Redesigning Logical Syntax with a Bit of Topology

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Joint work with
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31 May 2011

This talk is available at `http://cs.bath.ac.uk/ag/t/RDLS.pdf`
It requires Acrobat 9 or later

The Dream

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- No syntax, no symbols, no words.
- An alien could understand this proof.
- Is something like this possible for every proof?

The Reality

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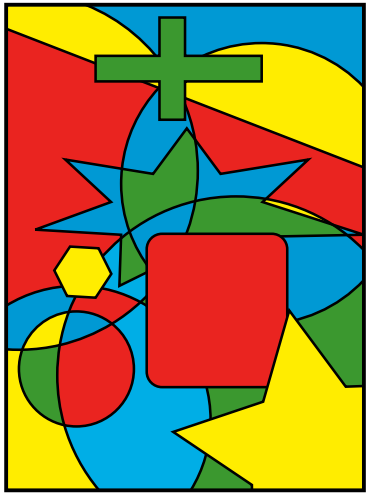
Lemma sumt_ctree_pick_rev : forall t t', sumt (ctree_pick_rev t t') = Color0.
Proof.
move=> t' t; rewrite /ctree_pick_rev; set cs0 : colseq := seq0.
have: Color0 +c sumt cs0 = Color0 by done.
elim: t cs0 {1 3}Color0 => [t1 Ht1 t2 Ht2 t3 Ht3|lf _|] et e //.
  move=> Het /=; set cprr := ctree_pick_rev_rec.
  case Det1: (cprr _ _ _ t1) => [|e1 et1].
    case Det2: (cprr _ _ _ t2) => [|e2 et2].
      by apply: Ht3; rewrite [Color3]lock /= -addcA addc_inv.
    by rewrite -Det2; apply: Ht2; rewrite [Color2]lock /= -addcA addc_inv.
  by rewrite -Det1; apply: Ht1; rewrite [Color1]lock /= -addcA addc_inv.
by move=> Het /=; case (ctree_mem t' (etrace (belast e et))).
Qed.
```



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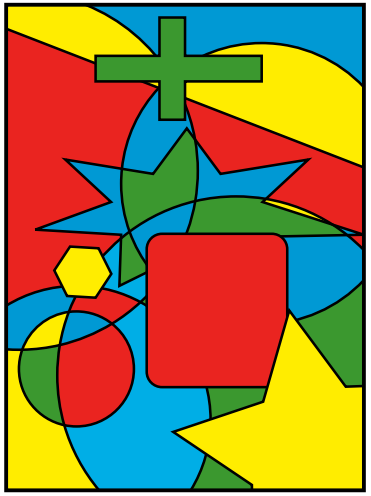

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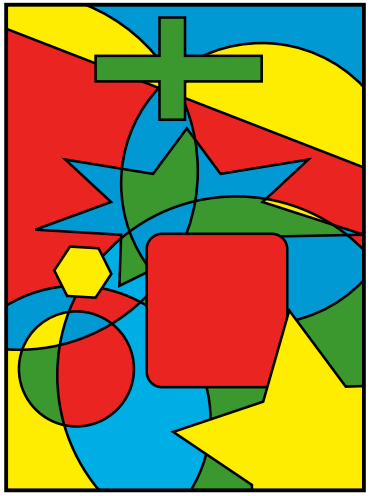

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Solving these is necessary for the universal mathematics database.

Outline of the Talk

- Strategy
- Proof Complexity and the Oddness of the Cut
- Open Deduction (Deep Inference)
  - Propositional Logic and System SKS
  - Examples
- Atomic Flows
  - Examples
  - Flow Reductions
- Normalisation
  - Cut Elimination: Experiments
  - Streamlining: Generalised Cut Elimination
  - The Path Breaker
  - Quasipolynomial Cut Elimination
  - Overview
- Conjecture
- Conclusion

Strategy:
We conserve the existing proof theory properties ...

Gentzen's major breakthrough (1930s):

- proofs can be analytic, *i.e.*, built in finitary ways,
- by time expensive algorithms,
- that nonetheless allow us to control and analyse them.

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$$

But Gentzen

- only knew classical logic, which is poor for algorithms;
- only wanted finiteness, while we want more: efficiency;
- had no idea of proof complexity.

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So:

- we want to keep proof size low (and possibly making it lower),
- but not too low (otherwise we probably don't have proof systems).

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We call this property locality.

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We envy the syntax-independence of proof complexity!

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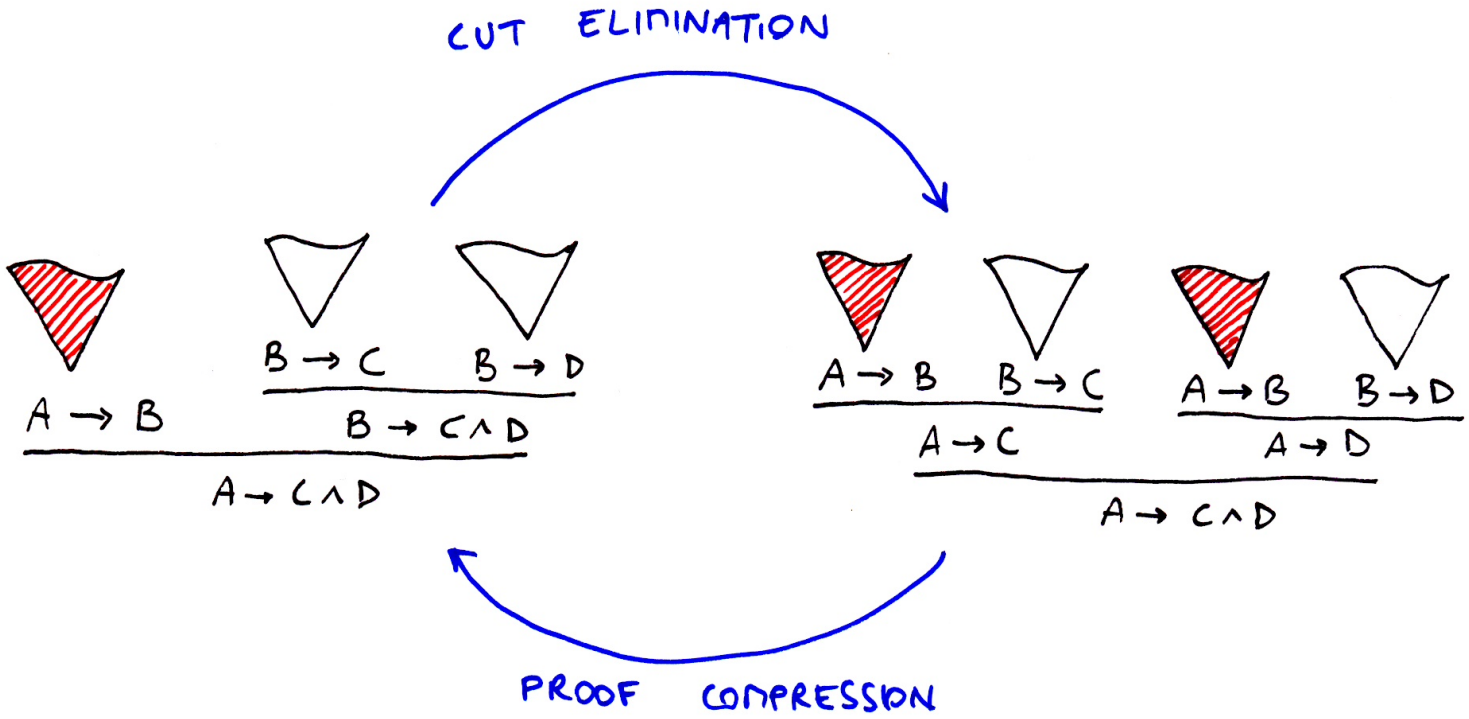
   Most studied, proof theory.

Idea of Cut Elimination



- Cuts are lifted and then eliminated against identity axioms.
- (Hyper-)exponential growth (in Gentzen).

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This talk answers YES to Question (1).

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- SKS is complete and implicationally complete for propositional logic.

Open Deduction

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(In [Guglielmi et al., 2010a].)

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$$\mathsf{m}\,\frac{\dfrac{a}{a \wedge a} \vee \dfrac{b}{b \wedge b}}{[a \vee b] \wedge [a \vee b]} \quad \wedge \quad \frac{a}{a \wedge a}$$

In Gentzen:

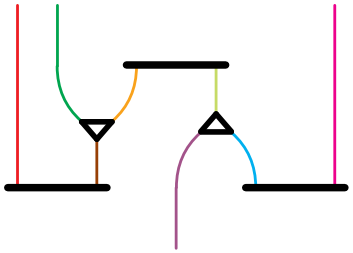
- no locality for (co)contraction (counterexample in [Brünnler, 2004]),
- no local reduction of cut into atomic form.

(Atomic) Flows

$$\mathsf{s}\,\dfrac{\mathsf{m}\,\dfrac{\dfrac{\mathsf{t}}{a \vee \bar{a}}}{[a \vee \mathsf{t}] \wedge [\mathsf{t} \vee \bar{a}]}}{\left[\mathsf{s}\,\dfrac{[a \vee \mathsf{t}] \wedge \bar{a}}{\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee \mathsf{t}} \quad \vee \quad \mathsf{t}\right]}$$

$$= \dfrac{\left(\mathsf{s}\,\dfrac{a \wedge \left[\bar{a} \vee \dfrac{\mathsf{t}}{\bar{a} \vee a}\right]}{a \wedge \dfrac{\dfrac{\bar{a} \vee \bar{a}}{\bar{a}}}{\mathsf{f}} \quad \vee \quad \dfrac{a}{a \wedge a}} \quad \wedge \quad \bar{a}\right)}{a \wedge \dfrac{a \wedge \bar{a}}{\mathsf{f}}}$$

$$\mathsf{m}\,\dfrac{\dfrac{a}{a \wedge a} \vee \dfrac{b}{b \wedge b}}{[a \vee b] \wedge [a \vee b]} \quad \wedge \quad \dfrac{a}{a \wedge a}$$



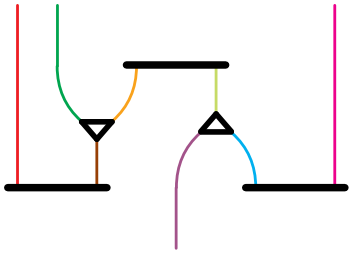
Below proofs, their (atomic) flows are shown:

(Atomic) Flows

$$\mathsf{s}\,\dfrac{\mathsf{m}\,\dfrac{\dfrac{\mathsf{t}}{a \vee \bar{a}}}{[a \vee \mathsf{t}] \wedge [\mathsf{t} \vee \bar{a}]}}{\left[\mathsf{s}\,\dfrac{[a \vee \mathsf{t}] \wedge \bar{a}}{\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee \mathsf{t}} \quad \vee \quad \mathsf{t}\right]}$$

$$= \dfrac{\left(\mathsf{s}\,\dfrac{a \wedge \left[\bar{a} \vee \dfrac{\mathsf{t}}{\bar{a} \vee a}\right]}{a \wedge \dfrac{\dfrac{\bar{a} \vee \bar{a}}{\bar{a}}}{\mathsf{f}} \quad \vee \quad \dfrac{a}{a \wedge a}} \quad \wedge \quad \bar{a}\right)}{a \wedge \dfrac{a \wedge \bar{a}}{\mathsf{f}}}$$

$$\mathsf{m}\,\dfrac{\dfrac{a}{a \wedge a} \vee \dfrac{b}{b \wedge b}}{[a \vee b] \wedge [a \vee b]} \quad \wedge \quad \dfrac{a}{a \wedge a}$$



Below proofs, their (atomic) flows are shown:

- only structural information is retained in flows;

(Atomic) Flows

$$\mathsf{s}\,\dfrac{\mathsf{m}\,\dfrac{\dfrac{\mathsf{t}}{a \vee \bar{a}}}{[a \vee \mathsf{t}] \wedge [\mathsf{t} \vee \bar{a}]}}{\left[\mathsf{s}\,\dfrac{[a \vee \mathsf{t}] \wedge \bar{a}}{\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee \mathsf{t}} \quad \vee \quad \mathsf{t}\right]}$$

$$= \dfrac{\left(\mathsf{s}\,\dfrac{a \wedge \left[\bar{a} \vee \dfrac{\mathsf{t}}{\bar{a} \vee a}\right]}{a \wedge \dfrac{\dfrac{\bar{a} \vee \bar{a}}{\bar{a}}}{\mathsf{f}} \quad \vee \quad \dfrac{a}{a \wedge a}} \quad \wedge \quad \bar{a}\right)}{a \wedge \dfrac{a \wedge \bar{a}}{\mathsf{f}}}$$

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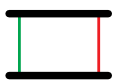

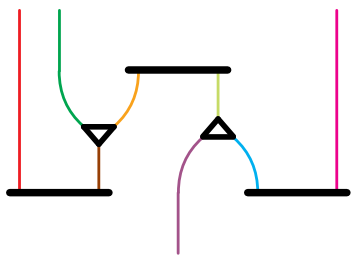

Below proofs, their (atomic) flows are shown:

- only structural information is retained in flows;
- logical information is lost;

(Atomic) Flows

$$\mathsf{s}\,\dfrac{\mathsf{m}\,\dfrac{\dfrac{\mathsf{t}}{a \vee \bar{a}}}{[a \vee \mathsf{t}] \wedge [\mathsf{t} \vee \bar{a}]}}{\left[\mathsf{s}\,\dfrac{[a \vee \mathsf{t}] \wedge \bar{a}}{\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee \mathsf{t}} \quad \vee \quad \mathsf{t}\right]}$$

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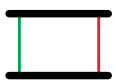
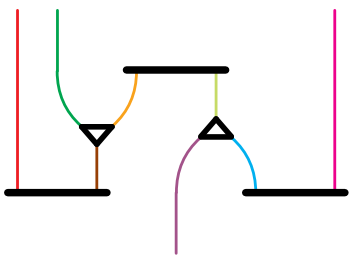
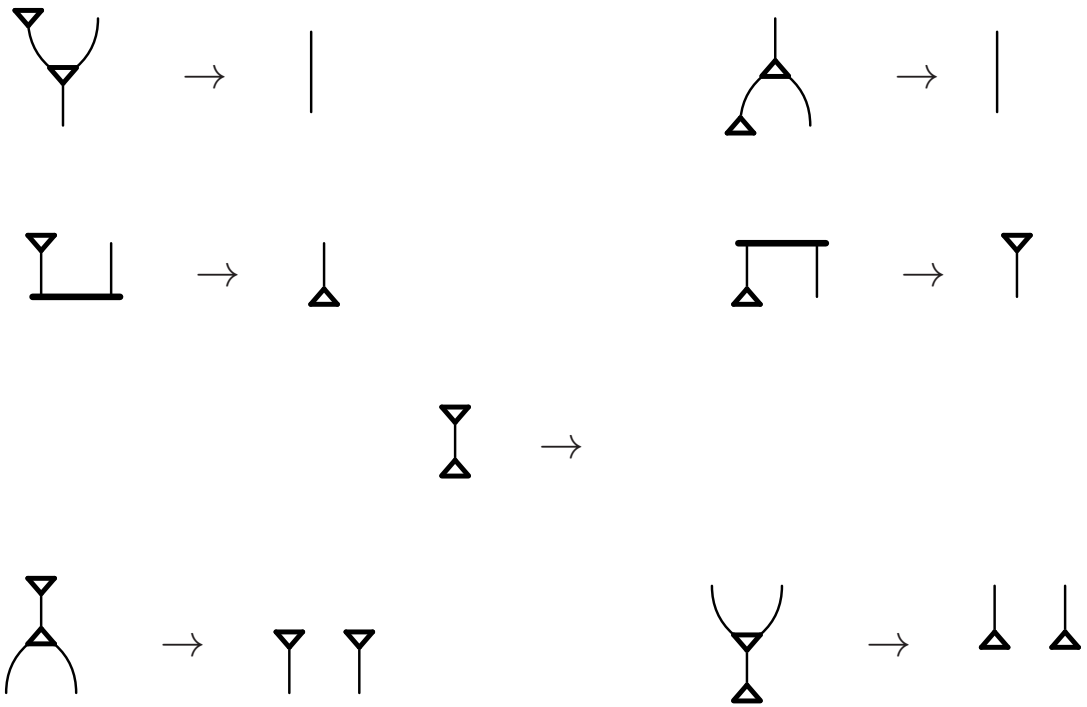




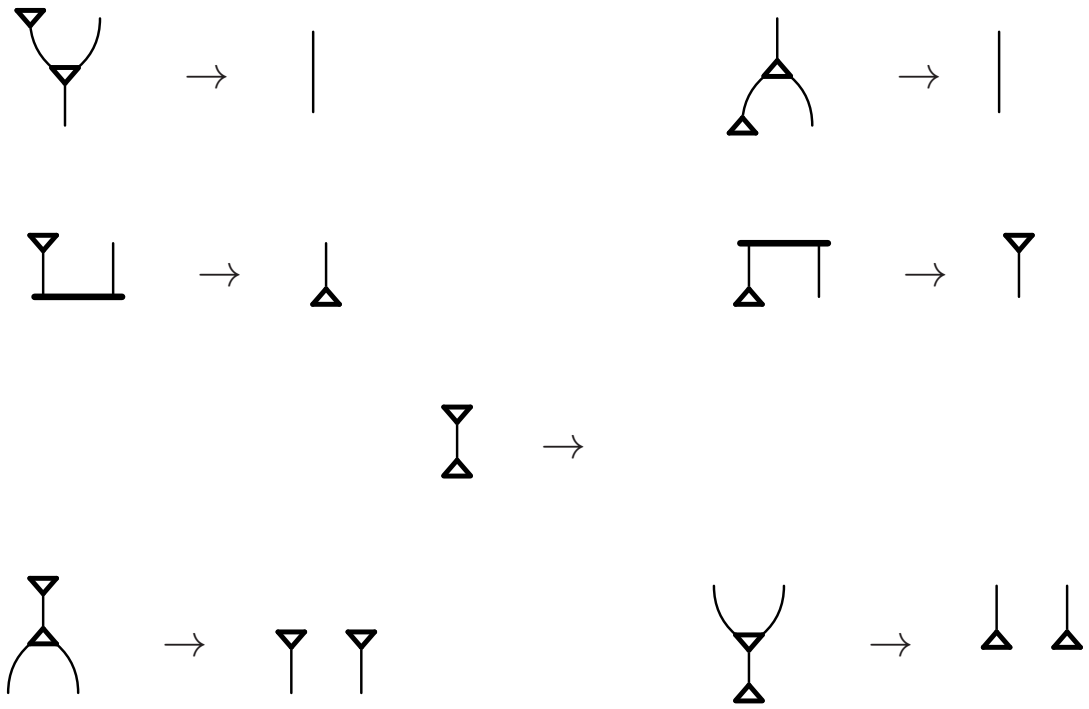
Below proofs, their (atomic) flows are shown:

- only structural information is retained in flows;
- logical information is lost;
- flow size is polynomially related to derivation size.

Flow Reductions: (Co)Weakening (1)

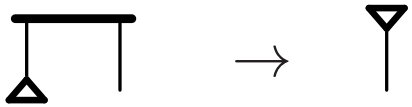


Flow Reductions: (Co)Weakening (1)



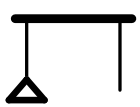
Each flow reduction corresponds to a correct proof reduction.

Flow Reductions: (Co)Weakening (2)

E.g.,  specifies that

$$\begin{array}{c} \Pi'' \Big\| \\ \xi\left\{\dfrac{\mathsf{t}}{a^{\epsilon} \vee \bar{a}}\right\} \\ \Phi \Big\| \\ \zeta\left\{\dfrac{a^{\epsilon}}{\mathsf{t}}\right\} \\ \Psi \Big\| \\ \alpha \end{array} \quad \text{becomes} \quad \begin{array}{c} \Pi'' \Big\| \\ \xi\left[\mathsf{t} \vee \dfrac{\mathsf{f}}{\bar{a}}\right] \\ \Phi\{a^{\epsilon}/\mathsf{t}\} \Big\| \\ \zeta\{\mathsf{t}\} \\ \Psi \Big\| \\ \alpha \end{array}$$

Flow Reductions: (Co)Weakening (2)

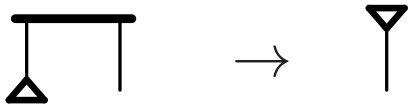
E.g.,  $\rightarrow$  specifies that

$$\begin{array}{c} \Pi'' \Vert \\ \xi\left\{\dfrac{\mathsf{t}}{a^{\epsilon} \vee \bar{a}}\right\} \\ \Phi \Vert \\ \zeta\left\{\dfrac{a^{\epsilon}}{\mathsf{t}}\right\} \\ \Psi \Vert \\ \alpha \end{array} \quad \text{becomes} \quad \begin{array}{c} \Pi'' \Vert \\ \xi\left[\mathsf{t} \vee \dfrac{\mathsf{f}}{\bar{a}}\right] \\ \Phi\{a^{\epsilon}/\mathsf{t}\} \Vert \\ \zeta\{\mathsf{t}\} \\ \Psi \Vert \\ \alpha \end{array}$$

We can operate on flow reductions instead than on derivations:

- much easier,

Flow Reductions: (Co)Weakening (2)

E.g.,  specifies that

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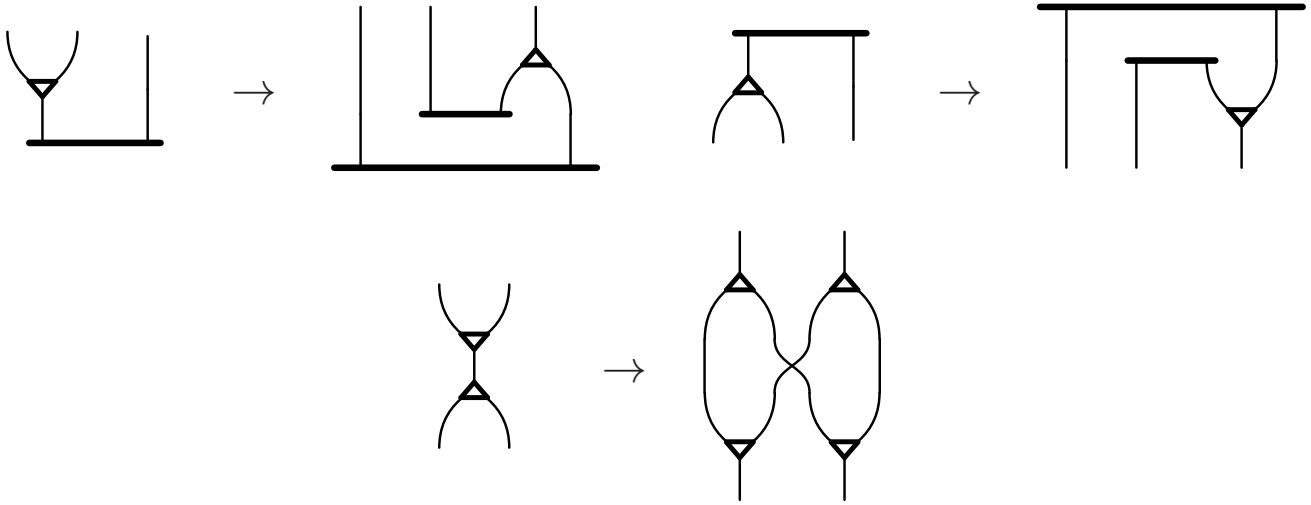
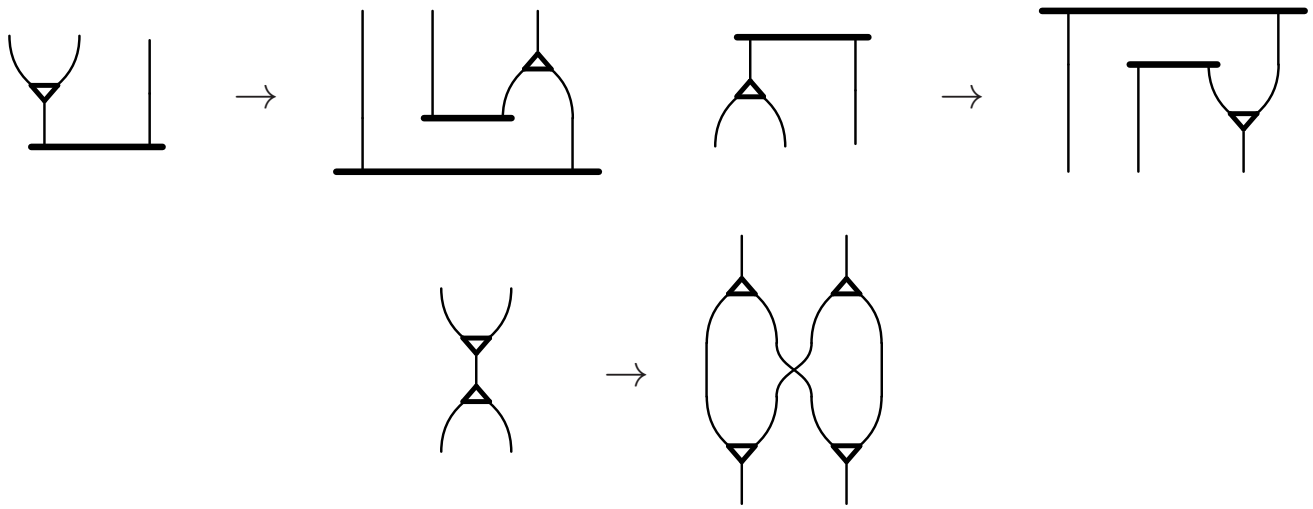
We can operate on flow reductions instead than on derivations:

- much easier,
- we get natural, syntax-independent induction measures.

Flow Reductions: (Co)Contraction

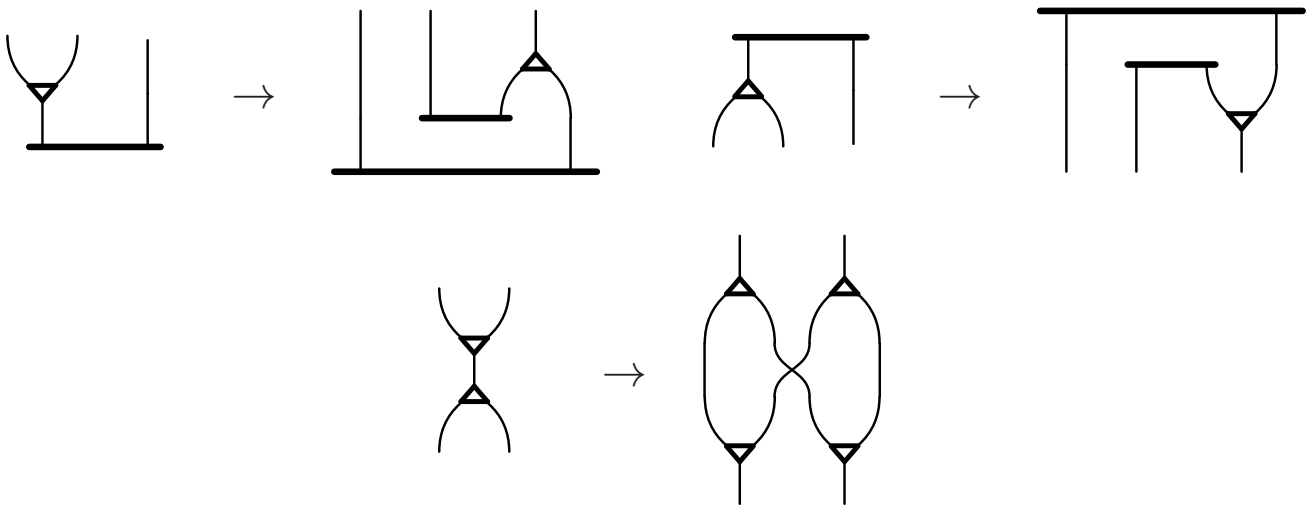


Flow Reductions: (Co)Contraction



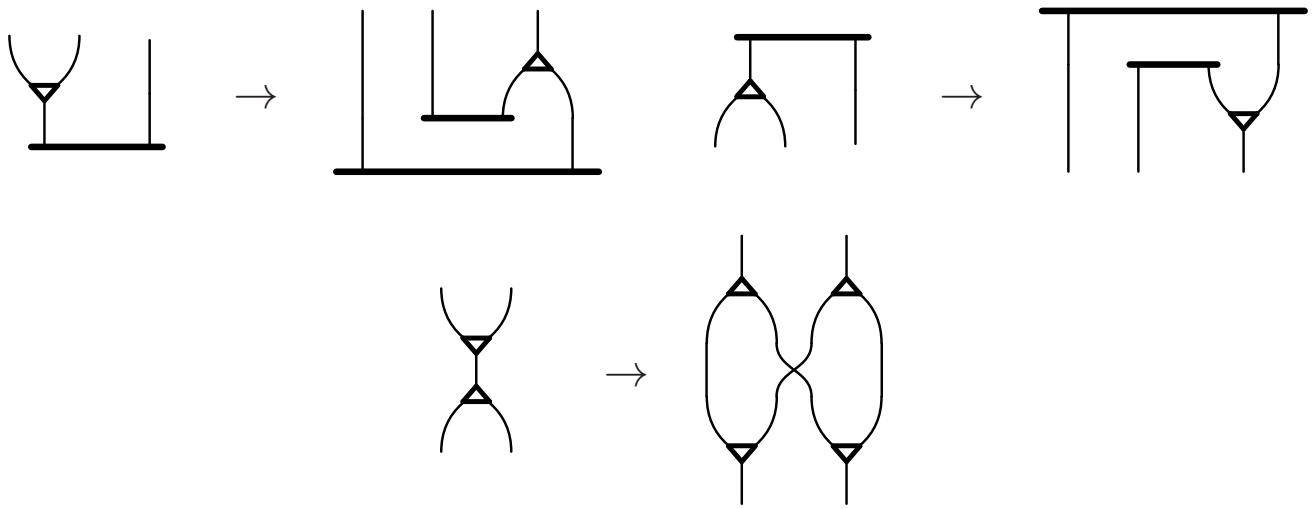
- These reductions conserve the number and length of paths.

Flow Reductions: (Co)Contraction



- These reductions conserve the number and length of paths.
- They can blow up a derivation exponentially.

Flow Reductions: (Co)Contraction



- These reductions conserve the number and length of paths.
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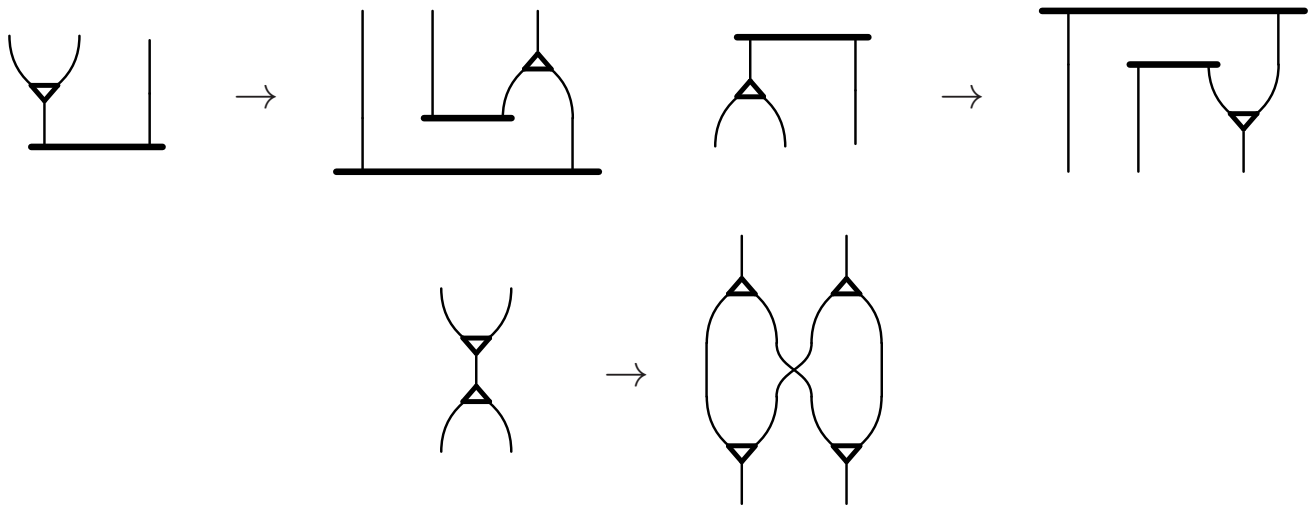
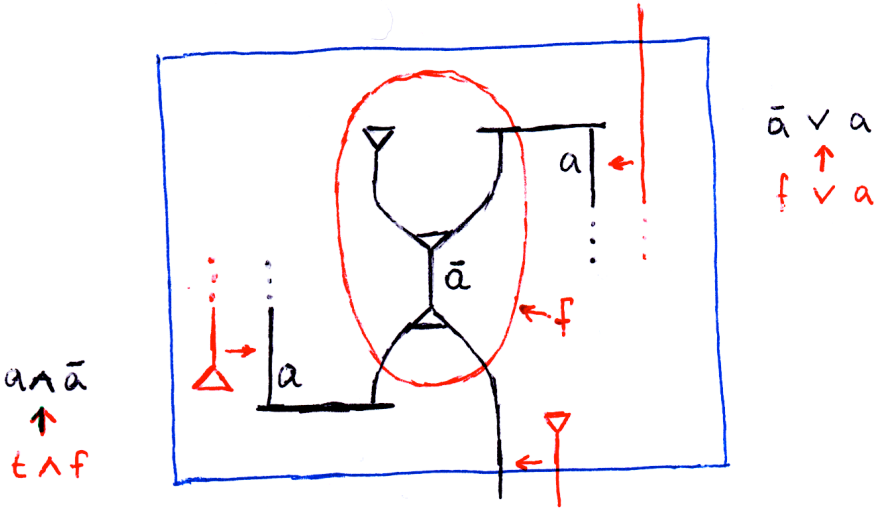
Flow Reductions: (Co)Contraction



- These reductions conserve the number and length of paths.
- They can blow up a derivation exponentially.
- It's a good thing: cocontraction is a new compression mechanism (dag-ness?).
- Open problem: does cocontraction yield exponential compression? Conjecture: yes.

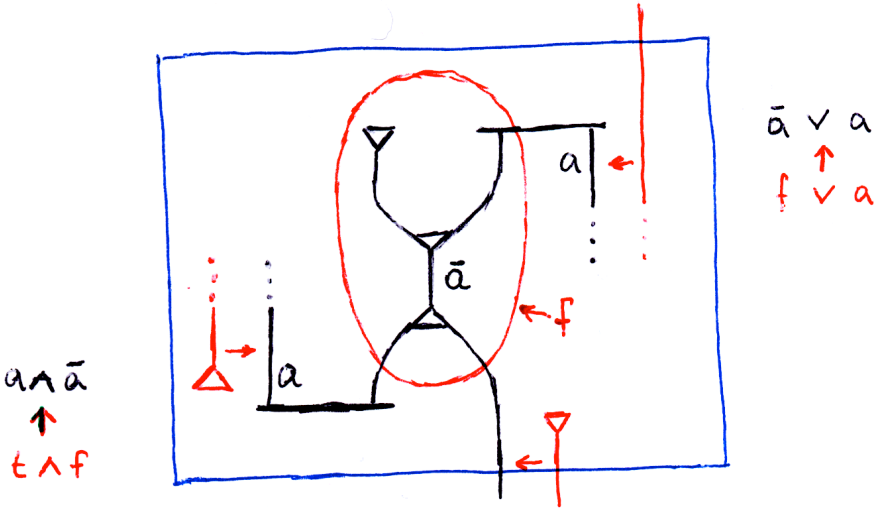
Cut Elimination by ‘Experiments’

Experiment:

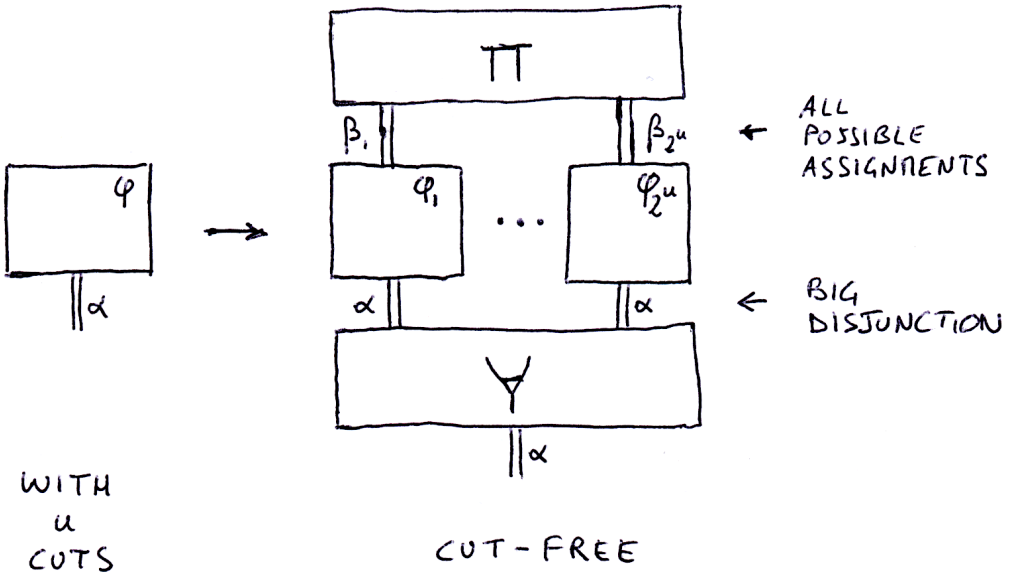


Cut Elimination by 'Experiments'

Experiment:

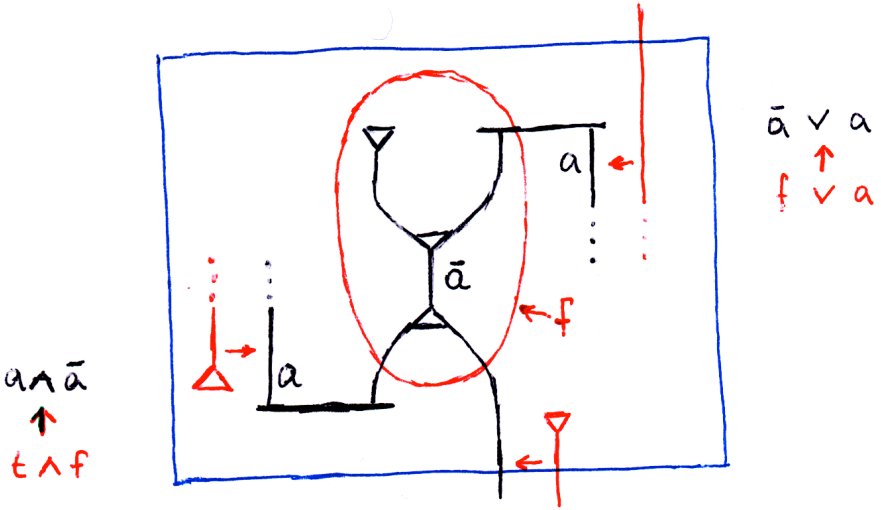


We do:

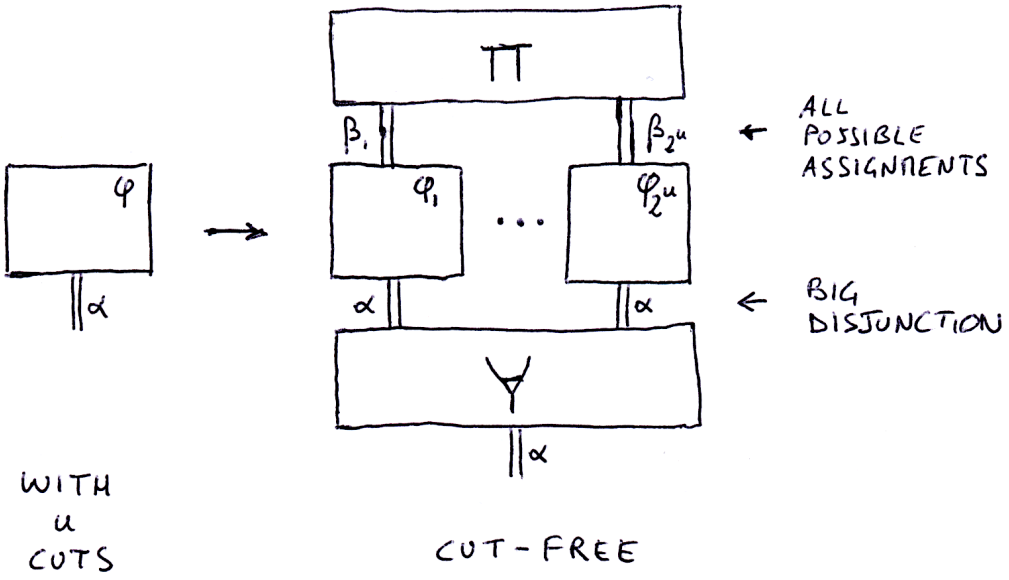


Cut Elimination by 'Experiments'

Experiment:

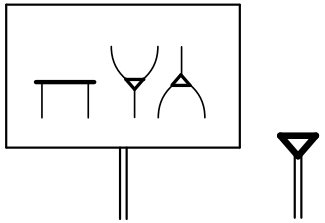


We do:



Simple, exponential cut elimination; proof generates 2^n experiments.

Generalising the Cut-Free Form

- Normalised proof: 

Generalising the Cut-Free Form

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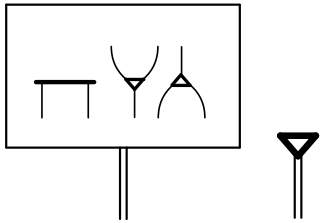
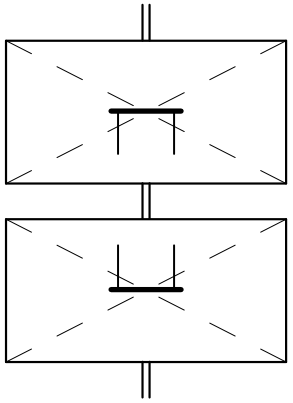
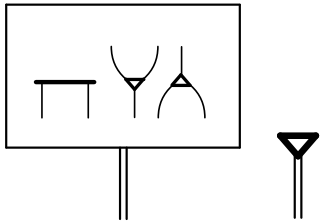
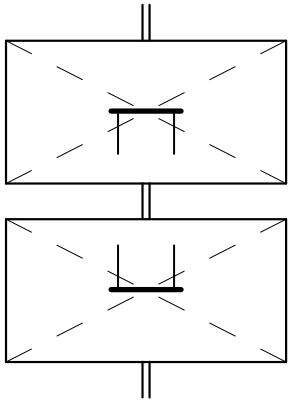


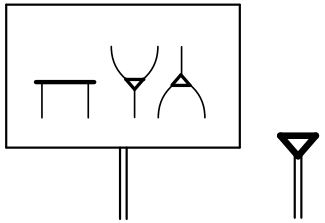
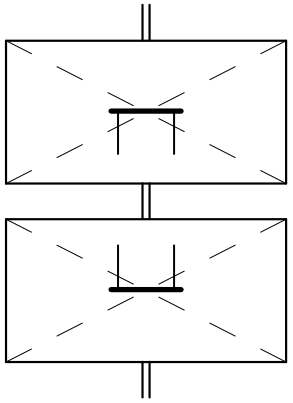
- Normalised derivation:



Generalising the Cut-Free Form

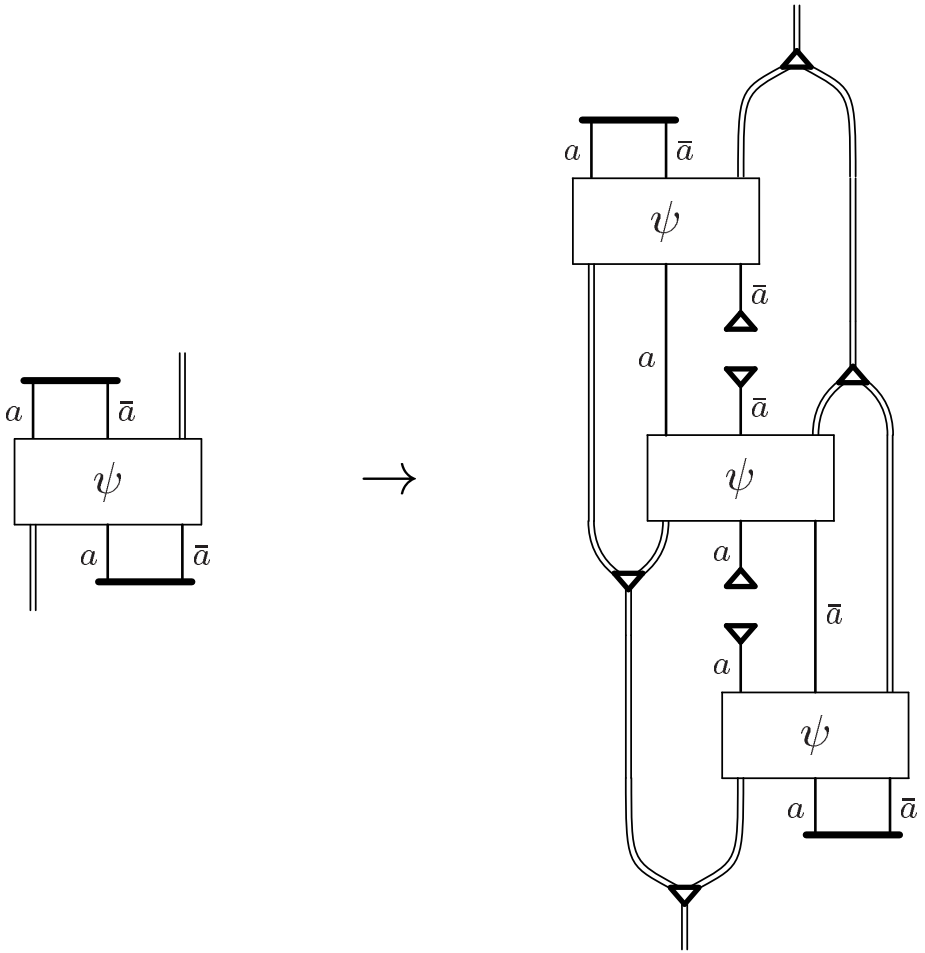
- Normalised proof: 
- Normalised derivation: 
- The symmetric form is called streamlined.
- Cut elimination is a corollary of streamlining.

Generalising the Cut-Free Form

- Normalised proof: 
- Normalised derivation: 
- The symmetric form is called streamlined.
- Cut elimination is a corollary of streamlining.
- We just need to break the paths between identities and cuts, and (co)weakenings do the rest.

How Do We Break Paths?

With the path breaker [Guglielmi et al., 2010b]:



Even if there is a path between identity and cut on the left, there is none on the right.

We Can Do This on Derivations, of Course

$$\begin{array}{c} A \\ \hline [a \vee \bar{a}] \wedge A \\ \Psi \Big\| \\ B \vee (a \wedge \bar{a}) \\ \hline B \end{array} \quad \longrightarrow \quad \begin{array}{c} A \\ \Big\| \{\mathsf{c}\uparrow, \mathsf{ai}\downarrow, =\} \\ (([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\ (\Psi \wedge A) \wedge A \Big\| \\ ([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\ \Phi_a \wedge A \Big\| \\ [B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\ [B \vee \Psi] \wedge A \Big\| \\ B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\ B \vee \Phi_{\bar{a}} \Big\| \\ B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\ B \vee [B \vee \Psi] \Big\| \\ B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\ \Big\| \{\mathsf{c}\downarrow, \mathsf{ai}\uparrow, =\} \\ B \end{array}$$

We Can Do This on Derivations, of Course

$$
\begin{array}{c}
A \\
\hline
[a \vee \bar{a}] \wedge A \\
{\scriptstyle \Psi}\Big\| \\
B \vee (a \wedge \bar{a}) \\
\hline
B
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
A \\
\Big\|{\scriptstyle \{\mathsf{c}\uparrow,\mathsf{ai}\downarrow,=\}} \\
(([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\
{\scriptstyle (\Psi \wedge A) \wedge A}\Big\| \\
([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\
{\scriptstyle \Phi_a \wedge A}\Big\| \\
[B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\
{\scriptstyle [B \vee \Psi] \wedge A}\Big\| \\
B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\
{\scriptstyle B \vee \Phi_{\bar{a}}}\Big\| \\
B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\
{\scriptstyle B \vee [B \vee \Psi]}\Big\| \\
B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\
\Big\|{\scriptstyle \{\mathsf{c}\downarrow,\mathsf{ai}\uparrow,=\}} \\
B
\end{array}
$$

- We can compose this as many times as there are paths between identities and cut.

We Can Do This on Derivations, of Course

$$
\begin{array}{c}
A \\
\hline
[a \vee \bar{a}] \wedge A \\
\Psi \Big\| \\
B \vee (a \wedge \bar{a}) \\
\hline
B
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
A \\
\Big\| \{\mathsf{c}\uparrow, \mathsf{ai}\downarrow, =\} \\
(([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\
(\Psi \wedge A) \wedge A \Big\| \\
([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\
\Phi_a \wedge A \Big\| \\
[B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\
[B \vee \Psi] \wedge A \Big\| \\
B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\
B \vee \Phi_{\bar{a}} \Big\| \\
B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\
B \vee [B \vee \Psi] \Big\| \\
B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\
\Big\| \{\mathsf{c}\downarrow, \mathsf{ai}\uparrow, =\} \\
B
\end{array}
$$

- We can compose this as many times as there are paths between identities and cut.
- We obtain a family of normalisers that only depends on n.

We Can Do This on Derivations, of Course

$$
\begin{array}{c}
A \\
\hline
[a \vee \bar{a}] \wedge A \\
\Psi \Big\| \\
B \vee (a \wedge \bar{a}) \\
\hline
B
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
A \\
\Big\| \{\mathsf{c}\uparrow, \mathsf{ai}\downarrow, =\} \\
(([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\
(\Psi \wedge A) \wedge A \Big\| \\
([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\
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[B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\
[B \vee \Psi] \wedge A \Big\| \\
B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\
B \vee \Phi_{\bar{a}} \Big\| \\
B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\
B \vee [B \vee \Psi] \Big\| \\
B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\
\Big\| \{\mathsf{c}\downarrow, \mathsf{ai}\uparrow, =\} \\
B
\end{array}
$$

- We can compose this as many times as there are paths between identities and cut.
- We obtain a family of normalisers that only depends on n.
- The construction is exponential.

We Can Do This on Derivations, of Course

$$
\begin{array}{c}
A \\ \hline
[a \vee \bar{a}] \wedge A \\
{\scriptstyle \Psi}\Big\| \\
B \vee (a \wedge \bar{a}) \\ \hline
B
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
A \\
\Big\| {\scriptstyle \{\mathsf{c}\uparrow, \mathsf{ai}\downarrow, =\}} \\
(([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\
{\scriptstyle (\Psi \wedge A) \wedge A}\Big\| \\
([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\
{\scriptstyle \Phi_a \wedge A}\Big\| \\
[B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\
{\scriptstyle [B \vee \Psi] \wedge A}\Big\| \\
B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\
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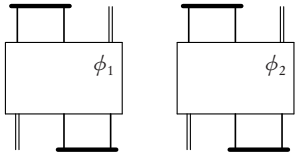
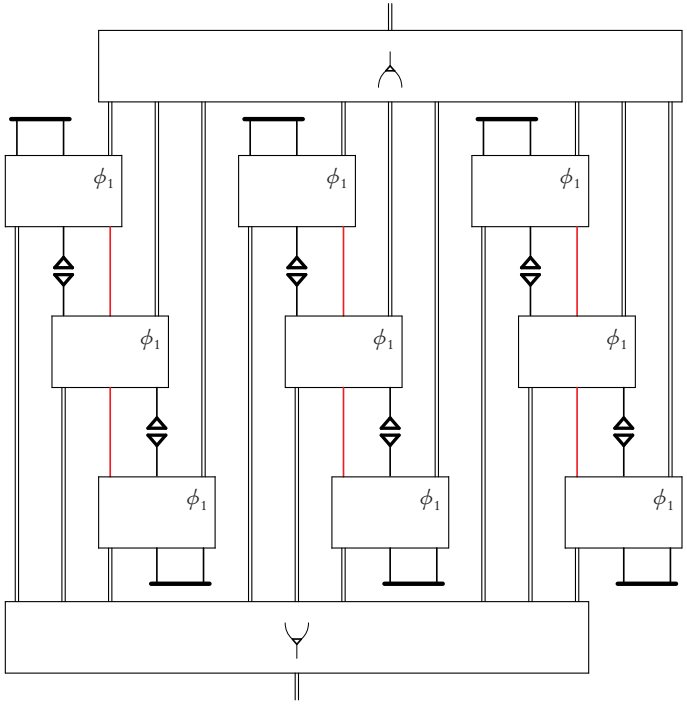
B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\
{\scriptstyle B \vee [B \vee \Psi]}\Big\| \\
B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\
\Big\| {\scriptstyle \{\mathsf{c}\downarrow, \mathsf{ai}\uparrow, =\}} \\
B
\end{array}
$$

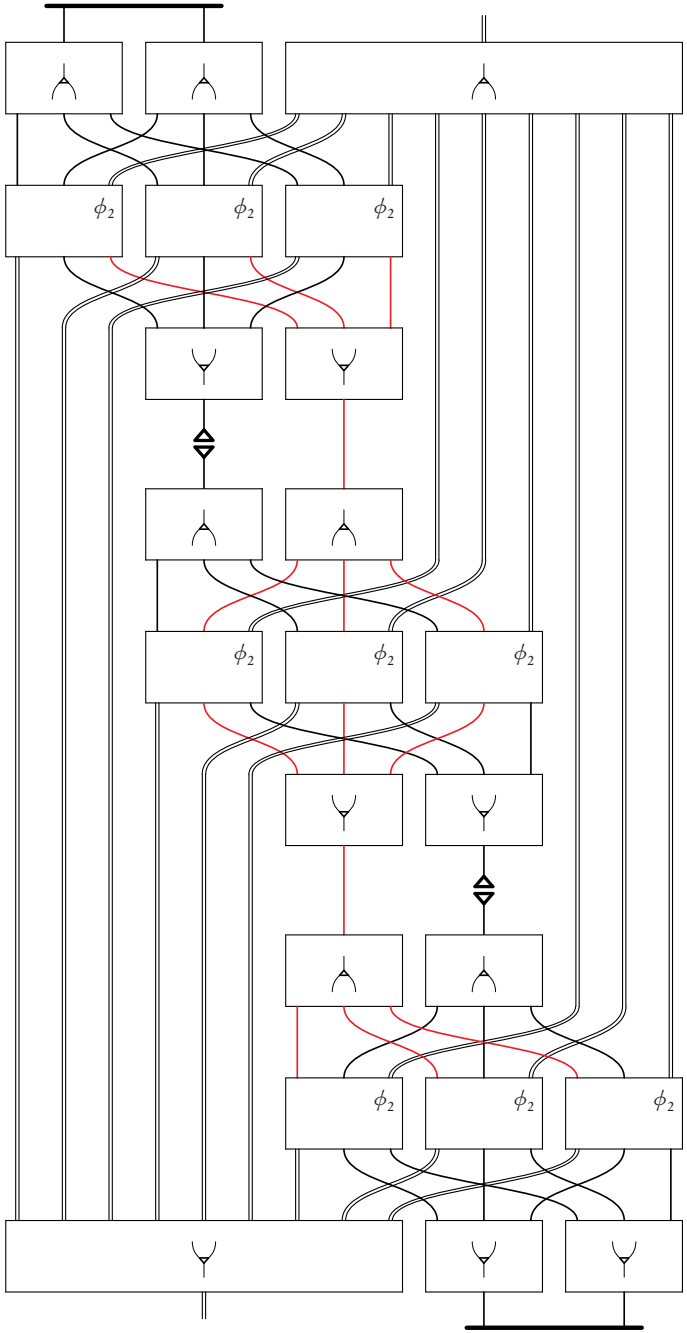
- We can compose this as many times as there are paths between identities and cut.
- We obtain a family of normalisers that only depends on n.
- The construction is exponential.
- Finding something like this is unthinkable without flows.

Example for $n = 2$

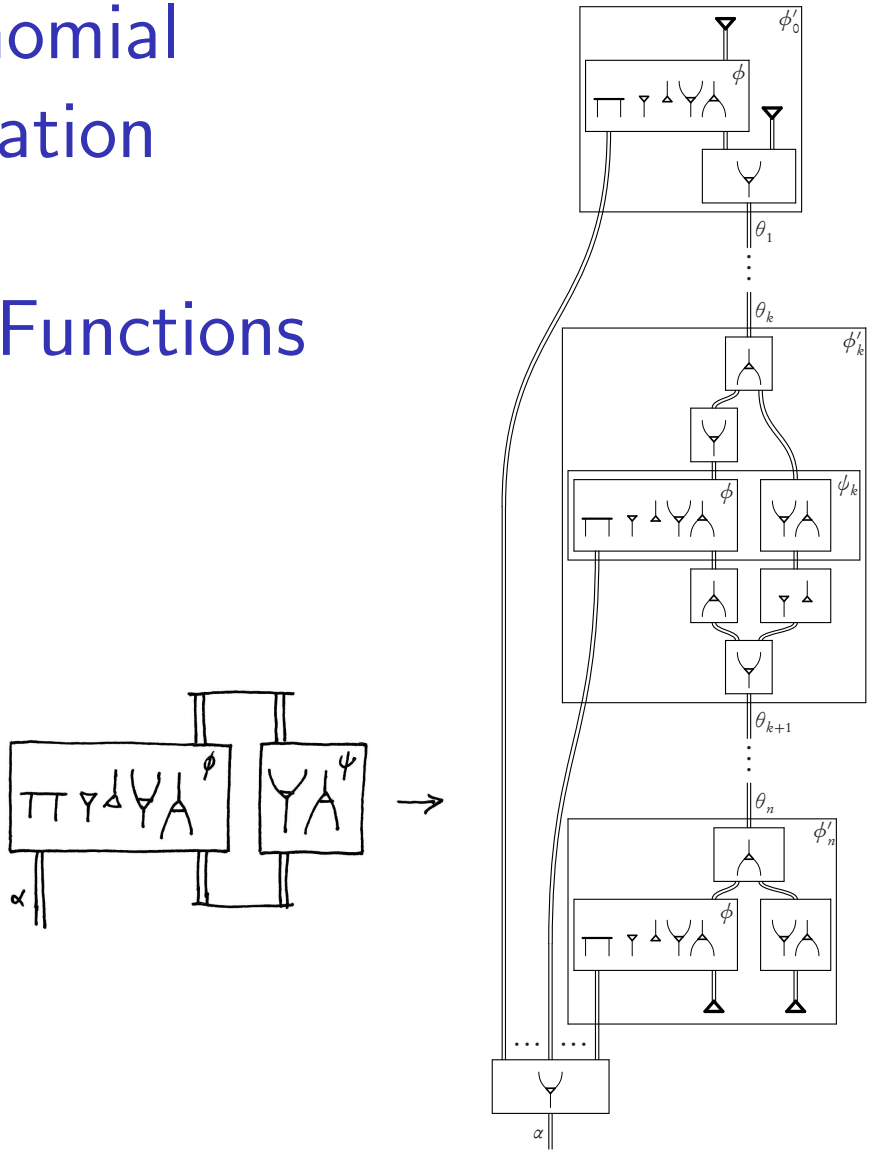


$\rightarrow$



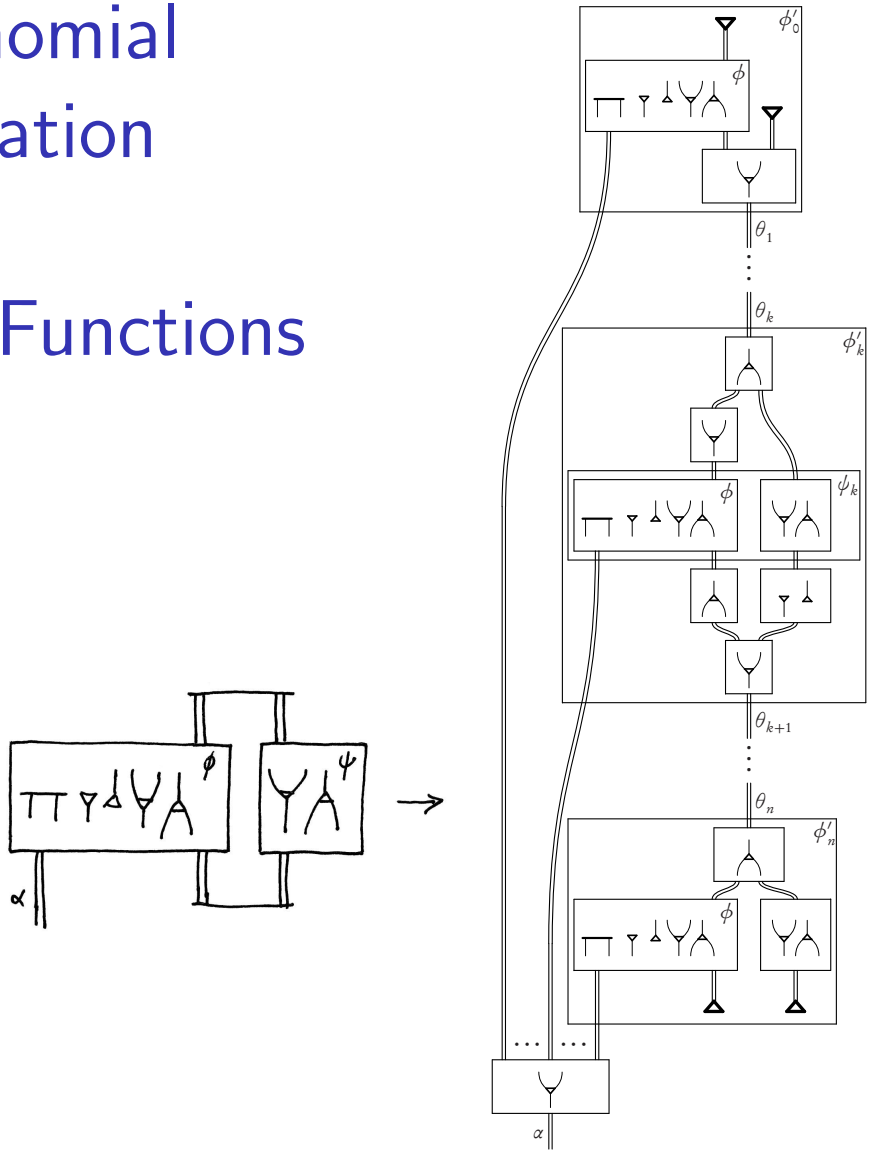


Quasipolynomial Cut Elimination by Threshold Functions



- Only $n+1$ copies of the proof are stitched together.

Quasipolynomial Cut Elimination by Threshold Functions



- Only $n+1$ copies of the proof are stitched together.
- Note local cocontraction (= better sharing, not available in Gentzen).

Normalisation
Overview

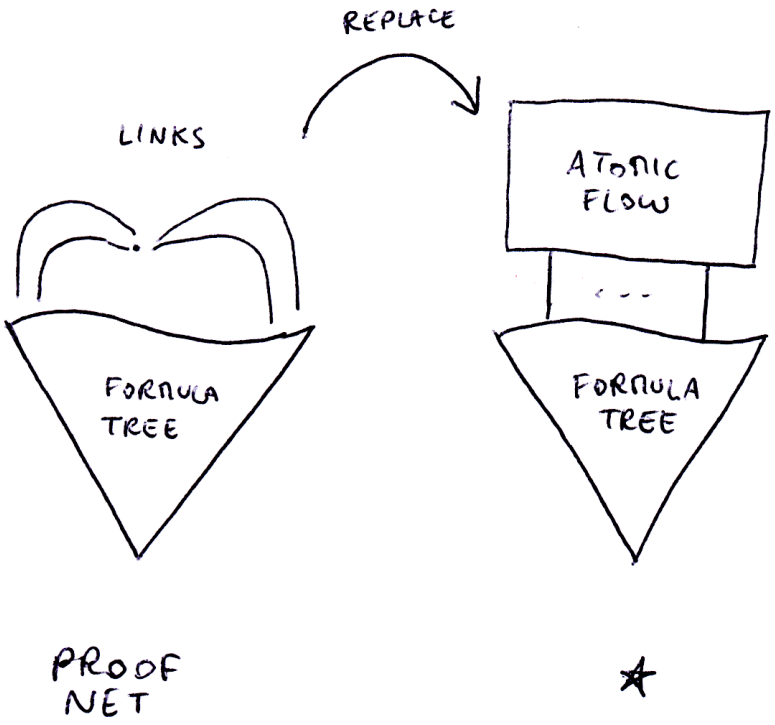
SYMMETRIC GENERALISATION

	CUT ELIMINATION	STREAMLINING
EXPONENTIAL	• SIMPLE EXPERIMENTS	• 'OPTIMISABLE' PROCEDURE (1) • 'PATH BREAKER' (2)
QUASIPOLYNOMIAL (I.E. $u^{O(\log u)}$)	• BY 'THRESHOLD FUNCTIONS' (3)	• THRESHOLD FUNCTIONS + PATH BREAKER (FORTHCOMING)

- None of these methods existed before atomic flows, none of them requires permutations or other syntactic devices.
- Quasipolynomial procedures are surprising.

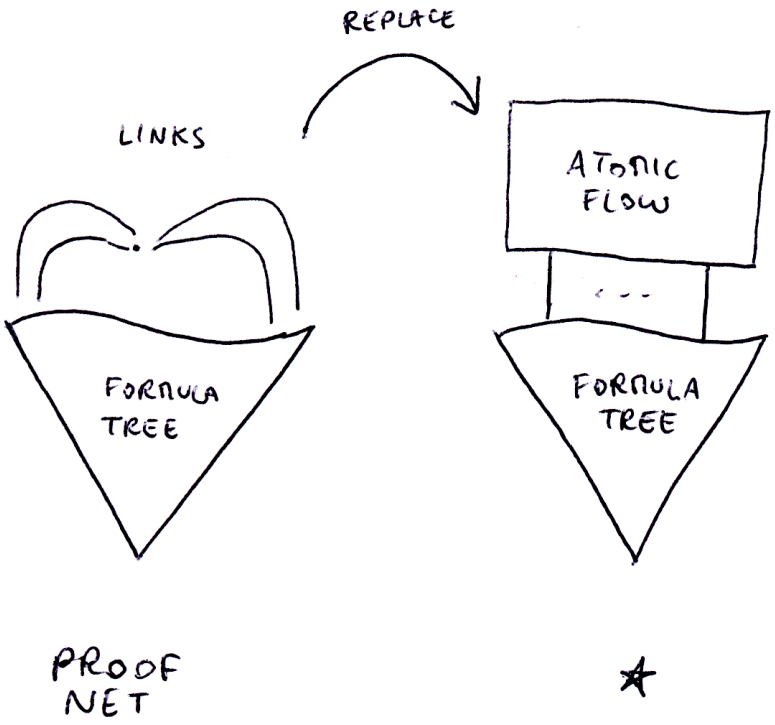
(1, 2) [Guglielmi et al., 2010b]; (3) [Bruscoli et al., 2010].

Conjecture



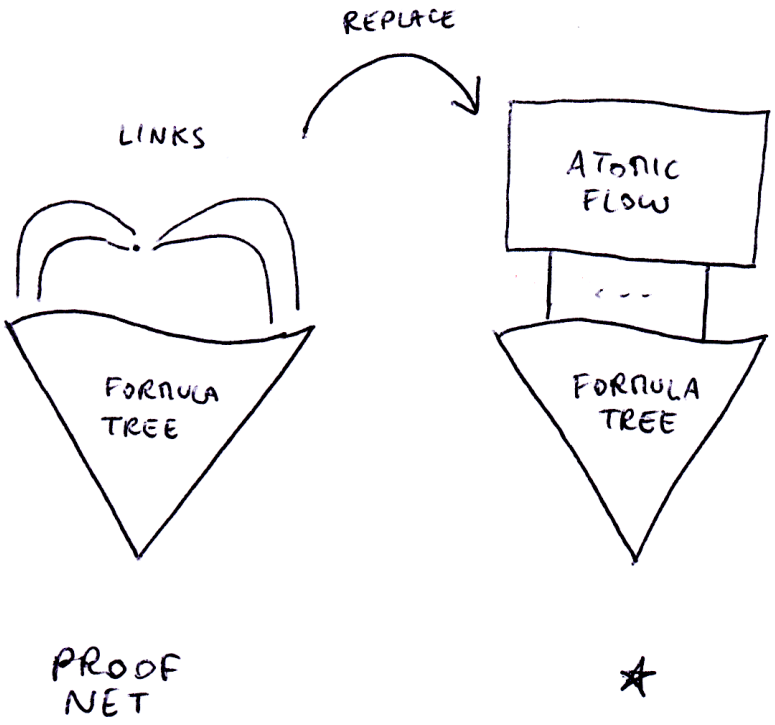
- We think that $*$ might make for a proof system.

Conjecture



- We think that $*$ might make for a proof system.
- If true, excellent bureaucracy-free formalism.

Conjecture



- We think that $*$ might make for a proof system.
- If true, excellent bureaucracy-free formalism.
- Note: if such a thing existed for proof nets, then coNP $=$ NP (because proof nets are [too?] small).

Conclusion

- Normalisation does not depend on logical rules.
- It only depends on structural information, *i.e.*, geometry.
- This is crucial progress for capturing the essence of proofs.

This talk is available at `http://cs.bath.ac.uk/ag/t/RDLS.pdf`

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