A Subatomic Proof System

Alessio Guglielmi
(joint work with Andrea Aler Tubella)

University of Bath

29th September 2014

This talk is available at `http://cs.bath.ac.uk/ag/t/SPS.pdf`
Deep inference web site: `http://alessio.guglielmi.name/res/cos/`

Outline

Proof systems with a single rule?

The proof system

Soundness and completeness

Final considerations

Proof systems with a single rule?

- Goal: generating propositional proofs by a single, linear, simple and regular inference rule scheme.
- Idea: consider atoms as self-dual, noncommutative binary logical relations.
- This means working with an extended language of formulae (atoms inside atoms, whatever that means) …
- …but if we only look for proofs of normal formulae, we only get normal proofs with the usual (deep inference) proof theory.

Is it interesting?

It is interesting for us because it tells us something about the geometry of normalisation of Formalism B.

The proof system

Idea: occurrences of an atom a are interpretations of more primitive expressions involving a noncommutative binary relation denoted by a.

- Formulae A and B in the relation a, in this order, are denoted by $A\ a\ B$.
- Formulae are built over the two units for disjunction and conjunction, respectively 0 and 1.

Example: the following two expressions are SA formulae:

$$\langle 0\ a\ 1\rangle \vee \langle 1\ a\ 0\rangle \qquad \langle 0\ b\ 1\rangle\ a\ \langle 1\ c\ \langle 1\ d\ 0\rangle\rangle \wedge 0 \wedge [\langle 0\ a\ 0\rangle \vee \langle 1\ b\ 1\rangle]$$

We call tame the formulae where atoms do not appear in the scope of other atoms (e.g., left) and wild the others (e.g., right).

The proof system (cont.)

Let $\mapsto$ be an interpretation map from tame SA formulae to ordinary formulae such that

$$\begin{aligned} 0\ a\ 0 &\mapsto 0 \\ 0\ a\ 1 &\mapsto a \\ 1\ a\ 0 &\mapsto \bar{a} \\ 1\ a\ 1 &\mapsto 1 \end{aligned}$$

where $\bar{a}$ denotes the negation of a.

Note

- self-duality: $\overline{A\ a\ B} \equiv \bar{A}\ a\ \bar{B}$
- noncommutativity: $A\ a\ B \not\equiv B\ a\ A$ whenever $A \not\equiv B$

Extend $\mapsto$ to all the tame SA formulae in the natural way. For example:

$$\langle 0\ a\ 1 \rangle \vee \langle 1\ a\ 0 \rangle \mapsto a \vee \bar{a} \qquad [0 \vee 0]\ a\ [1 \vee 1] \mapsto a$$

The proof system (cont.)

Consider the usual contraction rule for an atom:

$$\frac{a \vee a}{a}$$

We could obtain this rule via $\mapsto$ as follows:

$$\frac{\langle 0\ a\ 1\rangle \vee \langle 0\ a\ 1\rangle}{[0 \vee 0]\ a\ [1 \vee 1]} \mapsto \frac{a \vee a}{a} \qquad \text{and} \qquad \frac{\langle 1\ a\ 0\rangle \vee \langle 1\ a\ 0\rangle}{[1 \vee 1]\ a\ [0 \vee 0]} \mapsto \frac{\bar{a} \vee \bar{a}}{\bar{a}} \quad .$$

We might consider those rules as generated by the linear scheme

$$\frac{\langle A\ a\ C\rangle \vee \langle B\ a\ D\rangle}{[A \vee B]\ a\ [C \vee D]}$$

This scheme is typical of logical rules in deep inference!

The proof system (cont.)

$$\frac{\langle A\ a\ C\rangle \vee \langle B\ a\ D\rangle}{[A \vee B]\ a\ [C \vee D]}$$

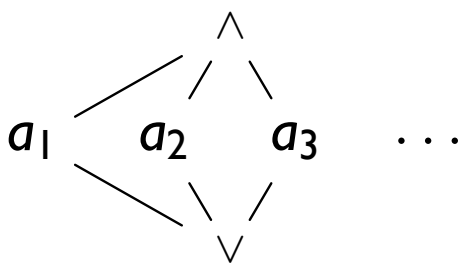
Is the scheme sound? Does it work for all rules?

Two more examples, identity and cut:

$$\frac{[0 \vee 1]\ a\ [1 \vee 0]}{\langle 0\ a\ 1\rangle \vee \langle 1\ a\ 0\rangle} \mapsto \frac{1}{a \vee \bar{a}} \qquad \text{and} \qquad \frac{\langle 0\ a\ 1\rangle \wedge \langle 1\ a\ 0\rangle}{(0 \wedge 1)\ a\ (1 \wedge 0)} \mapsto \frac{a \wedge \bar{a}}{0} \quad .$$

The proof system (cont.)

- Consider the partial order C of logical relations



- $>_C$ corresponds to implication, e.g.: $A \wedge B \Rightarrow A \vee B$ and $0 \wedge 1 \Rightarrow 0\ a\ 1 \Rightarrow 0 \vee 1$.
- On C we define the involution $\bar{\cdot}$ such that $\bar{\vee} = \wedge$ and $\bar{a}_i = a_i$.
- For each of element α of C we define the set $\mathsf{i}(\alpha) = \{\alpha, \bar{\alpha}\}$.

We define the (infinite) set of quadruples

$$\mathsf{Q}_C = \{\, \langle \alpha\beta\gamma\delta \rangle \mid \alpha \leq_C \delta, \delta \in \mathsf{i}(\alpha), \gamma \leq_C \beta, \beta \in \mathsf{i}(\gamma) \,\} \setminus \{\langle \vee\wedge\wedge\vee \rangle\}$$

System SA is the deep inference system whose only inference rule is

$$\frac{(A\ \alpha\ C)\ \beta\ (B\ \delta\ D)}{(A\ \beta\ B)\ \alpha\ (C\ \gamma\ D)}\ \langle \alpha\beta\gamma\delta \rangle \in \mathsf{Q}_C$$

Soundness and completeness

- Soundness: check that each inference rule instance involving tame formulae corresponds to a valid implication between premiss and conclusion.
- Completeness: show that each inference rule (on tame formulae) of a complete system for propositional logic, such as KS [1], can be represented by one or more rules of SA.

Since we have seen how to deal with identity and contraction above, suffice to see how we can represent weakening (with $\langle a, \wedge, \vee, a \rangle$), switch (with $\langle \vee, \wedge, \vee, \vee \rangle$ or $\langle \vee, \wedge, \wedge, \wedge \rangle$) and medial (with $\langle \wedge, \vee, \vee, \wedge \rangle$):

$$\frac{\langle 0\ a\ 1 \rangle \wedge \langle 0\ a\ 0 \rangle}{(0 \wedge 0)\ a\ [1 \vee 0]} \qquad \frac{\langle 1\ a\ 0 \rangle \wedge \langle 0\ a\ 0 \rangle}{(1 \wedge 0)\ a\ [0 \vee 0]} \qquad \frac{[A \vee C] \wedge [B \vee D]}{(A \wedge B) \vee [C \vee D]} \qquad \frac{(A \wedge C) \vee (B \wedge D)}{[A \vee B] \wedge [C \vee D]}$$

What about wild formulae and proofs?

What happens to proofs if we are only interested in tame formulae?

It is very easy to prove the following:
Proposition *If the conclusion of a proof in* SA *is a tame formula, then no wild formula appears in the proof.*

What we see as propositional logic proofs are just special observations, projections obtained from a more general and more regular collection of proofs.

Future work: see whether SA's regularity could be exploited for a better understanding of proof normalisation.

Future work: devise geometrical normalisation methods for SA.

Other logics?

If we had to interpret the SA contraction rule in linear logic [2] we would not be able to obtain contraction because in linear logic

$$\mathsf{I} \parr \mathsf{I} \not\equiv \mathsf{I}$$

as expected!

In fact the interpretation would be:

$$\frac{\langle \bot\ a\ \mathsf{I} \rangle \parr \langle \bot\ a\ \mathsf{I} \rangle}{[\bot \parr \bot]\ a\ [\mathsf{I} \parr \mathsf{I}]} \mapsto \frac{a \parr a}{0\ a\ [\mathsf{I} \parr \mathsf{I}]}$$

which does not correspond to any linear logic proof and which we could consider 'wild'.

Future work: controlling the 'resource consciousness' or 'substructurality' of a logic by an appropriate choice of unit equivalences and implementing them into the interpretation map.

Future work: specialising the proof systems and their proof theory by tuning the interpretation map.

References

[1] K. Brünnler and A. F. Tiu.
A local system for classical logic.
In R. Nieuwenhuis and A. Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*, volume 2250 of *Lecture Notes in Computer Science*, pages 347–361. Springer-Verlag, 2001.

[2] J.-Y. Girard.
Linear logic.
Theoretical Computer Science, 50:1–102, 1987.

[3] A. Guglielmi and T. Gundersen.
Normalisation control in deep inference via atomic flows.
Logical Methods in Computer Science, 4(1):9:1–36, 2008.

[4] A. Guglielmi, T. Gundersen, and L. Straßburger.
Breaking paths in atomic flows for classical logic.
In J.-P. Jouannaud, editor, *25th Annual IEEE Symposium on Logic in Computer Science (LICS)*, pages 284–293. IEEE, 2010.