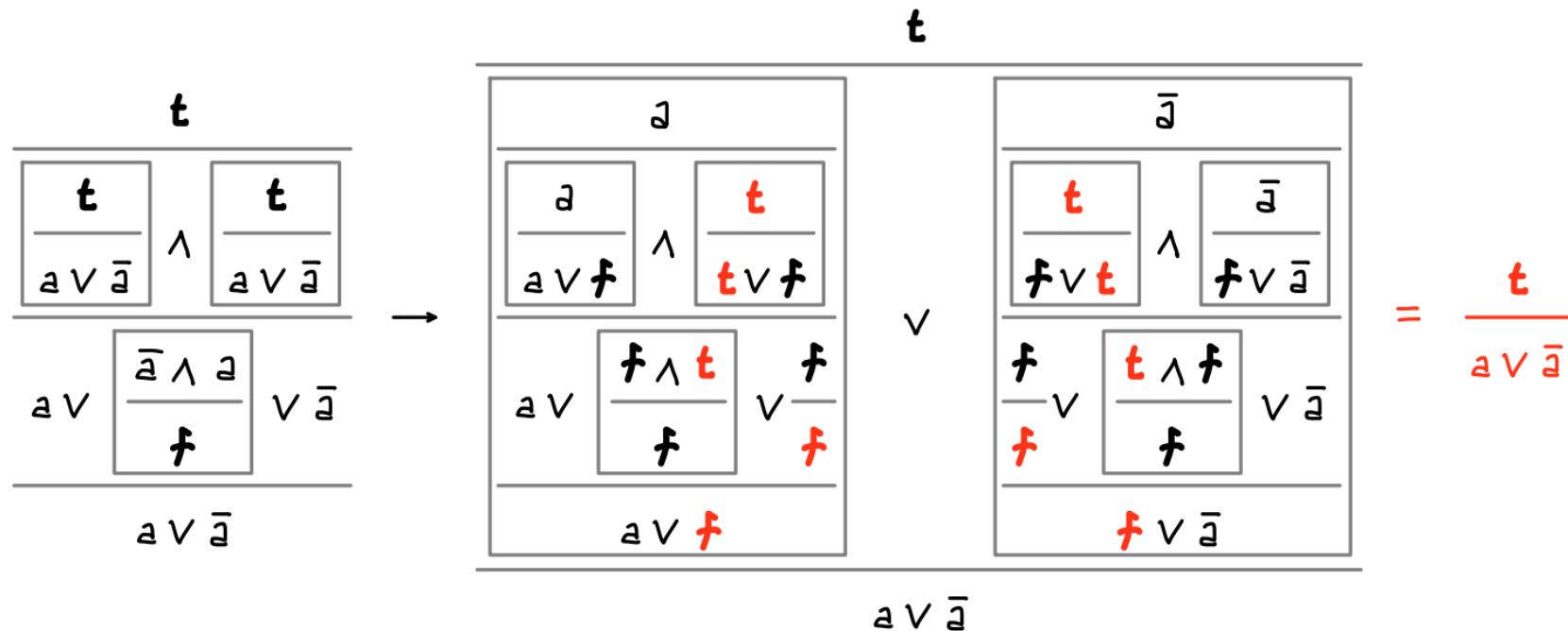
MOTIVATION: CANONICAL FORMS FOR PROOF SEMANTICS

Cut elimination for propositional classical logic, in deep inference. Example:



MOTIVATION: CANONICAL FORMS FOR PROOF SEMANTICS

Cut elimination for propositional classical logic, in deep inference. Example:



Propagate (co)weakenings. Still canonical.

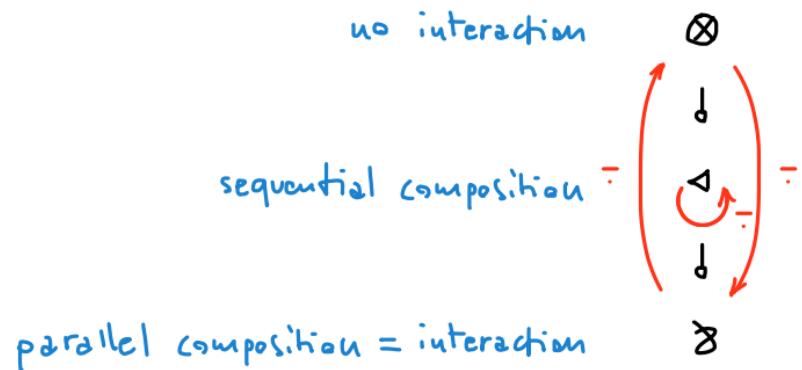
MOTIVATION: EXPRESS LOGICS THAT GENTZEN'S THEORY CANNOT EXPRESS

Example: BV = MLL + \bowtie , where \bowtie is self-dual non-commutative.

no interaction	\otimes
	\dashv
sequential composition	\bowtie
	\dashv
parallel composition = interaction	\wp

MOTIVATION: EXPRESS LOGICS THAT GENTZEN'S THEORY CANNOT EXPRESS

Example: BV = MLL + \bowtie , where \bowtie is self-dual non-commutative.



MOTIVATION: EXPRESS LOGICS THAT GENTZEN'S THEORY CANNOT EXPRESS

Example: BV = MLL + \bowtie , where \bowtie is self-dual non-commutative.

Question: Is there a proof system that generates all
the intermediate formulas between $A \otimes B \multimap \dots \multimap A \wp B$?

$$\vdash \left(\begin{array}{c} \otimes \\ \downarrow \\ \bowtie \\ \text{---} \\ \downarrow \\ \wp \end{array} \right) \vdash$$

MOTIVATION: EXPRESS LOGICS THAT GENTZEN'S THEORY CANNOT EXPRESS

Example: $BV = MLL + \triangleleft$, where \triangleleft is self-dual non-commutative.

Question: Is there a proof system that generates all the intermediate formulas between $A \otimes B \multimap \dots \multimap A \wp B$?

Yes:

$$\frac{(A \triangleleft B) \otimes (C \triangleleft D)}{(A \otimes C) \triangleleft (B \otimes D)}$$

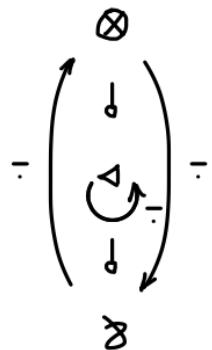
$$\frac{(A \wp B) \otimes (C \wp D)}{(A \otimes C) \wp (B \wp D)}$$

$$\frac{(A \wp B) \triangleleft (C \wp D)}{(A \triangleleft C) \wp (B \triangleleft D)}$$

+ commutativity (same shape)
and mirror images

$$\frac{(A \wp B) \triangleleft (C \wp D)}{(A \triangleleft C) \wp (B \triangleleft D)}$$

$$\frac{(A \triangleleft C) \wp (B \triangleleft D)}{(A \triangleleft C) \triangleleft (B \wp D)}$$



This follows from a combinatorial argument via relation webs [Guglielmi, ACTCL, 2007]. It can be generalised to linear logics with an arbitrary number of relations.

MOTIVATION: EXPRESS LOGICS THAT GENTZEN'S THEORY CANNOT EXPRESS

Example: $BV = MLL + \triangleleft$, where \triangleleft is self-dual non-commutative.

Question: Is there a proof system that generates all the intermediate formulas between $A \otimes B \multimap \dots \multimap A \wp B$?

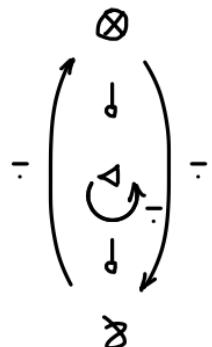
Yes:

$$\frac{(A \triangleleft B) \otimes (C \triangleleft D)}{(A \otimes C) \triangleleft (B \otimes D)}$$

$$\frac{(A \wp B) \otimes (C \wp D)}{(A \otimes C) \wp (B \wp D)}$$

$$\frac{(A \wp B) \triangleleft (C \wp D)}{(A \triangleleft C) \wp (B \triangleleft D)}$$

+ commutativity (same shape)
and mirror images



Theorems There is an analytic system for BV in deep inference [Guglielmi, ACM ToCL 2007] but not in Gentzen [Tiu, LMCS, 2006]. Applications in process algebras (many papers, see web).

MOTIVATION: EXPRESS LOGICS THAT GENTZEN'S THEORY CANNOT EXPRESS

Example: $BV = MLL + \triangleleft$, where \triangleleft is self-dual non-commutative.

Question: Is there a proof system that generates all the intermediate formulas between $A \otimes B \multimap \dots \multimap A \wp B$?

Yes:

$$\frac{\otimes\triangleleft}{(A \triangleleft B) \otimes (C \triangleleft D)} \quad \text{'saturates up'}$$

$$(A \otimes C) \triangleleft (B \otimes D)$$

$$\frac{\alpha\beta}{(A \beta B) \alpha (C \beta D)} \quad \frac{(A \wp B) \otimes (C \wp D)}{(A \otimes C) \wp (B \otimes D)}$$

This shape generates the rules.

$$\frac{(A \wp B) \triangleleft (C \wp D)}{(A \triangleleft C) \wp (B \triangleleft D)}$$

$$\triangleleft = \begin{pmatrix} \otimes & \downarrow \\ \downarrow & \circlearrowleft \\ \circlearrowright & \downarrow \end{pmatrix} = \begin{pmatrix} \wp & \uparrow \\ \uparrow & \circlearrowright \\ \circlearrowleft & \uparrow \end{pmatrix} = \wp$$

Theorems There is an analytic system for BV in deep inference [Guglielmi, ACM ToCL 2007] but not in Gentzen [Tiu, LMCS, 2006]. Applications in process algebras (many papers, see web).

MOTIVATION: EXPRESS LOGICS THAT GENTZEN'S THEORY CANNOT EXPRESS

Example: $BV = MLL + \triangleleft$, where \triangleleft is self-dual non-commutative.

Question: Is there a proof system that generates all the intermediate formulas between $A \otimes B \rightarrow \dots \rightarrow A \wp B$?

Yes:

$$\otimes \triangleleft \frac{(A \triangleleft B) \otimes (C \triangleleft D)}{(A \otimes C) \triangleleft (B \otimes D)}$$

'saturates up'

$$\begin{aligned} \triangleleft &= \triangleleft = \triangleleft & - \left(\begin{array}{c} \otimes \\ \downarrow \\ \triangleleft \\ \curvearrowright \\ \downarrow \end{array} \right) = \\ \triangleleft &= \triangleleft = \triangleleft & \wp = \wp = \wp \\ & & \wp \end{aligned}$$

$$\beta \triangleleft \frac{(A \beta B) \triangleleft (C \beta D)}{(A \triangleleft C) \beta (B \triangleleft D)}$$

$$\wp \otimes \frac{(A \wp B) \otimes (C \wp D)}{(A \otimes C) \wp (B \otimes D)}$$

'saturates down'

This shape generates the rules.

$$\wp \triangleleft \frac{(A \wp B) \triangleleft (C \wp D)}{(A \triangleleft C) \wp (B \triangleleft D)}$$

'saturates down'

Theorems There is an analytic system for BV in deep inference [Guglielmi, ACM ToCL 2007] but not in Gentzen [Tiu, LMCS, 2006]. Applications in process algebras (many papers, see web).

MOTIVATION: EXPRESS LOGICS THAT GENTZEN'S THEORY CANNOT EXPRESS

Example: $BV = MLL + \triangleleft$, where \triangleleft is self-dual non-commutative.

Question: Is there a proof system that generates all the intermediate formulas between $A \otimes B \rightarrow \dots \rightarrow A \wp B$?

Yes:

$$\otimes \triangleleft \frac{(A \triangleleft B) \otimes (C \triangleleft D)}{(A \otimes C) \triangleleft (B \otimes D)}$$

$$\wp \otimes \frac{(A \wp B) \otimes (C \wp D)}{(A \otimes C) \wp (B \otimes D)} \approx \otimes \wp$$

$$\wp \triangleleft \frac{(A \wp B) \triangleleft (C \wp D)}{(A \triangleleft C) \wp (B \triangleleft D)}$$

This shape generates the rules.

$$\begin{aligned} \hat{\otimes} &= \hat{\tau} = \otimes \\ \hat{\triangleleft} &= \hat{\triangleleft} = \triangleleft \\ \hat{\wp} &= \hat{\tau} = \wp \end{aligned}$$

$\vdash \left(\begin{array}{c} \otimes \\ \downarrow \\ \triangleleft \\ \curvearrowright \\ \downarrow \end{array} \right) = \wp$

Theorems There is an analytic system for BV in deep inference [Guglielmi, ACM ToCL 2007] but not in Gentzen [Tiu, LMCS, 2006]. Applications in process algebras (many papers, see web).

MOTIVATION: EXPRESS LOGICS THAT GENTZEN'S THEORY CANNOT EXPRESS

Example: $BV = MLL + \triangleleft$, where \triangleleft is self-dual non-commutative.

Question: Is there a proof system that generates all the intermediate formulas between $A \otimes B \rightarrow \dots \rightarrow A \wp B$?

Yes:

$$\otimes \triangleleft \frac{(A \triangleleft B) \otimes (C \triangleleft D)}{(A \otimes C) \triangleleft (B \otimes D)}$$

$$\alpha \beta \frac{\wedge}{\otimes} \frac{(A \beta B) \otimes (C \beta D)}{(A \otimes C) \beta (B \otimes D)} \quad \wp \otimes \frac{\vee}{\wp} \frac{(A \wp B) \otimes (C \wp D)}{(A \otimes C) \wp (B \otimes D)} \approx \otimes \wp$$

This shape generates the rules.

$$\wp \triangleleft \frac{\vee}{\wp} \frac{(A \wp B) \triangleleft (C \wp D)}{(A \triangleleft C) \wp (B \triangleleft D)}$$

Saturation can happen on the left, too, of course.

$$\begin{aligned} \hat{\otimes} &= \hat{\tau} = \otimes & \otimes \\ \hat{\triangleleft} &= \hat{\triangleleft} = \triangleleft & - \begin{pmatrix} \downarrow & & \\ & \triangleleft & \\ & \uparrow & - \\ \downarrow & & \end{pmatrix} - \\ \hat{\otimes} &= \hat{\tau} = \tau & \wp \\ \hat{\triangleleft} &= \hat{\triangleleft} = \triangleleft & \wp \end{aligned}$$

THE GENERATING SHAPE

binary-binary

$$\alpha \hat{\beta} \frac{(A \beta B) \times (C \hat{\beta} D)}{(A \times C) \beta (B \times D)}$$

THE GENERATING SHAPE

examples

$$\wedge\wedge \frac{\wedge(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

commutativity / associativity

binary-binary

$$\wedge\beta \frac{\wedge(A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}$$

$$\wedge\vee \frac{\wedge(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

switch (classical logic)

$$\otimes\tau \frac{\wedge(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

switch (linear logic)

$$\vee\wedge \frac{\wedge(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

medial

THE GENERATING SHAPE

examples

$$\wedge\wedge \frac{\wedge(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

binary-binary

$$\alpha\beta \frac{\wedge(A \beta B) \wedge (C \beta D)}{(A \alpha C) \beta (B \alpha D)}$$

$$\wedge\vee \frac{\wedge(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\otimes\otimes \frac{\wedge(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

$$\vee\wedge \frac{\wedge(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

binary-unary

$$\alpha\beta \frac{\wedge(\beta A) \alpha (\hat{\wedge} B)}{\beta(A \alpha B)}$$

THE GENERATING SHAPE

binary-binary

$$\alpha \beta \frac{\wedge (A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}$$

examples

$$\wedge \wedge \frac{\wedge (A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

$$\wedge \vee \frac{\wedge (A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\otimes ? \frac{\wedge (A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

$$\vee \wedge \frac{\wedge (A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

examples

$$\wedge \widehat{\diamond} \frac{\diamond A \wedge \square B}{\diamond (A \wedge B)}$$

co-k

binary-unary

$$\alpha \beta \widehat{\wedge} \frac{(\beta A) \alpha (\widehat{\wedge} B)}{\beta (A \alpha B)}$$

$$\otimes ? \frac{? A \otimes ! B}{?(A \otimes B)}$$

co-promotion

$$\vee \widehat{\exists}_\kappa \frac{\exists_\kappa A \vee \forall_\kappa B}{\exists_\kappa (A \vee B)}$$

quantifier shift

THE GENERATING SHAPE

binary-binary

$$\alpha \hat{\beta} \frac{(A \beta B) \alpha (C \hat{\beta} D)}{(A \alpha C) \beta (B \alpha D)}$$

examples

$$\wedge\wedge \frac{\wedge (A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

$$\wedge\vee \frac{\wedge (A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\otimes\otimes \frac{\wedge (A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

$$\vee\wedge \frac{\wedge (A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

examples

$$\wedge\hat{\wedge} \frac{\diamond A \wedge \square B}{\diamond(A \wedge B)}$$

$$\otimes?\hat{\wedge} \frac{?A \otimes !B}{?(A \otimes B)}$$

$$\vee\hat{\exists}_\kappa \frac{\exists_\kappa A \vee \forall_\kappa B}{\exists_\kappa (A \vee B)}$$

Unary-unary

$$\alpha \hat{\beta} \frac{\alpha \hat{\beta} A}{\beta \alpha A}$$

example

$$\exists_\kappa \hat{\forall}_y \frac{\exists_\kappa \forall_y A}{\forall_y \exists_\kappa A}$$

THE GENERATING SHAPE – SURPRISE! NON-LINEAR RULES ARE GENERATED

$$\alpha \beta \frac{\wedge \frac{(A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}}{\wedge a \frac{(\text{fat}) \wedge (\text{tat})}{(\text{f} \wedge \text{t}) \alpha (\text{t} \wedge \text{f})}}$$

$$\vee a \frac{\wedge \frac{(\text{fat}) \vee (\text{fat})}{(\text{f} \vee \text{f}) \alpha (\text{t} \vee \text{t})}}{\vee a \frac{(\text{fat}) \vee (\text{fat})}{(\text{f} \vee \text{f}) \alpha (\text{t} \vee \text{t})}}$$

THE GENERATING SHAPE – SURPRISE! NON-LINEAR RULES ARE GENERATED

$$\alpha \beta \frac{\wedge (A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}$$

$$\lambda a \frac{\wedge (f a t) \wedge (t a f)}{(f \wedge t) a (t \wedge f)} = \frac{a \wedge \bar{a}}{f}$$

if we take $\begin{cases} f a f = f \\ f a t = a \\ t a f = \bar{a} \\ t a t = t \end{cases}$

i.e., atoms are superpositions of truth values.

$$\vee a \frac{\wedge (f a t) \vee (f a t)}{(f \vee f) a (t \vee t)}$$

THE GENERATING SHAPE – SURPRISE! NON-LINEAR RULES ARE GENERATED

$$\alpha \beta \frac{\wedge (A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}$$

$$\Lambda a \frac{\wedge (f a t) \wedge (t a f)}{(f \wedge t) a (t \wedge f)} = \frac{a \wedge \bar{a}}{f}$$

if we take $\begin{cases} f a f = f \\ f a t = a \\ t a f = \bar{a} \\ t a t = t \end{cases}$

i.e., atoms are superpositions of truth values.

$$\vee a \frac{\wedge (f a t) \vee (f a t)}{(f \vee f) a (t \vee t)} = \frac{a \vee a}{a}$$

THE GENERATING SHAPE – SURPRISE! NON-LINEAR RULES ARE GENERATED

Except for unit equations, every rule for all mainstream logics gets generated.

This yields a uniform and general normalisation theory

[After Tubella-Guglielmi, ACM ToCL, 2018 + papers in preparation].

$$\alpha \beta \frac{\wedge \hat{(A \beta B)} \alpha (C \hat{\beta} D)}{(A \alpha C) \beta (B \alpha D)}$$

$$\lambda a \frac{\wedge (f a t) \wedge (\bar{t} a \bar{f})}{(f \wedge t) a (t \wedge \bar{f})} = \frac{a \wedge \bar{a}}{f}$$

if we take $\begin{cases} f a f = f \\ f a t = a \\ t a f = \bar{a} \\ t a t = t \end{cases}$

i.e., atoms are superpositions of truth values.

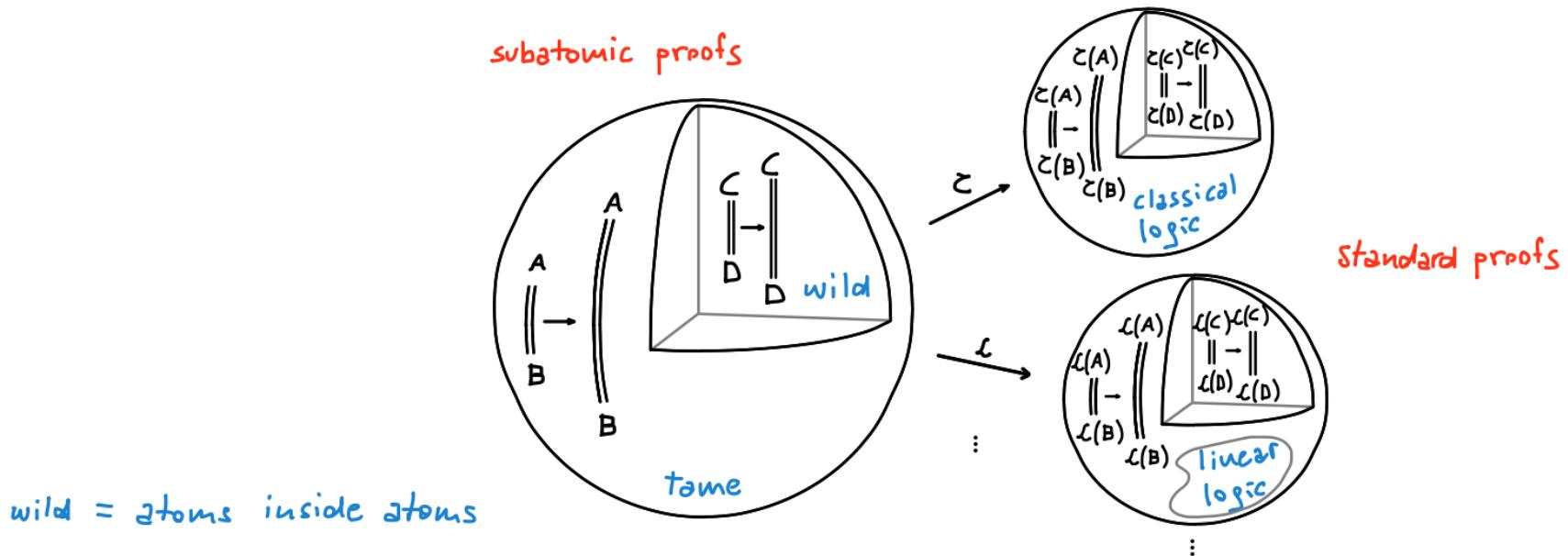
$$\vee a \frac{\wedge (f a t) \vee (f a \bar{t})}{(f \vee f) a (t \vee \bar{t})} = \frac{a \vee \bar{a}}{a}$$

THE GENERATING SHAPE – SURPRISE! NON-LINEAR RULES ARE GENERATED

Except for unit equations, every rule for all mainstream logics gets generated.

This yields a uniform and general normalisation theory

[After Tubella-Guglielmi, ACM ToCL, 2018 + papers in preparation].



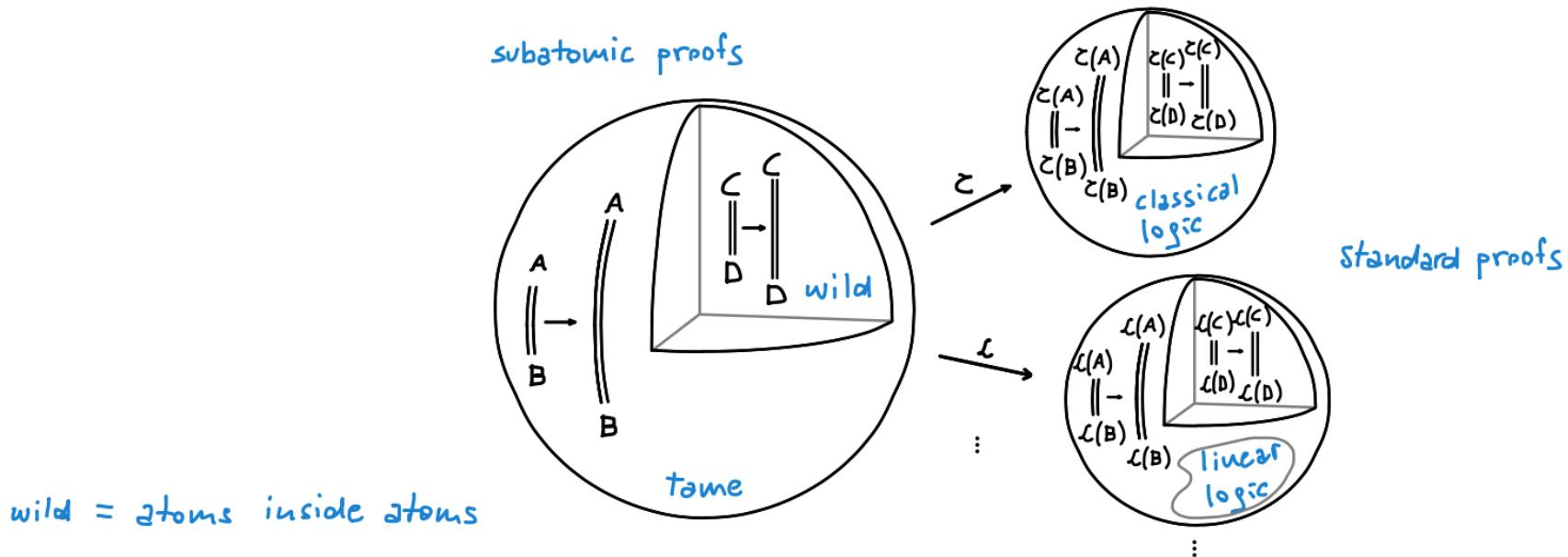
THE GENERATING SHAPE – SURPRISE! NON-LINEAR RULES ARE GENERATED

Except for unit equations, every rule for all mainstream logics gets generated.

Work in progress – very technical, almost done.

This yields a uniform and general normalisation theory

[After Tubella-Guglielmi, ACM ToCL, 2018 + papers in preparation].



QUESTIONS

Question 1 Why is the shape so successful?

QUESTIONS

Question 1 Why is the shape so successful?

Question 2 Does designing a proof system around the equation
 $[\tau \rightarrow x]B * [v \rightarrow x]B = [(\tau * v) \rightarrow x]B$, where $x \in \{v, \lambda, \dots\}$, make sense?

QUESTIONS

Question 1 Why is the shape so successful?

Question 2 Does designing a proof system around the equation
 $[\tau \rightarrow x]B * [v \rightarrow x]B = [(\tau * v) \rightarrow x]B$, where $x \in \{v, \lambda, \dots\}$, make sense?

Check the next example with induction and note:

$$* \overline{[(\) \rightarrow x]B} \frac{[\tau \rightarrow x]B * [v \rightarrow x]B}{[(\tau * v) \rightarrow x]B}$$

$$* \overline{[(\) \rightarrow x]B} \frac{[(\tau * v) \rightarrow x]B}{[\tau \rightarrow x]B * [v \rightarrow x]B}$$

EXAMPLE WITH INDUCTION

$A0$

formula Ax where every free x is substituted with 0

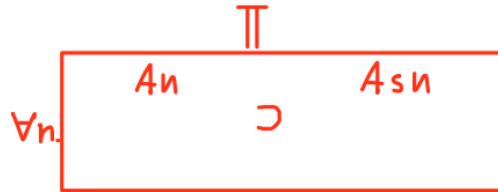
EXAMPLE WITH INDUCTION

Π
 A_0

proof of A_0 in deep inference

EXAMPLE WITH INDUCTION

ΠA_0



proof of $\forall n.(A_n \supset A_{sn})$, where s stands for the successor function

EXAMPLE WITH INDUCTION

$$\frac{\Pi}{A_0} \quad \frac{\Pi}{\forall n. A_n \supset A_{sn}}$$

$[0 \rightarrow x] A x$

$[0 \rightarrow x]$ is an indicated (formal) substitution

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[\lambda x] A x} \quad \forall n. \frac{\prod A_n}{[\lambda x] A x} \supset \frac{\prod A_{sn}}{[\lambda x] A x}$$

applying equational inference steps derived from equation

$$[t \rightarrow x] B = [t \rightarrow x] B,$$

where $[t \rightarrow x]$ is an actual substitution

EXAMPLE WITH INDUCTION

$$\frac{\prod A0}{[0 \rightarrow x] Ax}$$

Λ

$$\forall n. \frac{\prod An}{[\underline{n \rightarrow x}] Ax} \supset \frac{\prod Asn}{[\underline{sn \rightarrow x}] Ax}$$

EXAMPLE WITH INDUCTION

$$\frac{\Pi}{A_0} \quad \wedge$$
$$\frac{[0 \rightarrow x] A_x}{\Pi}$$

$$\frac{\Pi}{A_n} \quad \wedge$$
$$\frac{\forall n. \frac{A_n}{[n \rightarrow x] A_x} \supset \frac{A_{sn}}{[sn \rightarrow x] A_x}}{\Pi}$$

0Λ

$$\forall n. (n \supset s_n)$$

$$\rightarrow x A_x$$

applying an equational inference step derived
from equations

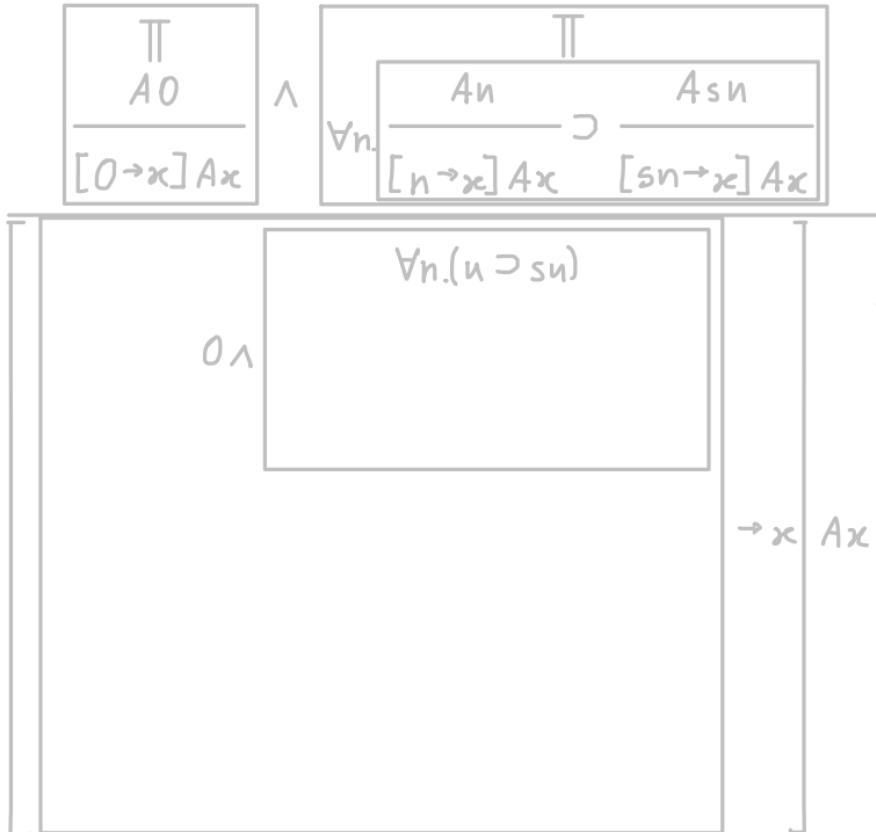
$$[\tau \rightarrow x] B * [v \rightarrow x] B = [(\tau * v) \rightarrow x] B$$

and

$$Q_y ([\tau \rightarrow x] B) = [Q_y \tau \rightarrow x] B,$$

where $*$ is any connective, Q is any
quantifier and y is not free in B

EXAMPLE WITH INDUCTION



applying an equational inference step derived
from equations

$$[\tau \rightarrow x] B * [\nu \rightarrow x] B = [(\tau * \nu) \rightarrow x] B$$

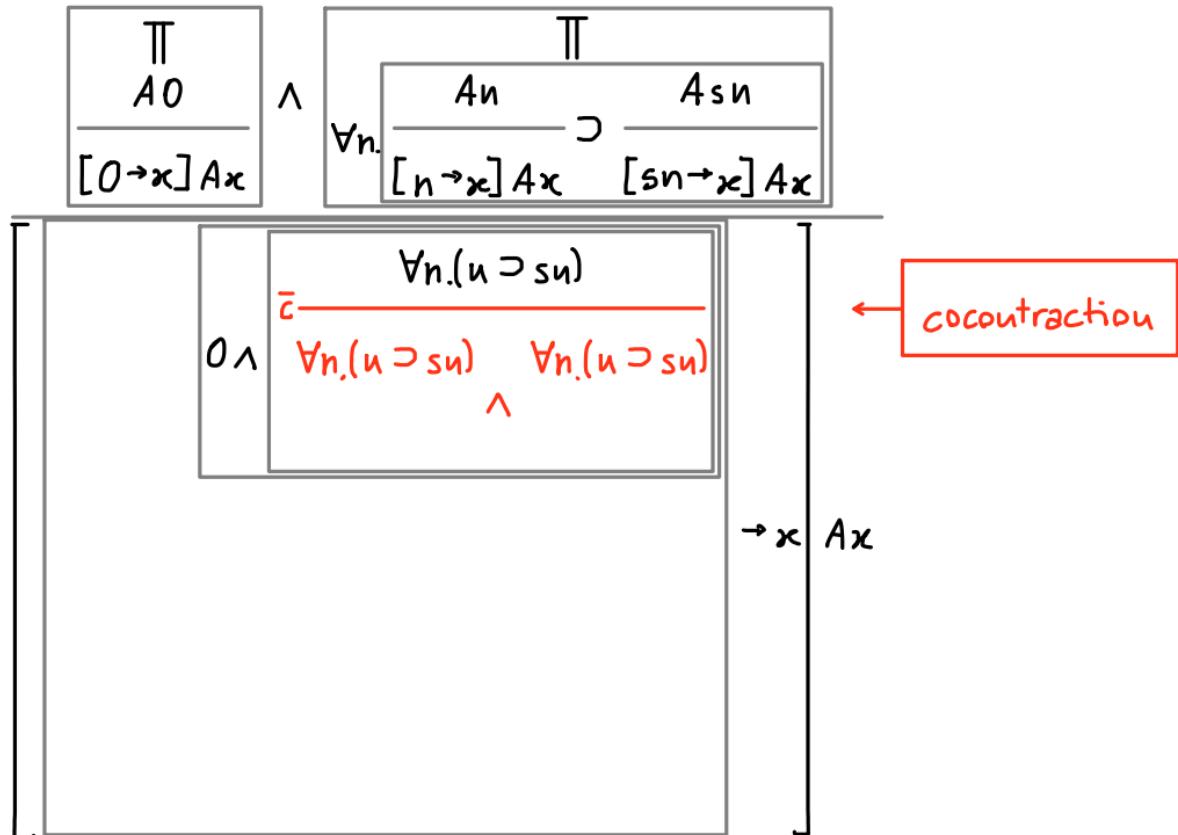
and

$$Qy([\tau \rightarrow x] B) = [Qy.\tau \rightarrow x] B,$$

where $*$ is any connective, Q is any quantifier and y is not free in B

could this provide a speed-up?

EXAMPLE WITH INDUCTION



EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A x}$$

$$\wedge \quad \frac{\prod \forall n. \frac{\prod A_n}{[n \rightarrow x] A x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A x}}{\prod [n \rightarrow x] A x}$$

[$0 \rightarrow p$,
 $s0 \rightarrow q$]

$$0 \wedge \frac{\bar{c} \quad \forall n. (n \supset s_n)}{\forall n. (n \supset s_n) \wedge \forall n. (n \supset s_n)}$$

$\rightarrow x A x$

p and q are not free
in $0 \wedge \forall n. (n \supset s_n)$, so
nothing changes

EXAMPLE WITH INDUCTION

$$\frac{\Pi}{A_0} \quad A_0$$

$$\wedge \quad \frac{\Pi}{\forall n. \frac{A_n}{[n \rightarrow x] A x} \supset \frac{A_{sn}}{[sn \rightarrow x] A x}}$$

$$\left[\begin{array}{c} [0 \rightarrow p, \\ s0 \rightarrow q] \quad 0 \wedge \\ \vdash \frac{\forall n. (n \supset sn)}{\forall n. (n \supset sn) \wedge \forall n. (n \supset sn)} \\ \qquad \qquad \qquad \text{P} \supset sP \quad \text{q} \supset sq \\ \end{array} \right] \rightarrow x A x$$

applying the rule

$$\frac{\forall y. B}{B}$$

(quantifiers only provide scope, not witnesses)

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A x}$$

$$\wedge \quad \frac{\prod \forall n. \frac{\prod A_n}{[n \rightarrow x] A x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A x}}{\prod [n \rightarrow x] A x}$$

$$\left[\begin{array}{l} [0 \rightarrow p, \\ s0 \rightarrow q] \quad 0 \wedge \quad \frac{\forall n. (n \supset sn)}{\forall n. (n \supset sn) \wedge \forall n. (n \supset sn)} \\ \qquad\qquad\qquad \frac{}{p \supset sp} \quad \frac{}{q \supset sq} \\ \hline 0 \wedge (0 \supset s0) \wedge (s0 \supset ss0) \end{array} \right] \rightarrow x A x$$

applying
 $[t \rightarrow x] B = [t \Rightarrow x] B$
(already seen)

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A x}$$

$$\wedge \quad \frac{\forall n. \frac{\prod A_n}{[n \rightarrow x] A x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A x}}{\prod [n \rightarrow x] A x}$$

$$\left[\begin{array}{c} [0 \rightarrow p, \\ s0 \rightarrow q] \quad 0 \wedge \quad \frac{\forall n. (n \supset sn)}{\wedge} \\ \bar{c} \quad \frac{\forall n. (n \supset sn)}{p \supset sp} \quad \frac{\forall n. (n \supset sn)}{q \supset sq} \\ 0 \wedge (0 \supset s0) \wedge (s0 \supset ss0) \\ \frac{\bar{0} \wedge \bar{0}}{\vdash f} \vee \frac{s0 \wedge \bar{s0}}{\vdash f} \vee \frac{ss0}{\vdash ss0} \end{array} \right] \rightarrow x A x$$

open deduction derivation
of switch instances —
note negation on terms

EXAMPLE WITH INDUCTION

$$\boxed{\begin{array}{c} \text{P} \\ A_0 \\ \hline [0 \rightarrow x] A_x \end{array}}$$

$$\forall n \frac{A_n}{[n \rightarrow x] A_x} \supset \frac{A_{sn}}{[sn \rightarrow x] A_x}$$

$[0 \rightarrow p,$ $s0 \rightarrow q]$	$\forall n.(n \supset s_n)$ $\bar{c} \quad \underline{\quad}$ $\forall n.(n \supset s_n) \quad \forall n.(n \supset s_n)$ $\underline{\quad \wedge \quad}$ $p \supset s_p \quad q \supset s_q$
	$0 \wedge (0 \supset s_0) \wedge (s_0 \supset ss_0)$ \parallel $\vdash \frac{0 \wedge \bar{0}}{f} \vee \vdash \frac{s_0 \wedge \bar{s_0}}{f} \vee ss_0$ <hr/> ss_0

applying

$$[t \rightarrow x] B = [t \Rightarrow x] B$$

(already seen)

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A_x}$$

$$\wedge \quad \frac{\forall n. \frac{\prod A_n}{[n \rightarrow x] A_x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A_x}}{\prod [n \rightarrow x] A_x}$$

$$\begin{array}{c}
 \boxed{[0 \rightarrow p, s0 \rightarrow q] \quad 0 \wedge \quad \frac{\forall n. (n \supset sn)}{\frac{\forall n. (n \supset sn) \wedge \forall n. (n \supset sn)}{\frac{p \supset sp}{}, \frac{q \supset sq}{}}}} \\
 \\
 \boxed{0 \wedge (0 \supset s0) \wedge (s0 \supset ss0) \quad \frac{}{\frac{0 \wedge \bar{0}}{\vdash f} \vee \frac{s0 \wedge \bar{s0}}{\vdash f} \vee ss0}{\vdash ss0}}
 \end{array}$$

A_{ss0}

cuts can be eliminated
as usual

EXAMPLE WITH INDUCTION

$$\frac{\prod A0}{[0 \rightarrow x] Ax}$$

$$\wedge \quad \frac{\prod \forall n. \frac{\prod An}{[n \rightarrow x] Ax} \supset \frac{\prod Asn}{[sn \rightarrow x] Ax}}{}$$

$$\boxed{\begin{array}{c} [0 \rightarrow p, \\ s0 \rightarrow q] \quad 0 \wedge \\ \quad \frac{\bar{c}}{\forall n. (n \supset sn)} \quad \frac{\bar{c}}{\forall n. (n \supset sn)} \wedge \frac{\bar{c}}{\forall n. (n \supset sn)} \\ \quad p \supset sp \quad q \supset sq \\ \hline 0 \wedge (0 \supset s0) \wedge (s0 \supset ss0) \\ \parallel \\ \frac{\bar{c} \quad \bar{c}}{f \quad f} \vee \frac{\bar{c} \quad \bar{c}}{f \quad f} \vee ss0 \\ \hline ss0 \end{array}}$$

$\rightarrow x \quad Ax$

we can also use an induction rule instead

$$\boxed{\begin{array}{c} [ss0 \rightarrow n] \quad 0 \wedge \forall n. (n \supset sn) \\ \hline \forall n. n \\ \hline ss0 \end{array}}$$

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A x}$$

$$\wedge \quad \frac{\forall n. \frac{\prod A_n}{[n \rightarrow x] A x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A x}}{\prod [n \rightarrow x] A x}$$

$$\left[\begin{array}{c} [0 \rightarrow p, \\ s0 \rightarrow q] \quad 0 \wedge \\ \bar{e} \quad \frac{\forall n. (n \supset sn)}{\forall n. (n \supset sn) \wedge \forall n. (n \supset sn)} \\ \qquad \qquad \qquad \wedge \\ \qquad \qquad \qquad p \supset sp \quad q \supset sq \\ \hline 0 \wedge (0 \supset s0) \wedge (s0 \supset ss0) \end{array} \right] \rightarrow x A x$$

$$\left[\begin{array}{c} \frac{\overline{0 \wedge \bar{0}} \quad \overline{s0 \wedge \bar{s0}}}{\text{f} \quad \text{f}} \vee \frac{\overline{s0 \wedge \bar{s0}}}{\text{f}} \vee ss0 \\ \hline ss0 \end{array} \right] Ass0$$

Could something like this lead to better quantification?
 A proof theory for the ε -calculus?