

TOWARDS A NEW PROOF THEORY OF QUANTIFICATION

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Bath

EECS Theory Seminar – Queen Mary, 16 July 2020

Talk available from my home page and at <http://cs.bath.ac.uk/ag/t/TANPTOQ.pdf>

All about deep inference at <http://alessio.guglielmi.name/res/cos>

26/7/20

MOTIVATION: SPEED-UPS FOR CUT-FREE PROOFS

Corollary of [Aguilera-Baaz, JSL, 2019]

Deep inference has a non-elementary speed-up over
cut-free tree-like Gentzen proofs of the predicate calculus.

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Deep inference has a non-elementary speed-up over cut-free tree-like Gentzen proofs of the predicate calculus.

$\vdash P_z,$	$\overline{P_z}$
$\vdash P_z,$	$\overline{P_z}, P_y$
$\vdash \overline{P_x}, P_z,$	$\overline{P_z}, P_y$
$\vdash \overline{P_x}, P_z,$	$\overline{P_z} \vee P_y$
$\vdash \overline{P_x} \vee P_z,$	$\overline{P_z} \vee P_y$
$\vdash \overline{P_x} \vee P_z,$	$\forall y. (\overline{P_z} \vee P_y)$
$\vdash \overline{P_x} \vee P_z,$	$\exists x. \forall y. (\overline{P_x} \vee P_y)$
$\vdash \forall y. (\overline{P_x} \vee P_y), \exists x. \forall y. (\overline{P_x} \vee P_y)$	
$\vdash \exists x. \forall y. (\overline{P_x} \vee P_y), \exists x. \forall y. (\overline{P_x} \vee P_y)$	
$\vdash \exists x. \forall y. (\overline{P_x} \vee P_y)$	

Gentzen's bureaucracy
demands contractions

MOTIVATION: SPEED-UPS FOR CUT-FREE PROOFS

Corollary of [Aguilera-Baaz, JSL, 2019]

Deep inference has a non-elementary speed-up over cut-free tree-like Gentzen proofs of the predicate calculus.

$$\frac{t}{\exists x. \overline{P}_x \vee \forall y. P_y}$$

$$\frac{\overline{P}_x \vee \forall y. P_y}{\exists x. \forall y. (\overline{P}_x \vee P_y)}$$

deep inference does not

Gentzen's bureaucracy
demands contractions

$$\frac{\vdash P_z, \quad \overline{P}_z}{\vdash P_z, \quad \overline{P}_z, P_y}$$

$$\frac{}{\vdash \overline{P}_x, P_z, \quad \overline{P}_z, P_y}$$

$$\frac{}{\vdash \overline{P}_x, P_z, \quad \overline{P}_z \vee P_y}$$

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$$\frac{}{\vdash \overline{P}_x \vee P_z, \quad \forall y. (\overline{P}_z \vee P_y)}$$

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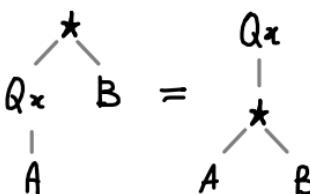
$$\frac{}{\vdash \exists x. \forall y. (\overline{P}_x \vee P_y), \quad \exists x. \forall y. (\overline{P}_x \vee P_y)}$$

$$\vdash \exists x. \forall y. (\overline{P}_x \vee P_y)$$

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t	
$\exists x. \overline{P}_x \vee \forall y. P_y$	
$\overline{P}_x \vee \forall y. P_y$ $\exists x. \quad$	success is due to quantifier shifts:  $A \vdash \star$ $B \vdash \star$ $Q_x \quad Q_z$ $A \quad B$
$\forall y. (\overline{P}_x \vee P_y)$	if x not free in B

$\vdash P_z, \quad \overline{P}_z$	
$\vdash P_z, \quad \overline{P}_z, P_y$	
$\vdash \overline{P}_x, P_z, \quad \overline{P}_z, P_y$	
$\vdash \overline{P}_x, P_z, \quad \overline{P}_z \vee P_y$	
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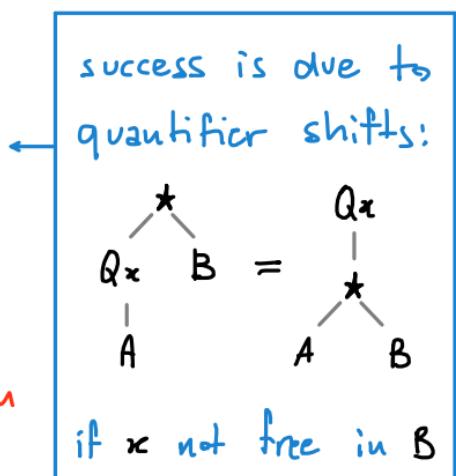
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Is there a good reason to do without quantifier shifts?

$$\frac{\vdash P_z, \quad \overline{P}_z}{\vdash P_z, \quad \overline{P}_z, P_y}$$

$$\frac{\vdash \overline{P}_x, P_z, \quad \overline{P}_z}{\vdash \overline{P}_x, P_z, \quad \overline{P}_z, P_y}$$

$$\frac{\vdash \overline{P}_x, P_z, \quad \overline{P}_z}{\vdash \overline{P}_x \vee P_z, \quad \overline{P}_z \vee P_y}$$

$$\frac{\vdash \overline{P}_x \vee P_z, \quad \overline{P}_z}{\vdash \overline{P}_x \vee P_z, \quad \forall y. (\overline{P}_z \vee P_y)}$$

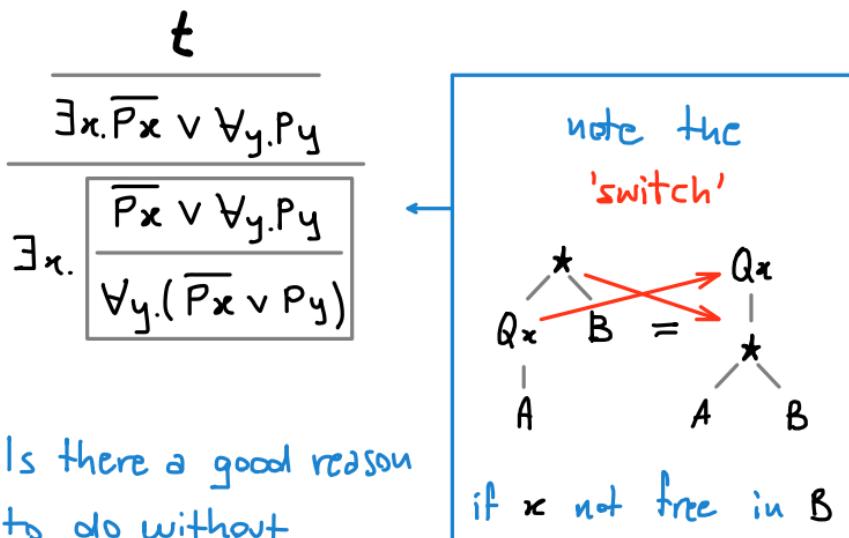
$$\frac{\vdash \overline{P}_x \vee P_z, \quad \exists x. \forall y. (\overline{P}_x \vee P_y)}{\vdash \forall y. (\overline{P}_x \vee P_y), \quad \exists x. \forall y. (\overline{P}_x \vee P_y)}$$

$$\frac{\vdash \exists x. \forall y. (\overline{P}_x \vee P_y), \quad \exists x. \forall y. (\overline{P}_x \vee P_y)}{\vdash \exists x. \forall y. (\overline{P}_x \vee P_y)}$$

MOTIVATION: SPEED-UPS FOR CUT-FREE PROOFS

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Is there a good reason
to do without
quantifier shifts?

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$\vdash \overline{Px}, Pz,$	$\overline{P}z, Py$
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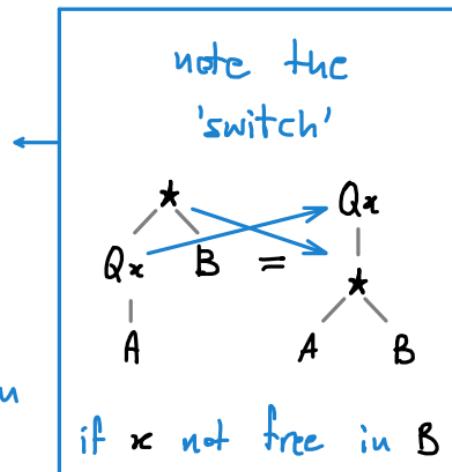
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t

$$\frac{\exists x. \overline{P}x \vee \forall y. Py}{\exists x. \overline{P}x \vee \forall y. Py}$$
$$\frac{}{\forall y. (\overline{P}x \vee Py)}$$



Is there a good reason
to do without
quantifier shifts?

Thanks to similar 'switch' mechanisms:

Theorems

The speed-up for cut-free propositional proofs is exponential.

[Bruscoli-Ciaglìelmi, ACM ToCL, 2009]

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Deep inference has a non-elementary speed-up over cut-free tree-like Gentzen proofs of the predicate calculus.

$$\frac{\begin{array}{c} t \\ \hline \exists x. \overline{Px} \vee \forall y. Py \end{array}}{\begin{array}{c} \overline{Px} \vee \forall y. Py \\ \hline \exists x. \boxed{\forall y. (\overline{Px} \vee Py)} \end{array}}$$

←

note the 'switch'

if x not free in B

Is there a good reason
to do without
quantifier shifts?

Thanks to similar 'switch' mechanisms:

Theorems

The speed-up for cut-free propositional proofs is exponential.

[Bruscoli-Cuglielmi, ACM ToCL, 2009]

Cut-elimination for propositional classical logic is quasi-polynomial.

[Jeřábek, JLC, 2009]

THE BIG PICTURE

More proofs → more compression + more canonical proofs.

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unique (natural) representatives
of equivalence classes



THE BIG PICTURE

More proofs → more compression + more canonical proofs.

How do we get more proofs without sacrificing normalisation?



THE BIG PICTURE

More proofs \rightarrow more compression + more canonical proofs.

How do we get more proofs without sacrificing normalisation?

Hyperssequents

Display calculus

Labelled sequents ...



add structure

Sequent calculus



remove structure

Deep inference

THE BIG PICTURE

More proofs → more compression + more canonical proofs.

How do we get more proofs without sacrificing normalisation?

Hyperssequents

Display calculus

Labelled sequents ...



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Deep inference



Statue: Michelangelo 3D-model; Jerry Fisher Elaboration: Alice Guglielmi

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More proofs → more compression + more canonical proofs.

How do we get more proofs without sacrificing normalisation?

Hyperssequents

Display calculus

Labeled sequent, ...



← add structure

Sequent calculus

remove structure →

Deep inference



Statue: Michelangelo 3D-model; Jerry Fisher Elaboration: Alice Guglielmi

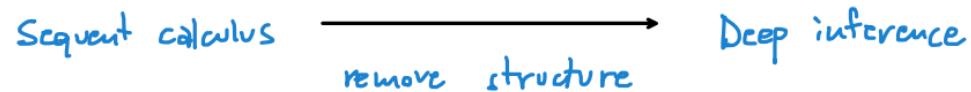
Let's not
do this.



THE BIG PICTURE

More proofs → more compression + more canonical proofs.

How do we get more proofs without sacrificing normalisation?



Two guiding principles:

- I A general shape of inference rules
(the 'switch').
→ simpler and regular
normalisation theory.



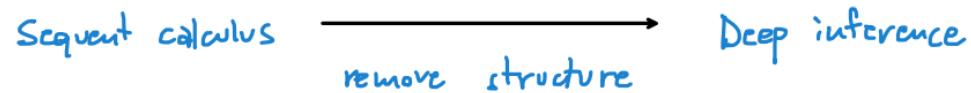
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THE BIG PICTURE

More proofs → more compression + more canonical proofs.

How do we get more proofs without sacrificing normalisation?



Two guiding principles:

1 A general shape of inference rules
(the 'switch'):

→ simpler and regular
normalisation theory.

2 A general notion of substitution:
→ factorisation mechanism.



Statue: Michelangelo 3D-model; Jerry Fisher Elaboration: Alice Guglielmi



THE GENERATING SHAPE

$$\alpha \hat{\beta} \frac{(A \beta B) \times (C \hat{\beta} D)}{(A \times C) \beta (B \times D)}$$

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saturation:
 $\hat{v} = \hat{\wedge} = \wedge,$
 $\hat{a} = a, \dots$

THE GENERATING SHAPE

$$\alpha \beta \frac{(A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}$$

saturation:
 $v = \lambda = v,$
 $a = a, \dots$

THE GENERATING SHAPE

$$\alpha \hat{\beta} \frac{(A \beta B) \times (C \beta D)}{(A \checkmark C) \beta (B \times D)}$$

saturation:
 $\check{v} = \check{x} = v,$
 $\check{a} = a, \dots$

THE GENERATING SHAPE

$$\alpha \hat{\beta} \frac{(A \hat{\beta} B) \times (C \beta D)}{(A \times C) \beta (B \times D)}$$

saturation:
 $\hat{v} = \hat{\wedge} = \wedge,$
 $\hat{a} = a, \dots$

THE GENERATING SHAPE

examples

$$\wedge\wedge \frac{\wedge(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

commutativity / associativity

$$\wedge\beta \frac{\wedge(A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}$$

$$\wedge\vee \frac{\wedge(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

switch (classical logic)

$$\otimes\tau \frac{\wedge(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

switch (linear logic)

$$\vee\wedge \frac{\wedge(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

medial

THE GENERATING SHAPE

examples

$$\wedge\wedge \frac{\wedge(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$
$$\wedge\vee \frac{\wedge(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$
$$\alpha\beta \frac{\wedge(A \beta B) \alpha (C \hat{\beta} D)}{(A \alpha C) \beta (B \alpha D)}$$
$$\otimes\tau \frac{\wedge(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$
$$\vee\wedge \frac{\wedge(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

Medial opens a new world: locality

medial

THE GENERATING SHAPE

examples

$$\wedge\wedge \frac{\wedge(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

$$\wedge\vee \frac{\wedge(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\otimes\tau \frac{\wedge(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

$$\vee\wedge \frac{\wedge(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\alpha\beta \frac{\wedge(A \beta B) \alpha (C \hat{\beta} D)}{(A \alpha C) \beta (B \alpha D)}$$

Medial opens a new world: locality

$$\vee\wedge \frac{\wedge(A \wedge B) \vee (A \wedge B)}{\begin{array}{c} A \vee A \\ \hline A \end{array} \wedge \begin{array}{c} B \vee B \\ \hline B \end{array}}$$

A big contraction can be replaced by atomic contractions.

medial

THE GENERATING SHAPE

examples

$$\wedge\wedge \frac{\wedge(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

$$\wedge\vee \frac{\wedge(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\otimes\wedge \frac{\wedge(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

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Medial opens a new world: locality

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A big contraction can be replaced by atomic contractions.

Locality = inference steps checked in constant time

→ topology drives normalisation

medial

THE GENERATING SHAPE

examples

$$\wedge\wedge \frac{\wedge(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

$$\wedge\vee \frac{\wedge(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\otimes\otimes \frac{\wedge(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

$$\vee\wedge \frac{\wedge(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\alpha\beta \frac{\wedge(A \beta B) \alpha (C \hat{\beta} D)}{(A \alpha C) \beta (B \alpha D)}$$

Example: work in progress on cut-elimination
for linear logic with a Kleene star-like operator

NORMALISATION OF CONTRACTIONS – DECOMPOSITION

vs. cocontraction	vs. promotion	vs. copromotion
direct		
external		

+ internal (trivial) + dual

Induction measure $\langle E, H \rangle$, lexicographic
 multiset $E = \{c_i\}_1^k$, clause energies
 ordering $c_i = \{e_{ij}\}_{j=1}^{n_i}$, energies
 $H = \{h_i\}_1^m$, contraction-heights

→ topology drives normalisation – atomic flows

medial

THE GENERATING SHAPE

examples

$$\wedge\wedge \frac{\wedge(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

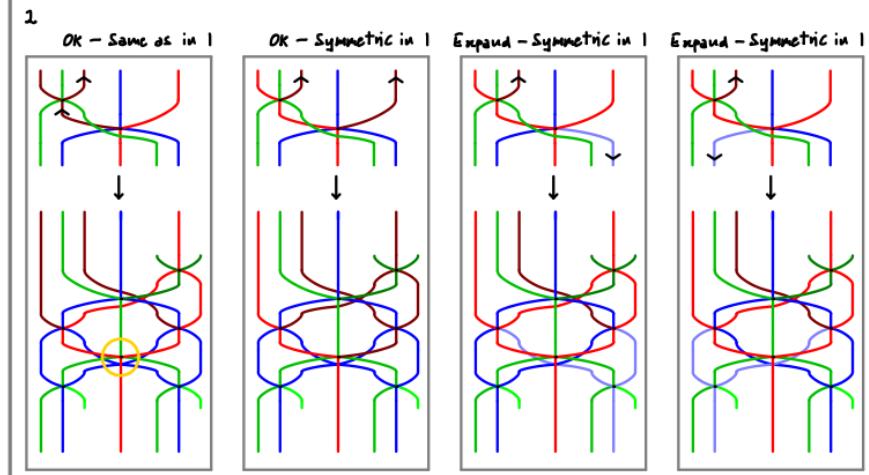
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$$\vee\wedge \frac{\wedge(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\alpha\beta \frac{\wedge(A \beta B) \alpha (C \hat{\beta} D)}{(A \alpha C) \beta (B \alpha D)}$$

Example: work in progress with Victoria Barrett
on a general normalisation theory of the shape



→ topology drives normalisation – atomic flows

medial

THE GENERATING SHAPE

examples

$$\wedge\wedge \frac{\wedge(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

binary-binary

$$\alpha\beta \frac{\wedge(A \beta B) \alpha(C \beta D)}{(A \alpha C) \beta(B \alpha D)}$$

$$\wedge\vee \frac{\wedge(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\otimes\tau \frac{\wedge(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

$$\vee\wedge \frac{\wedge(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

binary-unary

$$\alpha\beta \frac{\wedge(\beta A) \alpha(\hat{\beta} B)}{\beta(A \alpha B)}$$

THE GENERATING SHAPE

binary-binary

$$\alpha \beta \frac{\wedge (A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}$$

examples

$$\wedge \wedge \frac{\wedge (A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

$$\wedge \vee \frac{\wedge (A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\otimes ? \frac{\wedge (A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

$$\vee \wedge \frac{\wedge (A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

examples

$$\wedge \hat{\diamond} \frac{\diamond A \wedge \square B}{\diamond (A \wedge B)}$$

co-k

binary-unary

$$\alpha \beta \hat{\wedge} \frac{(\beta A) \alpha (\hat{\wedge} B)}{\beta (A \alpha B)}$$

$$\otimes ? \frac{? A \otimes ! B}{?(A \otimes B)}$$

co-promotion

$$\vee \hat{\exists}_n \frac{\exists_n A \vee \forall_n B}{\exists_n (A \vee B)}$$

quantifier shift

THE GENERATING SHAPE

binary-binary

$$\alpha \hat{\beta} \frac{(A \beta B) \alpha (C \hat{\beta} D)}{(A \alpha C) \beta (B \alpha D)}$$

examples

$$\wedge\wedge \frac{\wedge(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

$$\wedge\vee \frac{\wedge(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

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examples

$$\wedge\hat{\diamond} \frac{\diamond A \wedge \Box B}{\diamond(A \wedge B)}$$

binary-unary

$$\alpha \hat{\beta} \frac{\wedge(\beta A) \alpha (\hat{\beta} B)}{\beta(A \alpha B)}$$

$$\otimes ? \frac{?A \otimes !B}{?(A \otimes B)}$$

$$\vee \hat{\exists}_n \frac{\exists_n A \vee \forall_n B}{\exists_n(A \vee B)}$$

Unary-Unary

$$\alpha \hat{\beta} \frac{\wedge \alpha \hat{\beta} A}{\beta \alpha A}$$

example

$$\exists_n \hat{\forall}_y \frac{\exists_n. \forall_y. A}{\forall_y. \exists_n. A}$$

THE GENERATING SHAPE - SURPRISE! NON-LINEAR RULES ARE GENERATED

$$\alpha \beta \frac{\wedge \frac{(A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}}{\wedge a \frac{(\text{f at}) \wedge (\text{t a f})}{(\text{f} \wedge \text{t}) \alpha (\text{t} \wedge \text{f})}}$$

$$\vee a \frac{\wedge \frac{(\text{f at}) \vee (\text{f a t})}{(\text{f} \vee \text{f}) \alpha (\text{t} \vee \text{t})}}{\vee a}$$

THE GENERATING SHAPE - SURPRISE! NON-LINEAR RULES ARE GENERATED

$$\alpha \beta \frac{\wedge (A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}$$

$$\wedge a \frac{\wedge (f a t) \wedge (t a f)}{(f \wedge t) a (t \wedge f)} = \frac{a \wedge \bar{a}}{f}$$

if we take $\begin{cases} f a f = f \\ f a t = a \\ t a f = \bar{a} \\ t a t = t \end{cases}$

i.e., atoms are superpositions
of truth values.

$$\vee a \frac{\wedge (f a t) \vee (f a \bar{t})}{(f \vee f) a (t \vee \bar{t})}$$

THE GENERATING SHAPE – SURPRISE! NON-LINEAR RULES ARE GENERATED

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THE GENERATING SHAPE - SURPRISE! NON-LINEAR RULES ARE GENERATED

Every rule for all mainstream logics gets generated.

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$$\wedge a \frac{\wedge (f a t) \wedge (t a f)}{(f \wedge t) a (t \wedge f)} = \frac{a \wedge \bar{a}}{f}$$

if we take $\begin{cases} f a f = f \\ f a t = a \\ t a f = \bar{a} \\ t a t = t \end{cases}$

i.e., atoms are superpositions of truth values.

$$\vee a \frac{\wedge (f a t) \vee (f a t)}{(f \vee f) a (t \vee t)} = \frac{a \vee \bar{a}}{a}$$

THE GENERATING SHAPE - SURPRISE! NON-LINEAR RULES ARE GENERATED

Every rule for all mainstream logics gets generated.

The sources of compression have the same origin.

$$\alpha \beta \frac{\wedge (A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}$$

$$\lambda a \frac{\wedge (f a t) \wedge (t a f)}{(f \wedge t) a (t \wedge f)} = \frac{a \wedge \bar{a}}{f}$$

if we take

$$\begin{cases} f a f = f \\ f a t = a \\ t a f = \bar{a} \\ t a t = t \end{cases}$$

i.e., atoms are superpositions
of truth values.

$$\vee a \frac{\wedge (f a t) \vee (f a t)}{(f \vee f) a (t \vee t)} = \frac{a \vee \bar{a}}{a}$$

contraction

THE GENERATING SHAPE - SURPRISE! NON-LINEAR RULES ARE GENERATED

Every rule for all mainstream logics gets generated.

The sources of compression have the same origin.

This yields a uniform and general normalisation theory

[After Tubella-Guglielmi, ACM ToCL, 2018 + papers in preparation].

$$\alpha \beta \frac{\wedge (A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)}$$

$$\lambda a \frac{\wedge (f a t) \wedge (t a f)}{(f \wedge t) a (t \wedge f)} = \frac{a \wedge \bar{a}}{f}$$

if we take $\left\{ \begin{array}{l} f a f = f \\ f a t = a \\ t a f = \bar{a} \\ t a t = t \end{array} \right.$

i.e., atoms are superpositions
of truth values.

$$\vee a \frac{\wedge (f a t) \vee (f a \bar{t})}{(f \vee f) a (t \vee t)} = \frac{a \vee \bar{a}}{a}$$

THE GENERATING SHAPE – SURPRISE! NON-LINEAR RULES ARE GENERATED

Every rule for all mainstream logics gets generated.

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if we take $\begin{cases} f a f = f \\ f a t = a \\ t a f = \bar{a} \\ t a t = t \end{cases}$

i.e., atoms are superpositions of truth values.

$$\vee a \frac{\wedge (f a t) \vee (f a \bar{t})}{(f \vee f) a (t \vee \bar{t})} = \frac{a \vee \bar{a}}{a}$$

Negation is absorbed into the interpretation function.

THE GENERATING SHAPE

Every rule for all mainstream logics gets generated.

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$$\frac{\alpha \beta}{(A \beta B) \alpha (C \beta D)} \quad \frac{ba}{\wedge (A a B) b (C a D)} \quad \begin{array}{l} \text{Atoms inside atoms: decision trees} \\ \text{(work in progress with Chris Bennett)} \end{array}$$

THE GENERATING SHAPE

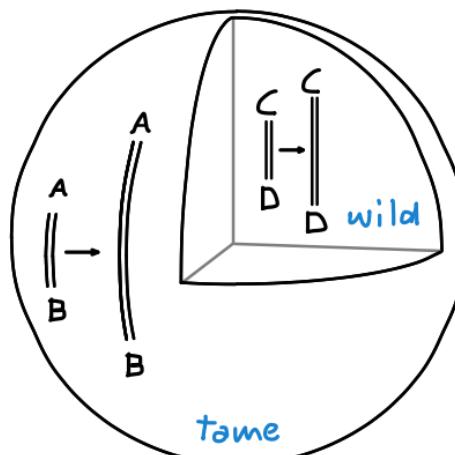
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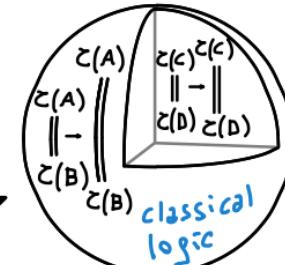
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subatomic proofs

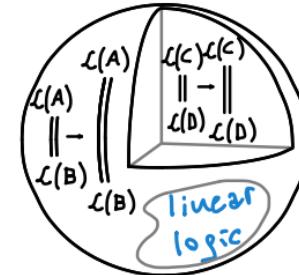


ζ



classical
logic

ι



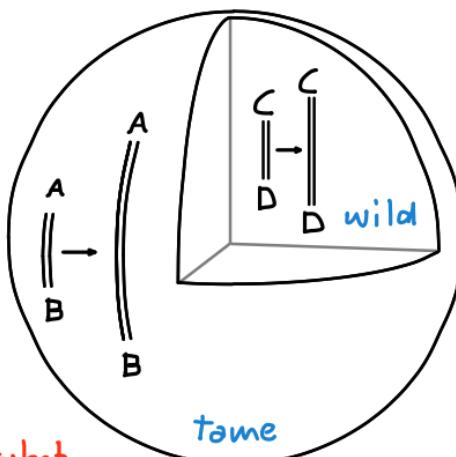
linear
logic

wild = atoms inside atoms

standard proofs

THE GENERATING SHAPE FOR QUANTIFICATION/SUBSTITUTION

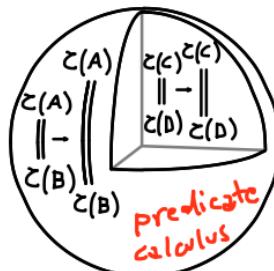
new quantification/substitution



\exists

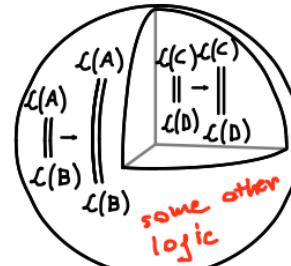
\forall

:



predicate
calculus

standard proofs



some other
logic

wild = + non-standard quantif./subst.

NORMALISATION FOR QUANTIFICATION/SUBSTITUTION

Quantification and substitution are similar, e.g.:

$$Qx. A * B = Qx.(A * B) \quad \text{if } x \text{ not free in } B$$

$$[t \rightarrow x] A * B = [t \rightarrow x] (A * B)$$

Quantification could be a special case of substitution.

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| $[q \rightarrow x] \psi$ should be defined for proofs φ and ψ .

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$$[q \rightarrow x] \psi = x$$

2 $\downarrow \quad \downarrow \quad \downarrow$ normalisation

$$[q' \rightarrow x] \psi' = x'$$

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$$[\varphi \rightarrow x]\psi = x$$

2 $\downarrow \quad \downarrow \quad \downarrow$ normalisation

$$[\varphi' \rightarrow x]\psi' = x'$$

Major difficulty: circular structures (cycles in the atomic flows) break locality.

EXAMPLE WITH INDUCTION

$A0$

formula Ax where every free x is substituted with 0

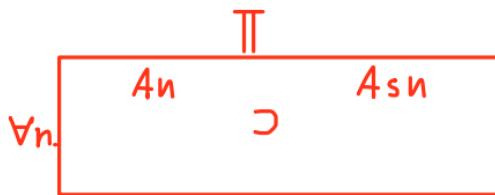
EXAMPLE WITH INDUCTION

Π
 A_0

proof of A_0 in deep inference

EXAMPLE WITH INDUCTION

Π
 A_0



proof of $\forall n.(A_n \supset A_{sn})$, where s stands for the successor function

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0 \quad \forall n. \boxed{A_n \quad \vdash \quad A_{sn}}}{\prod A_x}$$

[$0 \rightarrow x$] is an indicated (formal) substitution

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A x} \quad \forall n. \frac{\prod A_n}{[n \rightarrow x] A x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A x}$$

applying equational inference steps derived from equation

$$[t \rightarrow x] B = [t \rightarrow x] B,$$

where $[t \rightarrow x]$ is an actual substitution

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A x}$$

Λ

$$\forall n. \frac{\prod A_n}{[n \rightarrow x] A x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A x}$$

EXAMPLE WITH INDUCTION

$$\frac{\Pi}{A_0} \quad A_0$$

$$\wedge \quad \frac{\forall n. \frac{\Pi}{A_n} \quad A_{sn}}{[\bar{n} \rightarrow x] A_x \quad [\bar{s}n \rightarrow x] A_x}$$

0Λ

$$\forall n. (\bar{n} \supset \bar{s}n)$$

$\rightarrow x$ A_x

applying an equational inference step derived
from equations

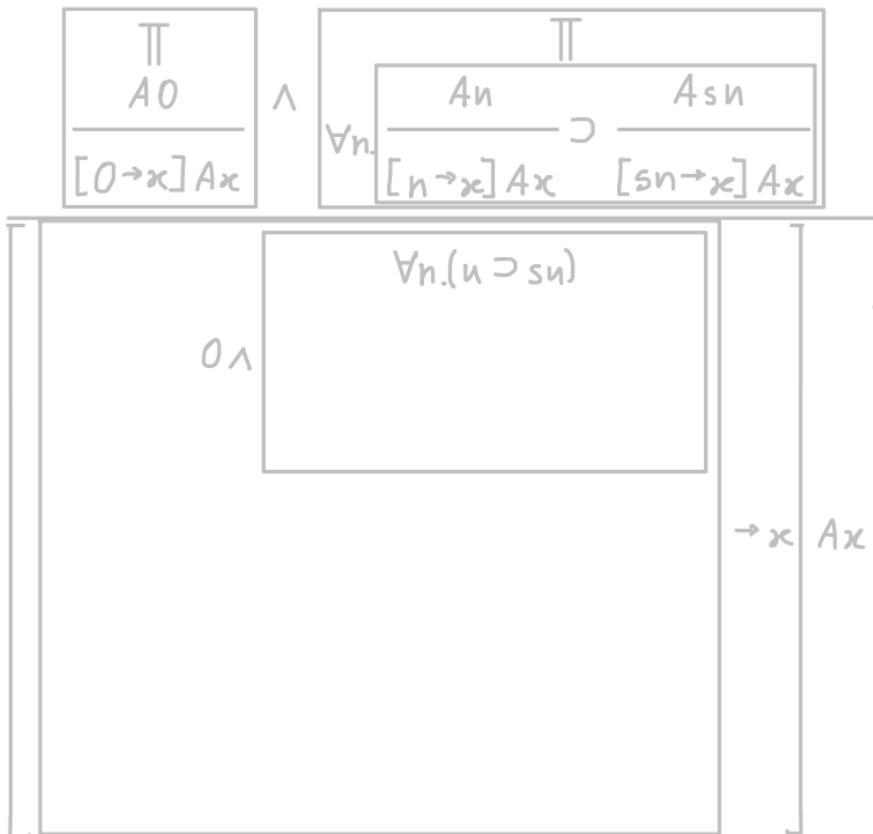
$$[\tau \rightarrow x] B * [\nu \rightarrow x] B = [(\tau * \nu) \rightarrow x] B$$

and

$$Qy.([\tau \rightarrow x] B) = [Qy.\tau \rightarrow x] B,$$

where * is any connective, Q is any quantifier and y is not free in B

EXAMPLE WITH INDUCTION



applying an equational inference step derived
from equations

$$[\tau \rightarrow x] B * [v \rightarrow x] B = [(\tau * v) \rightarrow x] B$$

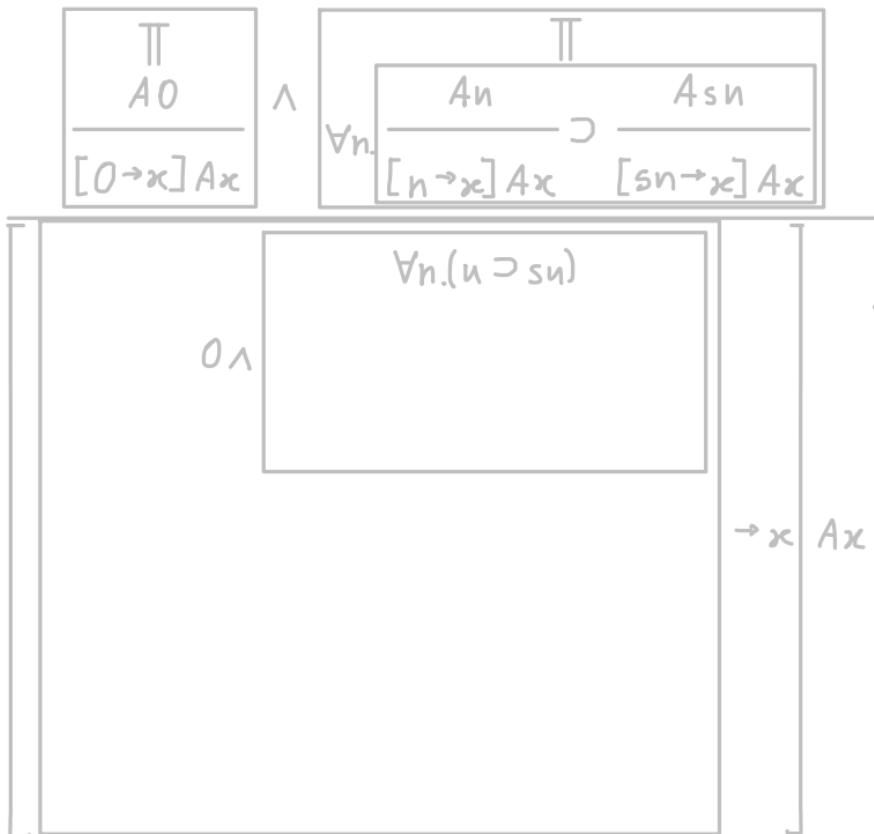
and

$$\textcircled{Qy.([\tau \rightarrow x] B) = [Qy.\tau \rightarrow x] B}$$

where $*$ is any connective, Q is any quantifier and y is not free in B

this satisfies the shape

EXAMPLE WITH INDUCTION



applying an equational inference step derived
from equations

$$[\tau \rightarrow x] B * [v \rightarrow x] B = [(\tau * v) \rightarrow x] B$$

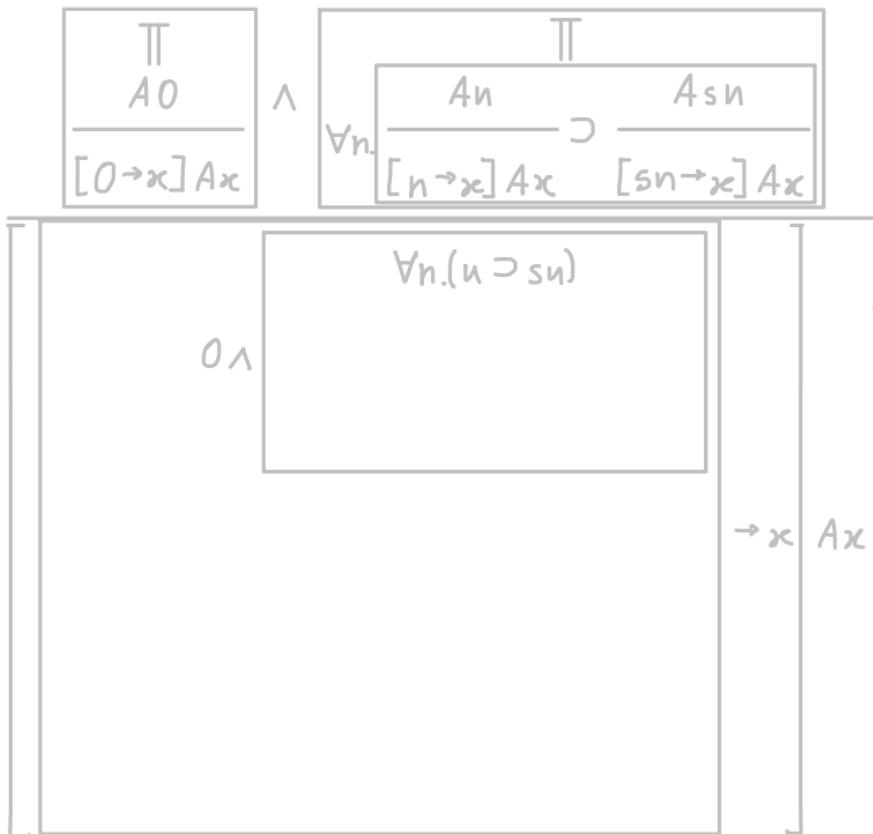
and

$$Qy.([\tau \rightarrow x] B) = [Qy.\tau \rightarrow x] B,$$

where $*$ is any connective, Q is any quantifier and y is not free in B

this satisfies the shape (arguably)

EXAMPLE WITH INDUCTION



applying an equational inference step derived
from equations

$$[\tau \rightarrow x] B * [v \rightarrow x] B = [(\tau * v) \rightarrow x] B$$

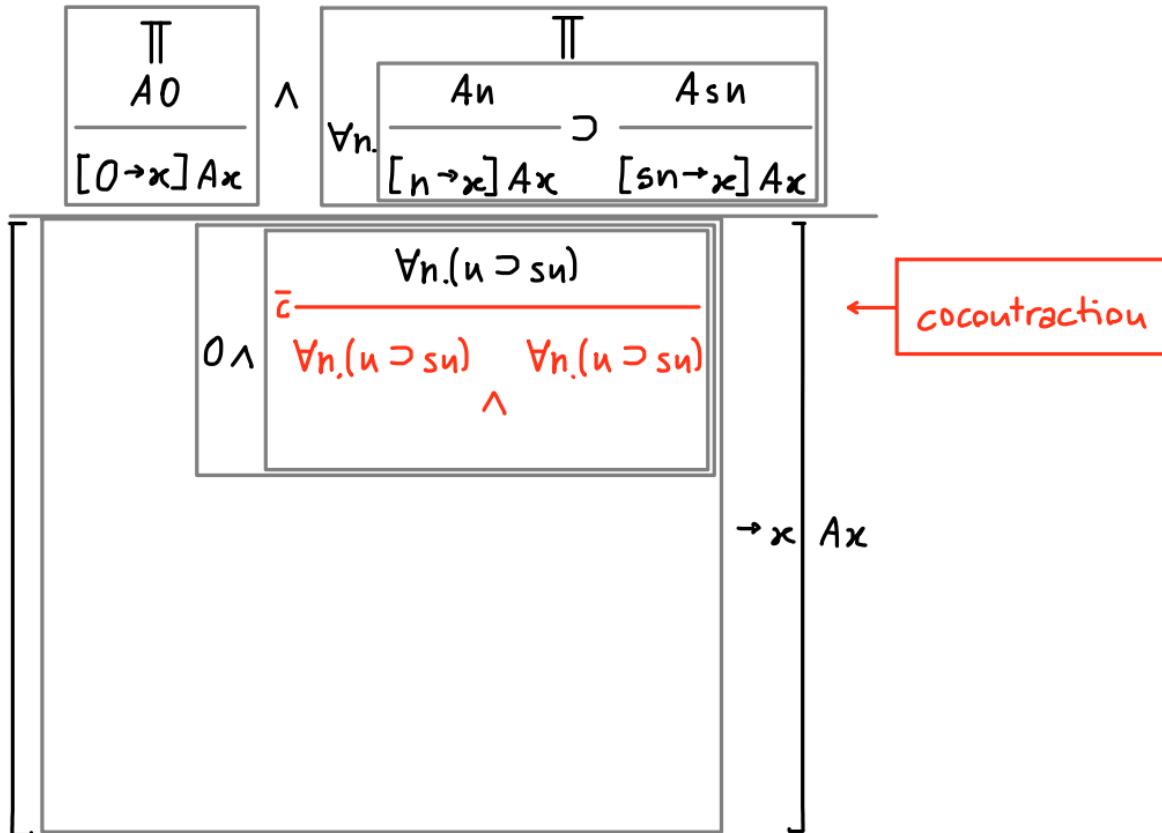
and

$$Qy.([\tau \rightarrow x] B) = [Qy.\tau \rightarrow x] B,$$

where $*$ is any connective, Q is any quantifier and y is not free in B

could this provide a speed-up?

EXAMPLE WITH INDUCTION



EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A_x}$$

$$\wedge \quad \frac{\forall n. \frac{\prod A_n}{[n \rightarrow x] A_x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A_x}}{\prod [n \rightarrow x] A_x}$$

$$\frac{[0 \rightarrow p, s0 \rightarrow q]}{0 \wedge \frac{\forall n. (n \supset sn)}{\forall n. (n \supset sn) \wedge \forall n. (n \supset sn)}}$$

p and q are not free
in $0 \wedge \forall n. (n \supset sn)$, so
nothing changes

$\rightarrow x A_x$

EXAMPLE WITH INDUCTION

$$\frac{\Pi}{A_0} \quad A_0$$

$$\wedge \quad \frac{\forall n. \frac{\Pi}{A_n} \quad A_{sn}}{[\bar{n} \rightarrow x] A_x \quad [sn \rightarrow x] A_x}$$

$$\left[\begin{array}{c} [0 \rightarrow p, \\ s0 \rightarrow q] \quad 0 \wedge \\ \vdash \frac{\forall n. (n \supset sn)}{\wedge \frac{\forall n. (n \supset sn) \quad \forall n. (n \supset sn)}{p \supset sp \quad q \supset sq}} \end{array} \right] \rightarrow x A_x$$

applying the rule

$$\frac{\forall y. B}{B}$$

(quantifiers only provide scope, not witnesses)

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A x}$$

$$\wedge \quad \frac{\prod \forall n. \frac{\prod A_n}{[n \rightarrow x] A x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A x}}{\prod [n \rightarrow x] A x}$$

$$\left[\begin{array}{c} [0 \rightarrow p, \\ s0 \rightarrow q] \quad 0 \wedge \quad \frac{\forall n. (n \supset sn)}{\forall n. (n \supset sn) \wedge \forall n. (n \supset sn)} \\ \hline p \supset sp \quad q \supset sq \end{array} \right] \xrightarrow{x} A x$$

$0 \wedge (0 \supset s0) \wedge (s0 \supset ss0)$

applying
 $[t \rightarrow x] B = [t \Rightarrow x] B$
(already seen)

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A_x}$$

$$\wedge \quad \frac{\forall n. \frac{\prod A_n}{[n \rightarrow x] A_x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A_x}}{\prod [n \rightarrow x] A_x}$$

$$\left[\begin{array}{l} [0 \rightarrow p, \\ s0 \rightarrow q] \quad 0 \wedge \quad \frac{\forall n. (n \supset sn)}{\frac{\forall n. (n \supset sn) \wedge \forall n. (n \supset sn)}{\frac{p \supset sp}{p \supset sp} \wedge \frac{q \supset sq}{q \supset sq}}} \\ 0 \wedge (0 \supset s0) \wedge (s0 \supset ss0) \\ \frac{\overline{0 \wedge \bar{0}} \vee \overline{s0 \wedge \bar{s0}}}{\frac{f \quad f}{ss0}} \end{array} \right] \rightarrow x A_x$$

open deduction derivation
of switch instances —
note negation on terms

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A_x}$$

$$\wedge \quad \frac{\forall n. \frac{\prod A_n}{[n \rightarrow x] A_x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A_x}}{\prod A_x}$$

$$\begin{array}{c}
 \boxed{[0 \rightarrow p, s0 \rightarrow q] \quad \boxed{0 \wedge \frac{\forall n. (n \supset sn)}{\wedge \frac{\forall n. (n \supset sn)}{p \supset sp} \quad \frac{\forall n. (n \supset sn)}{q \supset sq}}} \\
 \\
 \boxed{0 \wedge (0 \supset s0) \wedge (s0 \supset ss0) \quad \boxed{\frac{0 \wedge \bar{0}}{\vdash f} \vee \frac{s0 \wedge \bar{s0}}{\vdash f} \vee \frac{ss0}{\vdash ss0}} \\
 \\
 \boxed{Ass0}
 \end{array}$$

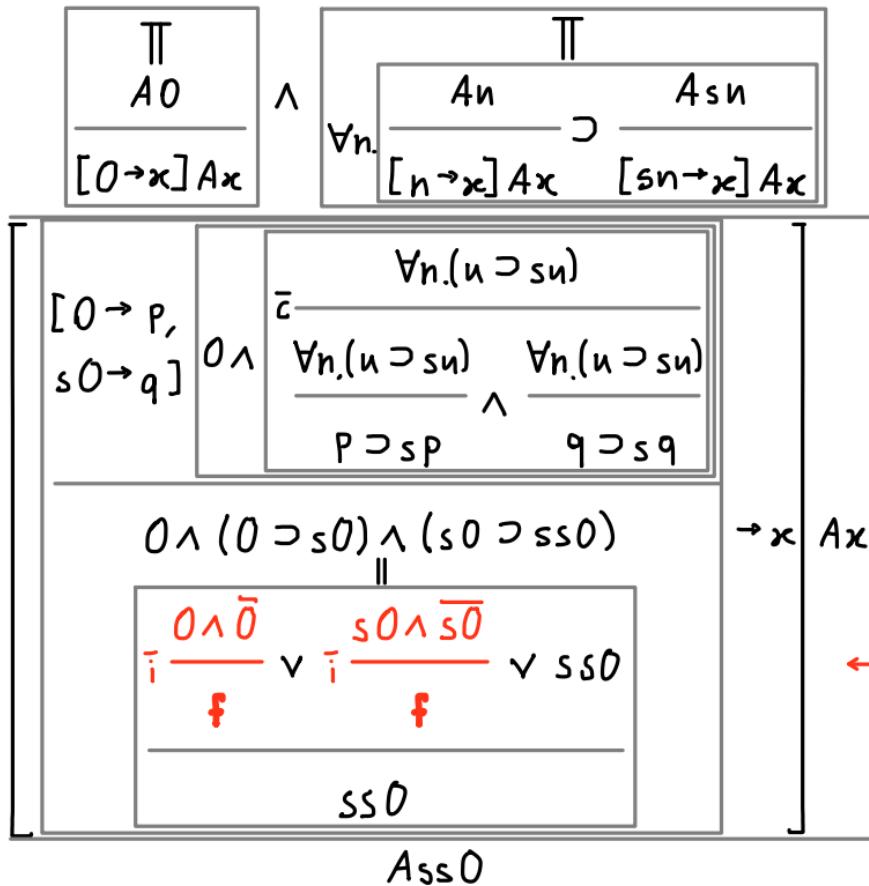
→ $x A_x$

applying

$$[t \rightarrow x] B = [t \Rightarrow x] B$$

(already seen)

EXAMPLE WITH INDUCTION



EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A x}$$

$$\wedge \quad \frac{\forall n. \frac{\prod A_n}{[n \rightarrow x] A x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A x}}{\prod A x}$$

$$\boxed{\begin{array}{c} [0 \rightarrow p, \\ s0 \rightarrow q] \quad 0 \wedge \quad \frac{\forall n. (n \supset sn)}{\frac{\forall n. (n \supset sn) \wedge \forall n. (n \supset sn)}{\frac{p \supset sp \quad q \supset sq}{p \supset q}}} \\ \hline 0 \wedge (0 \supset s0) \wedge (s0 \supset ss0) \\ \frac{0 \wedge \bar{0} \quad s0 \wedge \bar{s0}}{\frac{\bar{i} \quad \bar{i}}{f \quad f}} \vee \frac{ss0}{\vee ss0} \\ \hline ss0 \end{array}}$$

$\rightarrow x A x$

we can also use an induction rule instead

$$\boxed{\begin{array}{c} [ss0 \rightarrow n] \quad 0 \wedge \forall n. (n \supset sn) \\ \hline \forall n. n \\ \hline ss0 \end{array}}$$

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A x}$$

$$\wedge \quad \frac{\prod \forall n. \frac{\prod A_n}{[n \rightarrow x] A x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A x}}{\prod [n \rightarrow x] A x}$$

$$\begin{array}{c}
 \boxed{[0 \rightarrow p, s0 \rightarrow q] \quad \boxed{0 \wedge \frac{\forall n. (n \supset sn)}{\wedge \frac{\forall n. (n \supset sn)}{p \supset sp} \quad \frac{\forall n. (n \supset sn)}{q \supset sq}}} \\
 \\
 \boxed{0 \wedge (0 \supset s0) \wedge (s0 \supset ss0) \quad \boxed{\frac{0 \wedge \bar{0}}{\vdash f} \vee \frac{s0 \wedge \bar{s0}}{\vdash f} \vee ss0} \\
 \\
 \boxed{ss0} \\
 \\
 Ass0
 \end{array}$$

A lot can happen here as well - factorisation

EXAMPLE WITH INDUCTION

$$\frac{\prod A_0}{[0 \rightarrow x] A_x}$$

$$\wedge \quad \frac{\forall n. \frac{\prod A_n}{[n \rightarrow x] A_x} \supset \frac{\prod A_{sn}}{[sn \rightarrow x] A_x}}{\prod [n \rightarrow x] A_x}$$

$$\left[\begin{array}{l} [0 \rightarrow p, \\ s0 \rightarrow q] \quad 0 \wedge \\ \vdash \frac{\forall n. (n \supset sn)}{\forall n. (n \supset sn) \wedge \forall n. (n \supset sn)} \\ \qquad \qquad \qquad p \supset sp \quad q \supset sq \\ \hline 0 \wedge (0 \supset s0) \wedge (s0 \supset ss0) \\ \parallel \\ \vdash \frac{0 \wedge \bar{0}}{f} \vee \vdash \frac{s0 \wedge \bar{s0}}{f} \vee ss0 \\ \hline ss0 \\ \text{Ass0} \end{array} \right] \rightarrow x A_x$$

Could something like this lead to better quantification?
A proof theory for the ε -calculus?