Removing Syntax From Proof Theory

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Joint work with
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This talk is available at `http://cs.bath.ac.uk/ag/t/RSPT.pdf`

Outline

Proof Systems

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- Example, a Frege system:
  - Axioms:
    $$\begin{gathered} A \supset (B \supset A), \\ (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)), \\ (\neg B \supset \neg A) \supset ((\neg B \supset A) \supset B), \end{gathered}$$
    and rules (often just modus ponens, or cut): $\dfrac{A \quad A \supset B}{B}$.
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  All Frege systems are polynomially equivalent.
- Due to implicational completeness: if $A \supset B$ then A proves B.

Proof Complexity and the NP Vs. co-NP Problem

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  $\exists$ *super proof system*
  iff
  NP = co-NP

  where

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- $\exists$ optimal (polynomially simulating all others) proof system? 50/50; perhaps feasible.

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Known mechanisms:

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   Most studied, proof theory.

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1. Can we capture cut and analyticity independently of syntax?
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This talk answers YES to Question (1).

(Proof) System SKS [Brünnler and Tiu, 2001]

- Atomic rules:

$$\boxed{\begin{array}{ccc}
\mathsf{ai}{\downarrow}\dfrac{\mathsf{t}}{a \vee \bar{a}} & \mathsf{aw}{\downarrow}\dfrac{\mathsf{f}}{a} & \mathsf{ac}{\downarrow}\dfrac{a \vee a}{a} \\
\textit{identity} & \textit{weakening} & \textit{contraction} \\
\\
\mathsf{ai}{\uparrow}\dfrac{a \wedge \bar{a}}{\mathsf{f}} & \mathsf{aw}{\uparrow}\dfrac{a}{\mathsf{t}} & \mathsf{ac}{\uparrow}\dfrac{a}{a \wedge a} \\
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- Linear rules:

$$\begin{array}{cc}
\mathsf{s}\,\dfrac{A \wedge [B \vee C]}{(A \wedge B) \vee C} & \mathsf{m}\,\dfrac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]} \\
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identity *weakening* *contraction*

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cut *coweakening* *cocontraction*

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- Plus an '$=$' linear rule (associativity, commutativity, units).

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- SKS is complete and implicationally complete for propositional logic.

Examples in Open Deduction

- $$\mathsf{m}\,\dfrac{\dfrac{a}{a \wedge a} \vee \dfrac{b}{b \wedge b}}{\left[a \vee b\right] \wedge \left[a \vee b\right]} \quad \wedge \quad \dfrac{a}{a \wedge a}$$

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(In [Guglielmi et al., 2010a].)

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In Gentzen:

- no locality for (co)contraction (counterexample in [Brünnler, 2004]),
- no local reduction of cut into atomic form.

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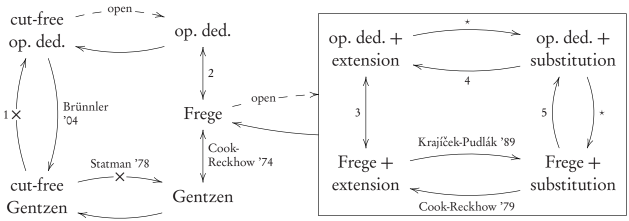
Locality $\rightarrow$ geometry $\rightarrow$ semantics of proofs.

Deep Inference and Proof Complexity



cut-free op. ded.

open

op. ded.

2

1 ×

Brünnler '04

Frege

open

Cook-Reckhow '74

cut-free Gentzen

Statman '78

×

Gentzen

op. ded. + extension

⋆

op. ded. + substitution

4

3

5

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$\longrightarrow =$ 'polynomially simulates'.

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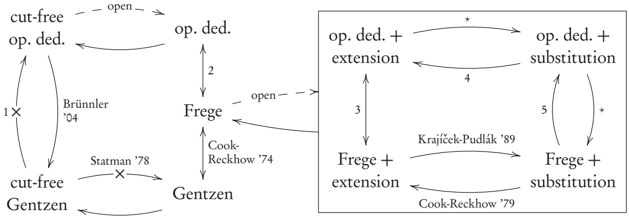

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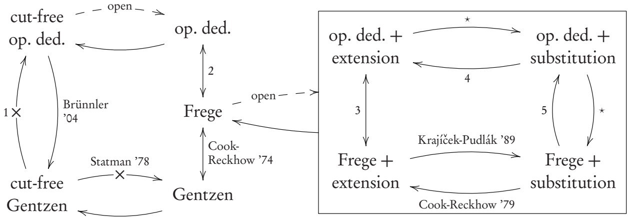

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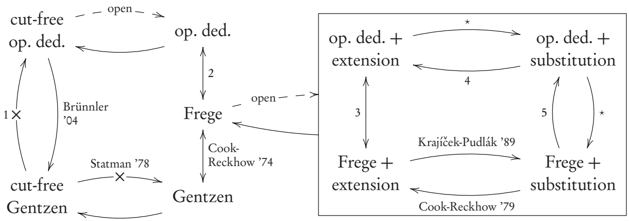

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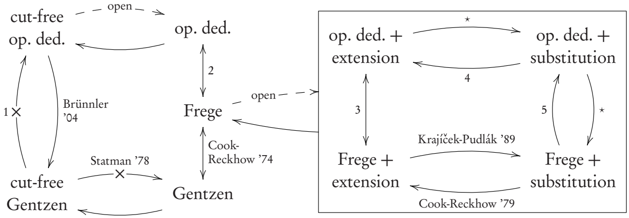

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Open deduction has as small proofs as the best formalisms

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and

cut elimination is quasipolynomial (instead of exponential).

(See [Jeřábek, 2009, Bruscoli and Guglielmi, 2009, Bruscoli et al., 2010]).

(Atomic) Flows

$$
{\scriptstyle\mathsf{s}}\,\frac{{\scriptstyle\mathsf{m}}\,\dfrac{\dfrac{\mathsf{t}}{a \vee \bar{a}}}{[a \vee \mathsf{t}] \wedge [\mathsf{t} \vee \bar{a}]}}{\left[{\scriptstyle\mathsf{s}}\,\dfrac{[a \vee \mathsf{t}] \wedge \bar{a}}{\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee \mathsf{t}} \vee \mathsf{t} \right]}
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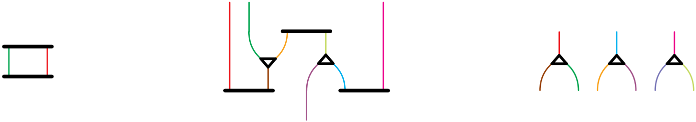

Below proofs, their (atomic) flows are shown:

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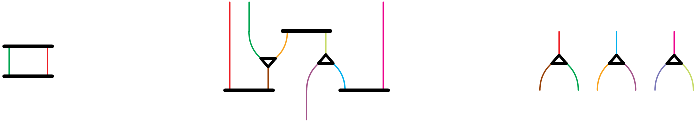

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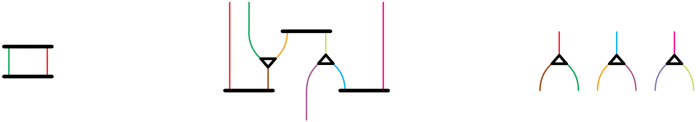

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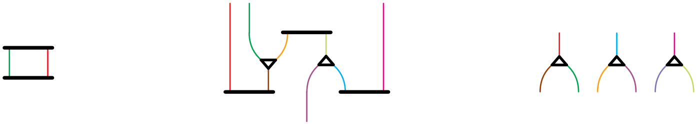
$$=\,\dfrac{\left(\mathsf{s}\,\dfrac{a\wedge\left[\bar a\vee\dfrac{\mathsf{t}}{\bar a\vee a}\right]}{\dfrac{a\wedge\dfrac{\bar a\vee\bar a}{\bar a}}{\mathsf{f}}\quad\vee\quad\dfrac{a}{a\wedge a}}\quad\wedge\quad\bar a\right)}{a\wedge\dfrac{a\wedge\bar a}{\mathsf{f}}}$$

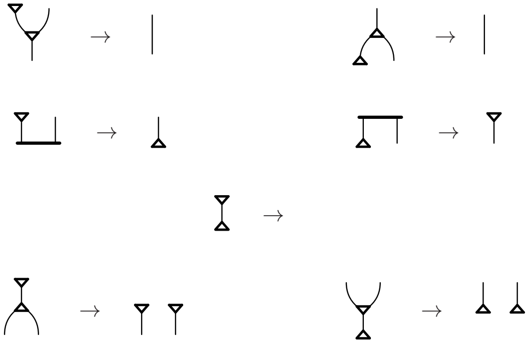
$$\mathsf{m}\,\dfrac{\dfrac{a}{a\wedge a}\vee\dfrac{b}{b\wedge b}}{[a\vee b]\wedge[a\vee b]}\quad\wedge\quad\dfrac{a}{a\wedge a}$$



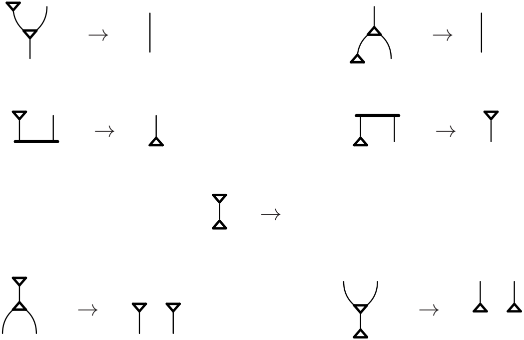
Below proofs, their (atomic) flows are shown:

- only structural information is retained in flows;
- logical information is lost;
- flow size is polynomially related to derivation size.

Flow Reductions: (Co)Weakening (1)



Flow Reductions: (Co)Weakening (1)



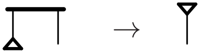
Each flow reduction corresponds to a correct proof reduction.

Flow Reductions: (Co)Weakening (2)

E.g.,  →  specifies that

$$\begin{array}{c} \Pi''\Big\Vert \\ \xi\left\{\dfrac{\mathsf{t}}{a^{\epsilon}\vee\bar{a}}\right\} \\ \Phi\Big\Vert \\ \zeta\left\{\dfrac{a^{\epsilon}}{\mathsf{t}}\right\} \\ \Psi\Big\Vert \\ \alpha \end{array} \qquad \text{becomes} \qquad \begin{array}{c} \Pi''\Big\Vert \\ \xi\left[\mathsf{t}\vee\dfrac{\mathsf{f}}{\bar{a}}\right] \\ \Phi\{a^{\epsilon}/\mathsf{t}\}\Big\Vert \\ \zeta\{\mathsf{t}\} \\ \Psi\Big\Vert \\ \alpha \end{array}$$

Flow Reductions: (Co)Weakening (2)

E.g.,  specifies that

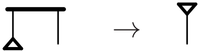
$$\begin{array}{c} \Pi''\Big\Vert \\ \xi\left\{\dfrac{\mathsf{t}}{a^{\epsilon}\vee\bar{a}}\right\} \\ \Phi\Big\Vert \\ \zeta\left\{\dfrac{a^{\epsilon}}{\mathsf{t}}\right\} \\ \Psi\Big\Vert \\ \alpha \end{array} \qquad \text{becomes} \qquad \begin{array}{c} \Pi''\Big\Vert \\ \xi\left[\mathsf{t}\vee\dfrac{\mathsf{f}}{\bar{a}}\right] \\ \Phi\{a^{\epsilon}/\mathsf{t}\}\Big\Vert \\ \zeta\{\mathsf{t}\} \\ \Psi\Big\Vert \\ \alpha \end{array}$$

We can operate on flow reductions instead than on derivations:

- much easier,

Flow Reductions: (Co)Weakening (2)

E.g.,



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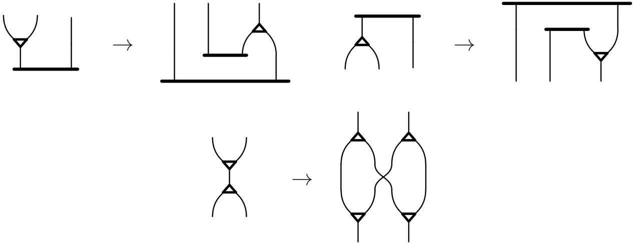
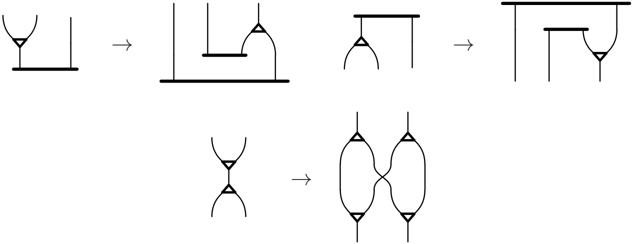
We can operate on flow reductions instead than on derivations:

- much easier,
- we get natural, syntax-independent induction measures.

Flow Reductions: (Co)Contraction

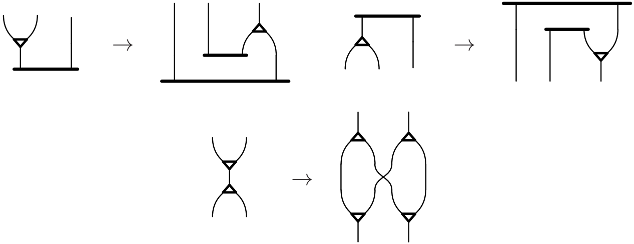


Flow Reductions: (Co)Contraction



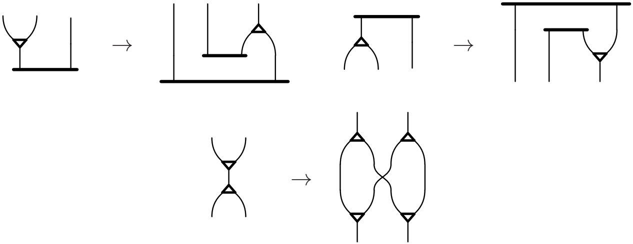
- These reductions conserve the number and length of paths.

Flow Reductions: (Co)Contraction



- These reductions conserve the number and length of paths.
- They can blow up a derivation exponentially.

Flow Reductions: (Co)Contraction



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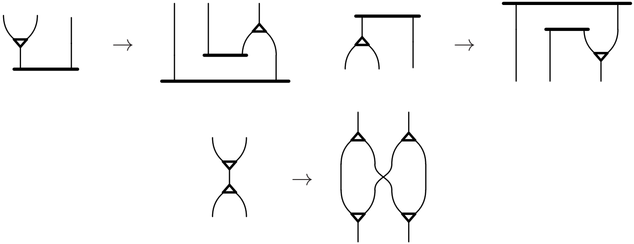
- It's a good thing: cocontraction is a new compression mechanism (dag-ness?).

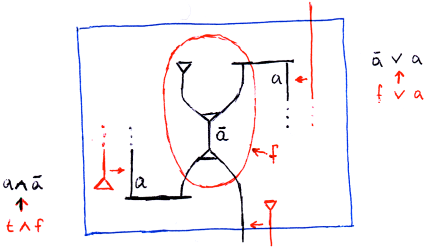
Flow Reductions: (Co)Contraction



- These reductions conserve the number and length of paths.
- They can blow up a derivation exponentially.
- It's a good thing: cocontraction is a new compression mechanism (dag-ness?).
- Open problem: does cocontraction yield exponential compression? Conjecture: yes.

Cut Elimination by ‘Experiments’

Experiment:



Cut Elimination by ‘Experiments’

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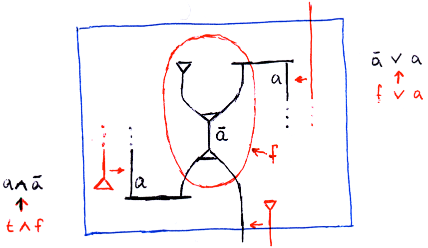
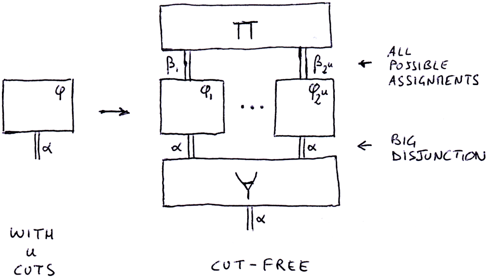
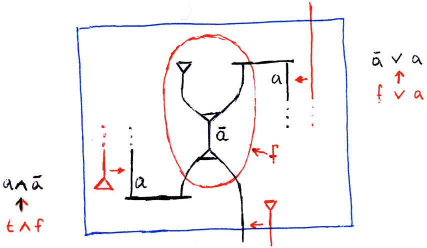


We do:



Cut Elimination by 'Experiments'

Experiment:



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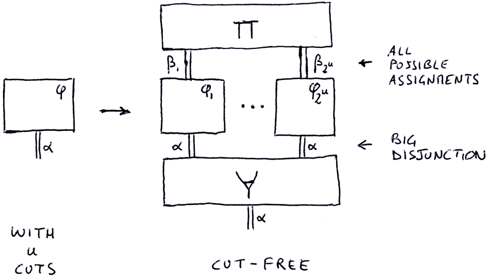
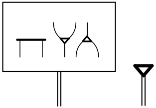


Simple, exponential cut elimination; proof generates 2^n experiments.

Generalising the Cut-Free Form

- Normalised proof:



Generalising the Cut-Free Form

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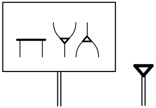
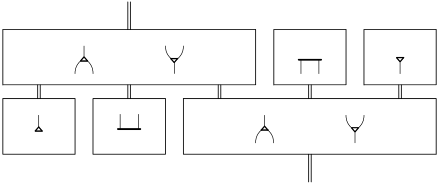


- Normalised derivation:



Generalising the Cut-Free Form

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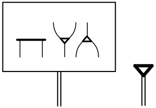

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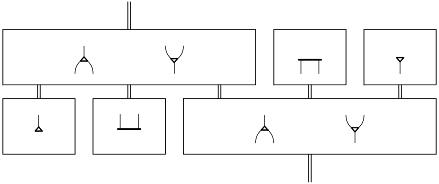

- The symmetric form is called streamlined.
- Cut elimination is a corollary of streamlining.

Generalising the Cut-Free Form

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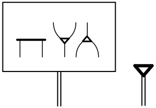

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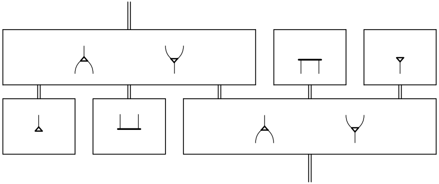
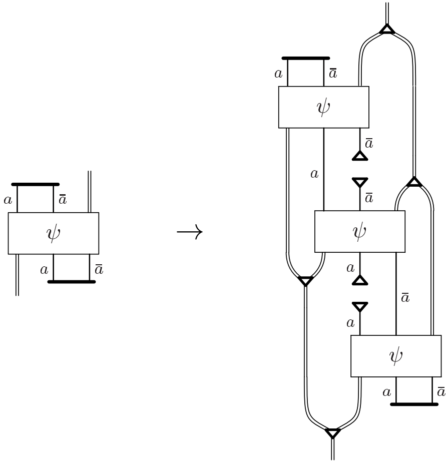


- The symmetric form is called streamlined.
- Cut elimination is a corollary of streamlining.
- We just need to break the paths between identities and cuts, and (co)weakenings do the rest.

How Do We Break Paths?

With the path breaker [Guglielmi et al., 2010b]:



Even if there is a path between identity and cut on the left, there is none on the right.

We Can Do This on Derivations, of Course

$$
\begin{array}{c}
A \\ \hline
[a \vee \bar{a}] \wedge A \\
\Psi \Big\| \\
B \vee (a \wedge \bar{a}) \\ \hline
B
\end{array}
\qquad \longrightarrow \qquad
\begin{array}{c}
A \\
\Big\| \{\mathsf{c}\!\uparrow, \mathsf{ai}\!\downarrow, =\} \\
(([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\
{\scriptstyle (\Psi \wedge A) \wedge A} \Big\| \\
([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\
{\scriptstyle \Phi_a \wedge A} \Big\| \\
[B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\
{\scriptstyle [B \vee \Psi] \wedge A} \Big\| \\
B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\
{\scriptstyle B \vee \Phi_{\bar{a}}} \Big\| \\
B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\
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B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\
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B
\end{array}
$$

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\Big\| {\scriptstyle \{\mathsf{c}\downarrow, \mathsf{ai}\uparrow, =\}} \\
B
\end{array}
$$

- We can compose this as many times as there are paths between identities and cut.

We Can Do This on Derivations, of Course

$$
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A \\ \hline
[a \vee \bar{a}] \wedge A \\
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B \vee (a \wedge \bar{a}) \\ \hline
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\end{array}
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$$

- We can compose this as many times as there are paths between identities and cut.
- We obtain a family of normalisers that only depends on n.

We Can Do This on Derivations, of Course

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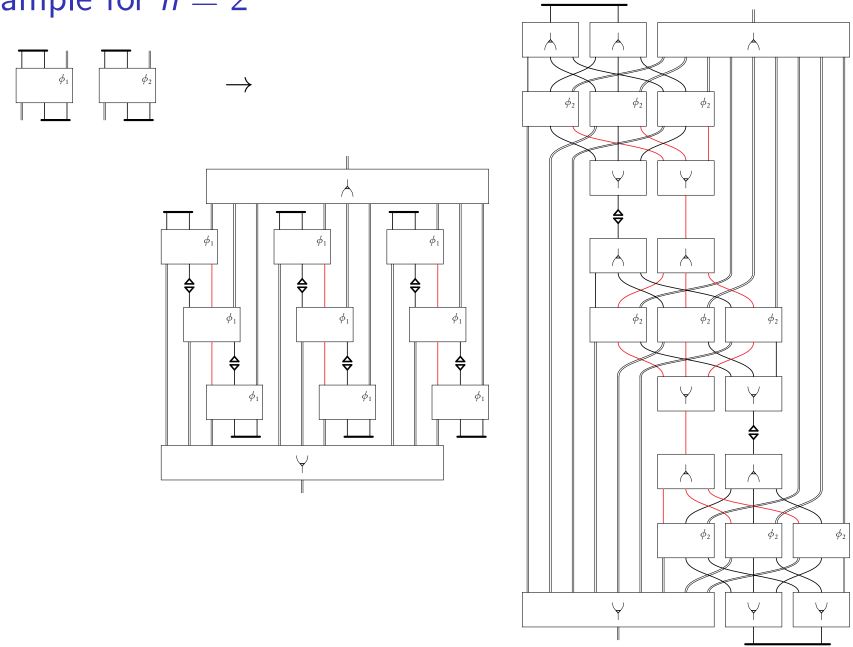
- We can compose this as many times as there are paths between identities and cut.
- We obtain a family of normalisers that only depends on n.
- The construction is exponential.

We Can Do This on Derivations, of Course

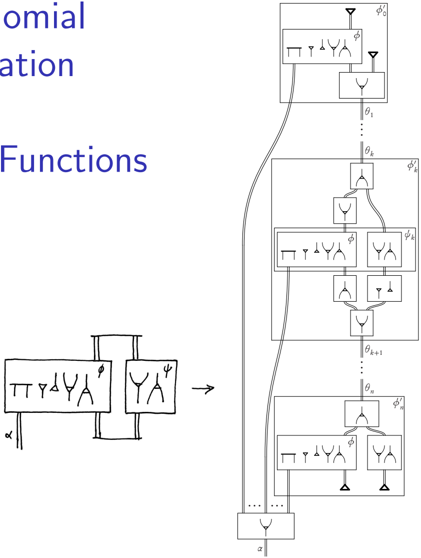
$$\begin{array}{c} A \\ \hline [a \vee \bar{a}] \wedge A \\ {\scriptstyle \Psi}\Big\| \\ B \vee (a \wedge \bar{a}) \\ \hline B \end{array} \qquad \rightarrow \qquad \begin{array}{c} A \\ \Big\|{\scriptstyle \{\mathsf{c}\uparrow, \mathsf{ai}\downarrow, =\}} \\ (([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\ {\scriptstyle (\Psi \wedge A) \wedge A}\Big\| \\ ([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\ {\scriptstyle \Phi_a \wedge A}\Big\| \\ [B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\ {\scriptstyle [B \vee \Psi] \wedge A}\Big\| \\ B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\ {\scriptstyle B \vee \Phi_{\bar{a}}}\Big\| \\ B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\ {\scriptstyle B \vee [B \vee \Psi]}\Big\| \\ B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\ \Big\|{\scriptstyle \{\mathsf{c}\downarrow, \mathsf{ai}\uparrow, =\}} \\ B \end{array}$$

- We can compose this as many times as there are paths between identities and cut.
- We obtain a family of normalisers that only depends on *n*.
- The construction is exponential.
- Finding something like this is unthinkable without flows.

Example for $n = 2$

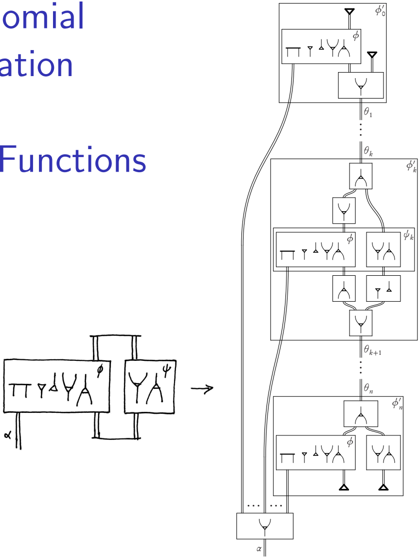


Quasipolynomial Cut Elimination by Threshold Functions



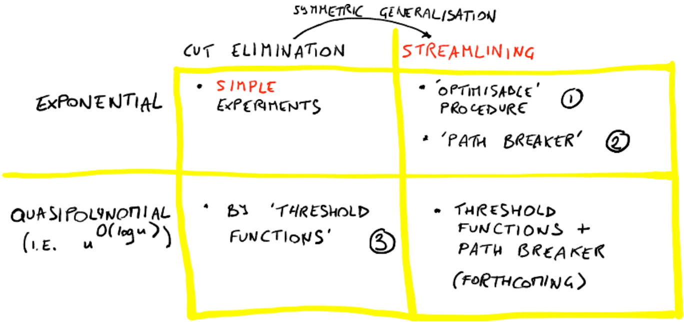
- Only $n+1$ copies of the proof are stitched together.

Quasipolynomial Cut Elimination by Threshold Functions



- Only $n+1$ copies of the proof are stitched together.
- Note local cocontraction (= better sharing, not available in Gentzen).

Normalisation
Overview



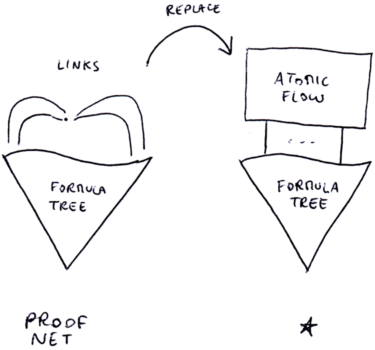
SYMMETRIC GENERALISATION

	CUT ELIMINATION	STREAMLINING
EXPONENTIAL	• SIMPLE EXPERIMENTS	• 'OPTIMISABLE' PROCEDURE ① • 'PATH BREAKER' ②
QUASIPOLYNOMIAL (I.E. $n^{O(\log n)}$)	• BY 'THRESHOLD FUNCTIONS' ③	• THRESHOLD FUNCTIONS + PATH BREAKER (FORTHCOMING)

- None of these methods existed before atomic flows, none of them requires permutations or other syntactic devices.
- Quasipolynomial procedures are surprising.

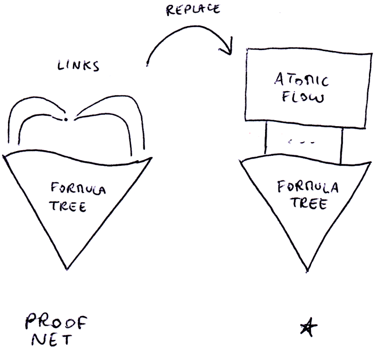
(1, 2) [Guglielmi et al., 2010b]; (3) [Bruscoli et al., 2010].

Conjecture



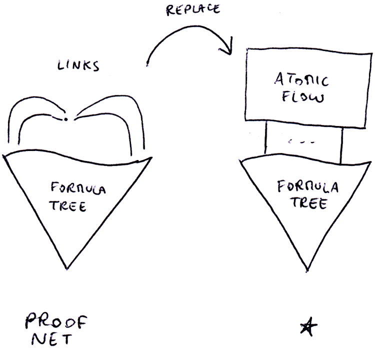
- We think that $*$ might make for a proof system.

Conjecture



- We think that $*$ might make for a proof system.
- If true, excellent bureaucracy-free formalism.

Conjecture



- We think that $*$ might make for a proof system.
- If true, excellent bureaucracy-free formalism.
- Note: if such a thing existed for proof nets, then coNP $=$ NP (because proof nets are [too?] small).

Conclusion

- Cut elimination does not depend on logical rules.
- It only depends on structural information, *i.e.*, geometry.
- Normalisation can be made robust.

This talk is available at `http://cs.bath.ac.uk/ag/t/RSPT.pdf`

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