PROPOSAL FOR A NEW APPROACH TO QUANTIFICATION

Alessio Guglielmi – Bath

Work in progress with Benjamin Ralph – Bath
provisionally approved (?) by Michel Parigot – CNRS, Paris

Second FISP Meeting – Paris 8-10 June 2017

Talk available from my home page and at http://cs.bath.ac.uk/ag/t/PFANATQ.pdf
All about deep inference at http://alessio.guglielmi.name/res/cos

ANALYTICITY IN GENTZEN'S PROOF THEORY

Find an analytic proof of: 'There are two irrationals x and y such that x^y is rational'.

ANALYTICITY IN GENTZEN'S PROOF THEORY

Rx^y

Find an analytic proof of: 'There are two irrationals x and y such that x^y is rational'.

Analyticity in Gentzen's Proof Theory

$$\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$$

Find an analytic proof of: 'There are two irrationals x and y such that x^y is rational'.

Analyticity in Gentzen's Proof Theory

$$\bar{R}\sqrt{2},\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})$$

Find an analytic proof of: 'There are two irrationals x and y such that x^{y} is rational'.

ANALYTICITY IN GENTZEN'S PROOF THEORY

$$\neg_R\;\dfrac{R\sqrt{2}^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}}{\vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}}$$

$$w_R\;\dfrac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\bar{R}\sqrt{2} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}}$$

$$w_R\;\dfrac{R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}}{R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}}$$

$$2w_L\;\dfrac{\vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}}
\qquad
w_L\;\dfrac{\bar{R}\sqrt{2} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}}$$

$$\wedge_R\;\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \qquad \bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}\sqrt{2}}$$

$$w_L\;\dfrac{R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}}$$

$$\wedge_R\;\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}\sqrt{2} \qquad \bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}\sqrt{2} \wedge R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}}$$

$$\exists_R\;\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}\sqrt{2} \wedge R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists y.\,(\bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}y \wedge R(\sqrt{2}^{\sqrt{2}})^{y})}$$

$$\exists_R\;\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists y.\,(\bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}y \wedge R(\sqrt{2}^{\sqrt{2}})^{y})}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}$$

$$\wedge_R\;\dfrac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \qquad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}$$

$$w_R\;\dfrac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}$$

$$w_L\;\dfrac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}$$

$$\wedge_R\;\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y}) \qquad \bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}$$

$$\exists_R\;\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists y.\,(\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^{y}), \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}$$

$$\exists_R\;\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists y.\,(\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^{y}), \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y}), \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}$$

$$c_R\;\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y}), \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}$$

There are two irrationals x and y such that x^{y} is rational.

ANALYTICITY IN GENTZEN'S PROOF THEORY

$$
{\scriptstyle \wedge_R}\dfrac{
{\scriptstyle w_L}\dfrac{{\scriptstyle w_R}\dfrac{{\scriptstyle \wedge_R}\dfrac{\bar R\sqrt2\vdash\bar R\sqrt2 \qquad \bar R\sqrt2\vdash\bar R\sqrt2}{\bar R\sqrt2\vdash\bar R\sqrt2\wedge\bar R\sqrt2}}{\bar R\sqrt2\vdash\bar R\sqrt2\wedge\bar R\sqrt2,\ \exists x.\exists y.(\bar Rx\wedge\bar Ry\wedge Rx^y)}}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash\bar R\sqrt2\wedge\bar R\sqrt2,\ \exists x.\exists y.(\bar Rx\wedge\bar Ry\wedge Rx^y)}
\qquad
{\scriptstyle \exists_R}\dfrac{{\scriptstyle \exists_R}\dfrac{{\scriptstyle \wedge_R}\dfrac{
{\scriptstyle \wedge_R}\dfrac{
{\scriptstyle 2w_L}\dfrac{{\scriptstyle \neg_R}\dfrac{R\sqrt2^{\sqrt2}\vdash R\sqrt2^{\sqrt2}}{\vdash R\sqrt2^{\sqrt2},\ \bar R\sqrt2^{\sqrt2}}}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash R\sqrt2^{\sqrt2},\ \bar R\sqrt2^{\sqrt2}}
\qquad
{\scriptstyle w_L}\dfrac{{\scriptstyle w_R}\dfrac{\bar R\sqrt2\vdash\bar R\sqrt2}{\bar R\sqrt2\vdash R\sqrt2^{\sqrt2},\ \bar R\sqrt2}}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash R\sqrt2^{\sqrt2},\ \bar R\sqrt2}
}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash R\sqrt2^{\sqrt2},\ \bar R\sqrt2^{\sqrt2}\wedge\bar R\sqrt2}
\qquad
{\scriptstyle w_L}\dfrac{{\scriptstyle w_R}\dfrac{R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash R(\sqrt2^{\sqrt2})^{\sqrt2}}{R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash R\sqrt2^{\sqrt2},\ R(\sqrt2^{\sqrt2})^{\sqrt2}}}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash R\sqrt2^{\sqrt2},\ R(\sqrt2^{\sqrt2})^{\sqrt2}}
}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash R\sqrt2^{\sqrt2},\ \bar R\sqrt2^{\sqrt2}\wedge\bar R\sqrt2\wedge R{\color{red}(\sqrt2^{\sqrt2})^{\sqrt2}}}
}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash R\sqrt2^{\sqrt2},\ \exists y.(\bar R\sqrt2^{\sqrt2}\wedge\bar Ry\wedge R(\sqrt2^{\sqrt2})^y)}
}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash R\sqrt2^{\sqrt2},\ \exists x.\exists y.(\bar Rx\wedge\bar Ry\wedge Rx^y)}
}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash\bar R\sqrt2\wedge\bar R\sqrt2\wedge R{\color{red}\sqrt2^{\sqrt2}},\ \exists x.\exists y.(\bar Rx\wedge\bar Ry\wedge Rx^y)}
$$

$$
{\scriptstyle c_R}\dfrac{{\scriptstyle \exists_R}\dfrac{{\scriptstyle \exists_R}\dfrac{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash\bar R\sqrt2\wedge\bar R\sqrt2\wedge R{\color{red}\sqrt2^{\sqrt2}},\ \exists x.\exists y.(\bar Rx\wedge\bar Ry\wedge Rx^y)}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash\exists y.(\bar R\sqrt2\wedge\bar Ry\wedge R\sqrt2^{y}),\ \exists x.\exists y.(\bar Rx\wedge\bar Ry\wedge Rx^y)}}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash\exists x.\exists y.(\bar Rx\wedge\bar Ry\wedge Rx^y),\ \exists x.\exists y.(\bar Rx\wedge\bar Ry\wedge Rx^y)}}{\bar R\sqrt2,\ R(\sqrt2^{\sqrt2})^{\sqrt2}\vdash\exists x.\exists y.(\bar Rx\wedge\bar Ry\wedge Rx^y)}
$$

There are two irrationals x and y such that x^y is rational.
Proof: if ${\color{red}\sqrt2^{\sqrt2}}$ is rational, fine, otherwise take ${\color{red}(\sqrt2^{\sqrt2})^{\sqrt2}}$.

What is wrong with analyticity in Gentzen's proof theory

$$
\begin{prooftree}
\AxiomC{$R\sqrt{2}^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}$}
\RightLabel{$\neg_R$}
\UnaryInfC{$\vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}$}
\RightLabel{$2w_L$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}$}
\AxiomC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}$}
\RightLabel{w_R}
\UnaryInfC{$\bar{R}\sqrt{2} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}$}
\RightLabel{w_L}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}\sqrt{2}$}
\AxiomC{$R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$}
\RightLabel{w_R}
\UnaryInfC{$R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$}
\RightLabel{w_L}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}\sqrt{2} \wedge R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists y.(\bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}y \wedge R(\sqrt{2}^{\sqrt{2}})^{y})$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})$}
\AxiomC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}$}
\AxiomC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}$}
\RightLabel{w_R}
\UnaryInfC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})$}
\RightLabel{w_L}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists y.(\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^{y}), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y}), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})$}
\RightLabel{c_R}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})$}
\end{prooftree}
$$

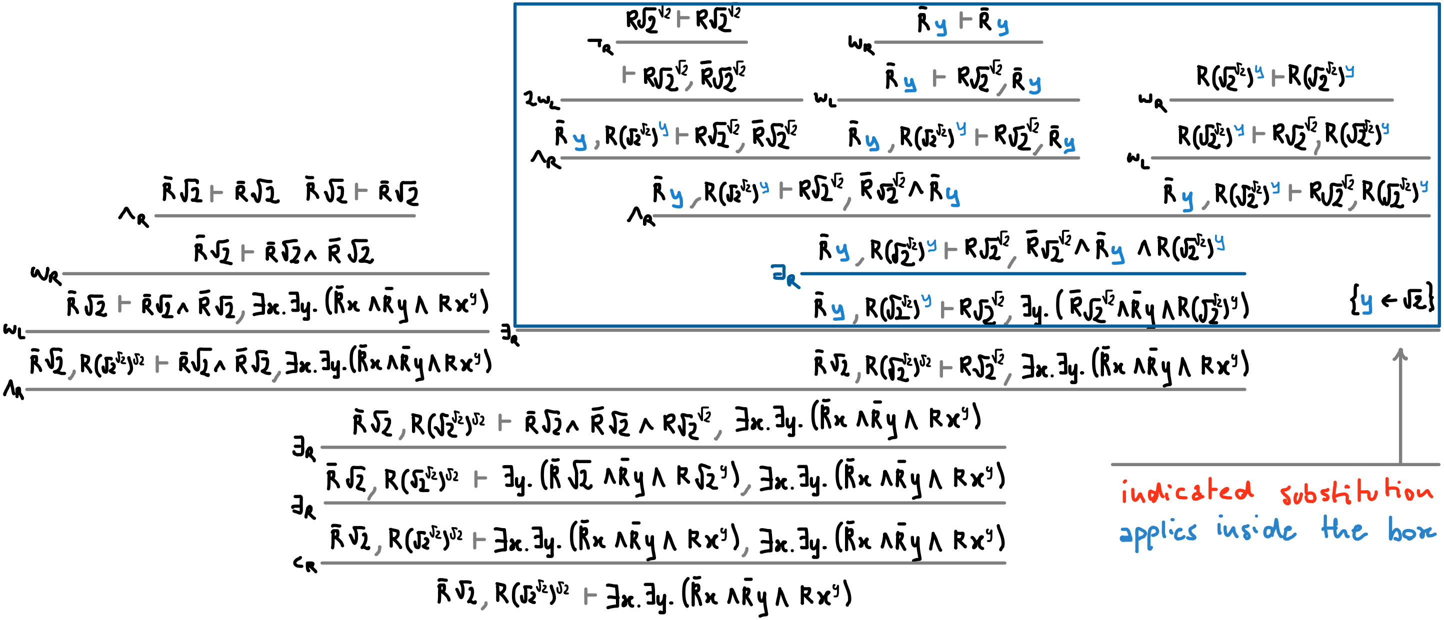
Infinite choice of witnesses: no real subformula property; no real analyticity.

GETTING REAL ANALYTICITY

$$
\begin{prooftree}
\AxiomC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}$}
\AxiomC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}$}
\RightLabel{w_R}
\UnaryInfC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{w_L}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2}, \exists y.(\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{c_R}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\end{prooftree}
$$

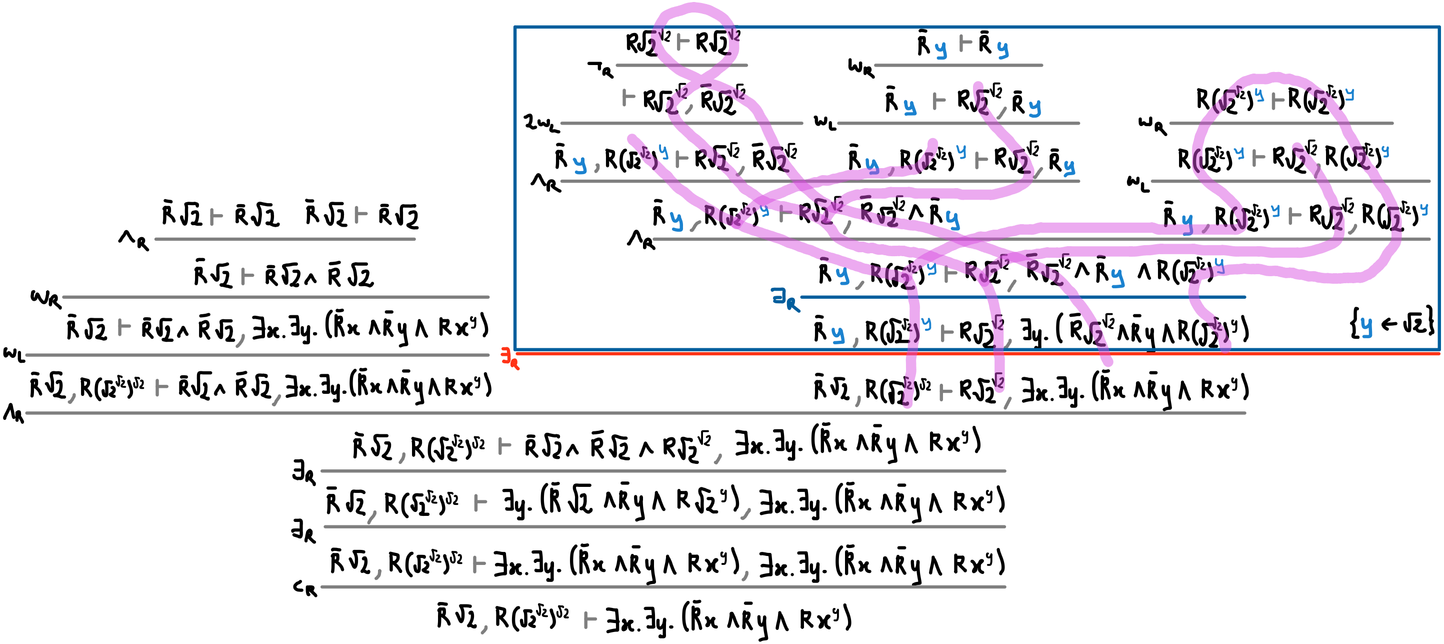
$$
\begin{prooftree}
\AxiomC{$R\sqrt{2}^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}$}
\RightLabel{$\neg_R$}
\UnaryInfC{$\vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}$}
\RightLabel{$2w_L$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}$}
\AxiomC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}$}
\RightLabel{w_R}
\UnaryInfC{$\bar{R}\sqrt{2} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}$}
\RightLabel{w_L}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}\sqrt{2}$}
\AxiomC{$R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$}
\RightLabel{w_R}
\UnaryInfC{$R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$}
\RightLabel{w_L}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}\sqrt{2} \wedge R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists y.(\bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}y \wedge R(\sqrt{2}^{\sqrt{2}})^y)$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\end{prooftree}
$$

GETTING REAL ANALYTICITY



indicated substitution applies inside the box

GETTING REAL ANALYTICITY



$$\dfrac{R\sqrt{2}^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}}{\vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}}\;\neg_R$$

$$\dfrac{\vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}}{\bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}}\;2w_L$$

$$\dfrac{\bar{R}y \vdash \bar{R}y}{\bar{R}y \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}y}\;w_R$$

$$\dfrac{\bar{R}y \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}y}{\bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}y}\;w_L$$

$$\dfrac{\bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \qquad \bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}y}{\bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}y}\;\wedge_R$$

$$\dfrac{R(\sqrt{2}^{\sqrt{2}})^y \vdash R(\sqrt{2}^{\sqrt{2}})^y}{R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^y}\;w_R$$

$$\dfrac{R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^y}{\bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^y}\;w_L$$

$$\dfrac{\bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}y \qquad \bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, R(\sqrt{2}^{\sqrt{2}})^y}{\bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}y \wedge R(\sqrt{2}^{\sqrt{2}})^y}\;\wedge_R$$

$$\dfrac{\bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}y \wedge R(\sqrt{2}^{\sqrt{2}})^y}{\bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, \exists y.(\bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}y \wedge R(\sqrt{2}^{\sqrt{2}})^y)}\;\exists_R$$

$\{y \leftarrow \sqrt{2}\}$

$$\dfrac{\bar{R}y, R(\sqrt{2}^{\sqrt{2}})^y \vdash R\sqrt{2}^{\sqrt{2}}, \exists y.(\bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}y \wedge R(\sqrt{2}^{\sqrt{2}})^y)}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\;\exists_R$$

$$\dfrac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \qquad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}\;\wedge_R$$

$$\dfrac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\;w_R$$

$$\dfrac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\;w_L$$

$$\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y) \qquad \bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\;\wedge_R$$

$$\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists y.(\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\;\exists_R$$

$$\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists y.(\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\;\exists_R$$

$$\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\;c_R$$

GETTING REAL ANALYTICITY

$$\wedge_R\;\dfrac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \qquad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}$$

$$w_R\;\dfrac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2},\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

$$w_L\;\dfrac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2},\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2},\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2},\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

$$\neg_R\;\dfrac{Rx \vdash Rx}{\vdash Rx,\ \bar{R}x}$$

$$2w_L\;\dfrac{\vdash Rx,\ \bar{R}x}{\bar{R}y,\ Rx^y \vdash Rx,\ \bar{R}x}$$

$$w_R\;\dfrac{\bar{R}y \vdash \bar{R}y}{\bar{R}y \vdash Rx,\ \bar{R}y}$$

$$w_L\;\dfrac{\bar{R}y \vdash Rx,\ \bar{R}y}{\bar{R}y,\ Rx^y \vdash Rx,\ \bar{R}y}$$

$$\wedge_R\;\dfrac{\bar{R}y,\ Rx^y \vdash Rx,\ \bar{R}x \qquad \bar{R}y,\ Rx^y \vdash Rx,\ \bar{R}y}{\bar{R}y,\ Rx^y \vdash Rx,\ \bar{R}x \wedge \bar{R}y}$$

$$w_R\;\dfrac{Rx^y \vdash Rx^y}{Rx^y \vdash Rx,\ Rx^y}$$

$$w_L\;\dfrac{Rx^y \vdash Rx,\ Rx^y}{\bar{R}y,\ Rx^y \vdash Rx,\ Rx^y}$$

$$\wedge_R\;\dfrac{\bar{R}y,\ Rx^y \vdash Rx,\ \bar{R}x \wedge \bar{R}y \qquad \bar{R}y,\ Rx^y \vdash Rx,\ Rx^y}{\bar{R}y,\ Rx^y \vdash Rx,\ \bar{R}x \wedge \bar{R}y \wedge Rx^y}$$

$$\exists_R\;\dfrac{\bar{R}y,\ Rx^y \vdash Rx,\ \bar{R}x \wedge \bar{R}y \wedge Rx^y}{\bar{R}y,\ Rx^y \vdash Rx,\ \exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \qquad \{y \leftarrow \sqrt{2}\}$$

$$\exists_R\;\dfrac{\bar{R}y,\ Rx^y \vdash Rx,\ \exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2},\ Rx^{\sqrt{2}} \vdash Rx,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \qquad \{x \leftarrow \sqrt{2}^{\sqrt{2}}\}$$

$$\wedge_R\;\dfrac{\bar{R}\sqrt{2},\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2},\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y) \qquad \bar{R}\sqrt{2},\ Rx^{\sqrt{2}} \vdash Rx,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2},\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\sqrt{2}},\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

$$\exists_R\;\dfrac{\bar{R}\sqrt{2},\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\sqrt{2}},\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2},\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists y.(\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^y),\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

$$\exists_R\;\dfrac{\bar{R}\sqrt{2},\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists y.(\bar{R}\sqrt{2} \wedge \bar{R}y \wedge R\sqrt{2}^y),\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2},\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y),\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

$$c_R\;\dfrac{\bar{R}\sqrt{2},\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y),\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2},\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

GETTING REAL ANALYTICITY

$$
\begin{prooftree}
\AxiomC{$R x \vdash R x$}
\RightLabel{$\neg_R$}
\UnaryInfC{$\vdash R x, \bar{R} x$}
\RightLabel{$2w_L$}
\UnaryInfC{$\bar{R} y,\ R x^y \vdash R x, \bar{R} x$}
\AxiomC{$\bar{R} y \vdash \bar{R} y$}
\RightLabel{w_R}
\UnaryInfC{$\bar{R} y \vdash R x, \bar{R} y$}
\RightLabel{w_L}
\UnaryInfC{$\bar{R} y,\ R x^y \vdash R x, \bar{R} y$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R} y,\ R x^y \vdash R x, \bar{R} x \wedge \bar{R} y$}
\AxiomC{$R x^y \vdash R x^y$}
\RightLabel{w_R}
\UnaryInfC{$R x^y \vdash R x,\ R x^y$}
\RightLabel{w_L}
\UnaryInfC{$\bar{R} y,\ R x^y \vdash R x,\ R x^y$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R} y,\ R x^y \vdash R x, \bar{R} x \wedge \bar{R} y \wedge R x^y$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R} y,\ R x^y \vdash R x, \exists y.(\bar{R} x \wedge \bar{R} y \wedge R x^y)$}
\RightLabel{$\{y \leftarrow \sqrt{2}\}$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}\sqrt{2},\ R x^{\sqrt{2}} \vdash R x, \exists x.\exists y.(\bar{R} x \wedge \bar{R} y \wedge R x^y)$}
\RightLabel{$\{x \leftarrow \sqrt{2}^{\sqrt{2}}\}$}
\AxiomC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}$}
\AxiomC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}$}
\RightLabel{w_R}
\UnaryInfC{$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.(\bar{R} x \wedge \bar{R} y \wedge R x^y)$}
\RightLabel{w_L}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x.\exists y.(\bar{R} x \wedge \bar{R} y \wedge R x^y)$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\sqrt{2}}, \exists x.\exists y.(\bar{R} x \wedge \bar{R} y \wedge R x^y)$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists y.(\bar{R}\sqrt{2} \wedge \bar{R} y \wedge R\sqrt{2}^y), \exists x.\exists y.(\bar{R} x \wedge \bar{R} y \wedge R x^y)$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R} x \wedge \bar{R} y \wedge R x^y), \exists x.\exists y.(\bar{R} x \wedge \bar{R} y \wedge R x^y)$}
\RightLabel{c_R}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R} x \wedge \bar{R} y \wedge R x^y)$}
\end{prooftree}
$$

GETTING REAL ANALYTICITY

$$\dfrac{Rx \vdash Rx}{\vdash Rx\,,\ \bar{R}x}\ \neg_R$$

$$\dfrac{\vdash Rx\,,\ \bar{R}x}{\bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ \bar{R}x}\ 2w_L$$

$$\dfrac{\bar{R}y \vdash \bar{R}y}{\bar{R}y \vdash Rx\,,\ \bar{R}y}\ w_R$$

$$\dfrac{\bar{R}y \vdash Rx\,,\ \bar{R}y}{\bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ \bar{R}y}\ w_L$$

$$\dfrac{\bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ \bar{R}x \qquad \bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ \bar{R}y}{\bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ \bar{R}x \wedge \bar{R}y}\ \wedge_R$$

$$\dfrac{Rx^{y} \vdash Rx^{y}}{Rx^{y} \vdash Rx\,,\ Rx^{y}}\ w_R$$

$$\dfrac{Rx^{y} \vdash Rx\,,\ Rx^{y}}{\bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ Rx^{y}}\ w_L$$

$$\dfrac{\bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ \bar{R}x \wedge \bar{R}y \qquad \bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ Rx^{y}}{\bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ \bar{R}x \wedge \bar{R}y \wedge Rx^{y}}\ \wedge_R$$

$$\dfrac{\bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ \bar{R}x \wedge \bar{R}y \wedge Rx^{y}}{\bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ \exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}\ \exists_R \qquad \{y \leftarrow z\}$$

$$\dfrac{\bar{R}y\,,\ Rx^{y} \vdash Rx\,,\ \exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\bar{R}z\,,\ Rx^{z} \vdash Rx\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}\ \exists_R \qquad \{x \leftarrow \sqrt{2}^{z}\}$$

$$\dfrac{\bar{R}z \vdash \bar{R}z \qquad \bar{R}z \vdash \bar{R}z}{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z}\ \wedge_R$$

$$\dfrac{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z}{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}\ w_R$$

$$\dfrac{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\bar{R}z\,,\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}z \wedge \bar{R}z\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}\ w_L$$

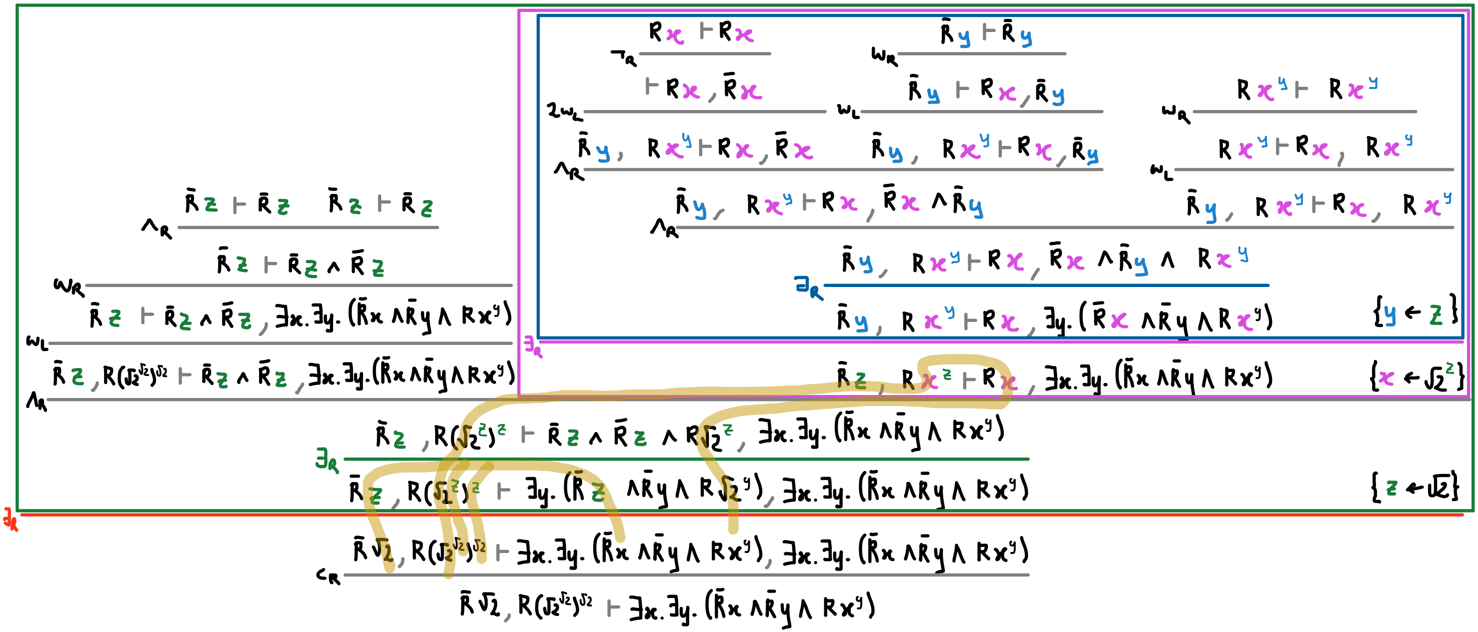
$$\dfrac{\bar{R}z\,,\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}z \wedge \bar{R}z\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y}) \qquad \bar{R}z\,,\ Rx^{z} \vdash Rx\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\bar{R}z\,,\ R(\sqrt{2}^{z})^{z} \vdash \bar{R}z \wedge \bar{R}z \wedge R\sqrt{2}^{z}\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}\ \wedge_R$$

$$\dfrac{\bar{R}z\,,\ R(\sqrt{2}^{z})^{z} \vdash \bar{R}z \wedge \bar{R}z \wedge R\sqrt{2}^{z}\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\bar{R}z\,,\ R(\sqrt{2}^{z})^{z} \vdash \exists y.(\bar{R}z \wedge \bar{R}y \wedge R\sqrt{2}^{y})\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}\ \exists_R \qquad \{z \leftarrow \sqrt{2}\}$$

$$\dfrac{\bar{R}z\,,\ R(\sqrt{2}^{z})^{z} \vdash \exists y.(\bar{R}z \wedge \bar{R}y \wedge R\sqrt{2}^{y})\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\bar{R}\sqrt{2}\,,\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}\ \exists_R$$

$$\dfrac{\bar{R}\sqrt{2}\,,\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})\,,\ \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\bar{R}\sqrt{2}\,,\ R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}\ c_R$$

GETTING REAL ANALYTICITY



$$\dfrac{Rx \vdash Rx}{\vdash Rx, \bar{R}x}\,\neg_R$$

$$\dfrac{\vdash Rx, \bar{R}x}{\bar{R}y, Rx^y \vdash Rx, \bar{R}x}\,2w_L$$

$$\dfrac{\bar{R}y \vdash \bar{R}y}{\bar{R}y \vdash Rx, \bar{R}y}\,w_R$$

$$\dfrac{\bar{R}y \vdash Rx, \bar{R}y}{\bar{R}y, Rx^y \vdash Rx, \bar{R}y}\,w_L$$

$$\dfrac{\bar{R}y, Rx^y \vdash Rx, \bar{R}x \qquad \bar{R}y, Rx^y \vdash Rx, \bar{R}y}{\bar{R}y, Rx^y \vdash Rx, \bar{R}x \wedge \bar{R}y}\,\wedge_R$$

$$\dfrac{Rx^y \vdash Rx^y}{Rx^y \vdash Rx, Rx^y}\,w_R$$

$$\dfrac{Rx^y \vdash Rx, Rx^y}{\bar{R}y, Rx^y \vdash Rx, Rx^y}\,w_L$$

$$\dfrac{\bar{R}y, Rx^y \vdash Rx, \bar{R}x \wedge \bar{R}y \qquad \bar{R}y, Rx^y \vdash Rx, Rx^y}{\bar{R}y, Rx^y \vdash Rx, \bar{R}x \wedge \bar{R}y \wedge Rx^y}\,\wedge_R$$

$$\dfrac{\bar{R}y, Rx^y \vdash Rx, \bar{R}x \wedge \bar{R}y \wedge Rx^y}{\bar{R}y, Rx^y \vdash Rx, \exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\,\exists_R \qquad \{y \leftarrow z\}$$

$$\dfrac{\bar{R}y, Rx^y \vdash Rx, \exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}z, Rx^z \vdash Rx, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\,\exists_R \qquad \{x \leftarrow \sqrt{2}^z\}$$

$$\dfrac{\bar{R}z \vdash \bar{R}z \qquad \bar{R}z \vdash \bar{R}z}{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z}\,\wedge_R$$

$$\dfrac{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z}{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\,w_R$$

$$\dfrac{\bar{R}z \vdash \bar{R}z \wedge \bar{R}z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}z, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}z \wedge \bar{R}z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\,w_L$$

$$\dfrac{\bar{R}z, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}z \wedge \bar{R}z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y) \qquad \bar{R}z, Rx^z \vdash Rx, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}z, R(\sqrt{2}^z)^z \vdash \bar{R}z \wedge \bar{R}z \wedge R\sqrt{2}^z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\,\wedge_R$$

$$\dfrac{\bar{R}z, R(\sqrt{2}^z)^z \vdash \bar{R}z \wedge \bar{R}z \wedge R\sqrt{2}^z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}z, R(\sqrt{2}^z)^z \vdash \exists y.(\bar{R}z \wedge \bar{R}y \wedge R\sqrt{2}^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\,\exists_R \qquad \{z \leftarrow \sqrt{2}\}$$

$$\dfrac{\bar{R}z, R(\sqrt{2}^z)^z \vdash \exists y.(\bar{R}z \wedge \bar{R}y \wedge R\sqrt{2}^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\,\exists_R$$

$$\dfrac{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)}\,c_R$$

GETTING REAL ANALYTICITY

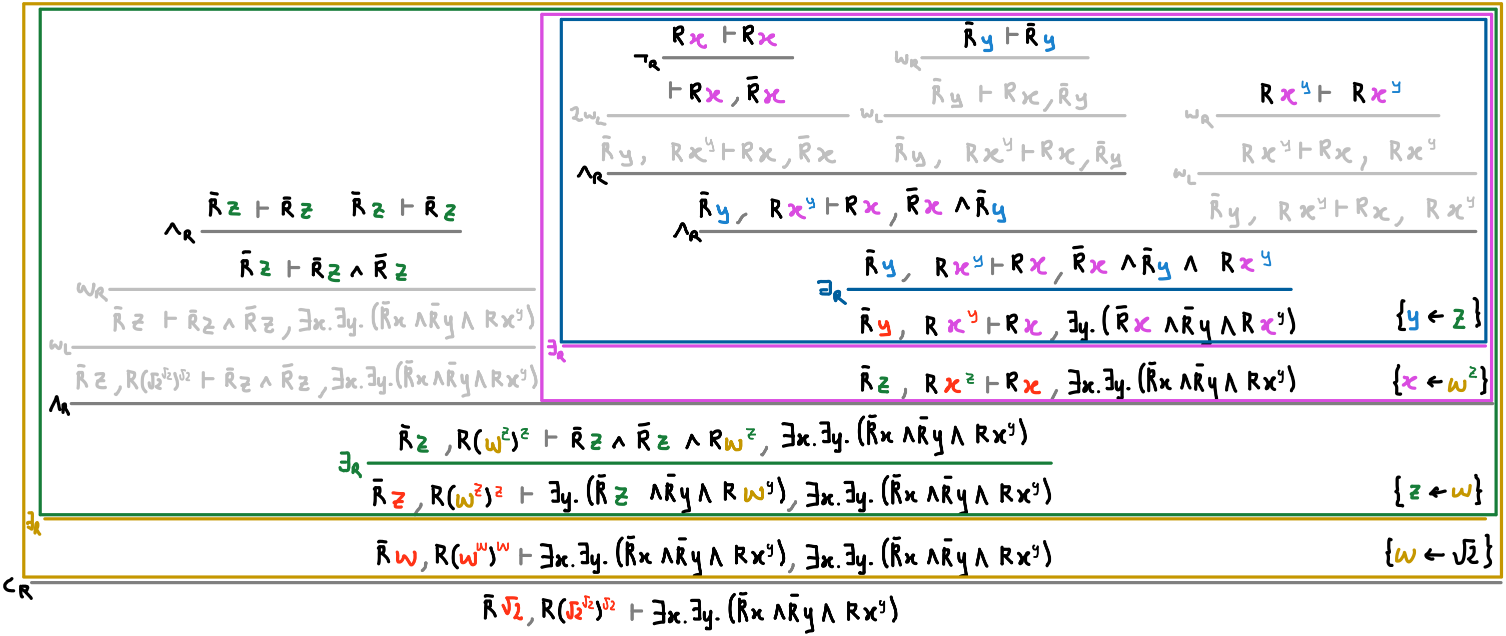
$$
\begin{prooftree}
\AxiomC{$Rx \vdash Rx$}
\RightLabel{$\neg_R$}
\UnaryInfC{$\vdash Rx, \bar{R}x$}
\AxiomC{$\bar{R}y \vdash \bar{R}y$}
\RightLabel{w_R}
\UnaryInfC{$\bar{R}y \vdash Rx, \bar{R}y$}
\RightLabel{w_L}
\BinaryInfC{$\bar{R}y, Rx^y \vdash Rx, \bar{R}x \quad \bar{R}y, Rx^y \vdash Rx, \bar{R}y$}
\RightLabel{$2w_L$}
\AxiomC{$Rx^y \vdash Rx^y$}
\RightLabel{w_R}
\UnaryInfC{$Rx^y \vdash Rx, Rx^y$}
\RightLabel{w_L}
\UnaryInfC{$\bar{R}y, Rx^y \vdash Rx, Rx^y$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}y, Rx^y \vdash Rx, \bar{R}x \wedge \bar{R}y \quad \bar{R}y, Rx^y \vdash Rx, Rx^y$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}y, Rx^y \vdash Rx, \bar{R}x \wedge \bar{R}y \wedge Rx^y$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}y, Rx^y \vdash Rx, \exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{$\{y \leftarrow z\}$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}z, Rx^z \vdash Rx, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{$\{x \leftarrow w^z\}$}
\AxiomC{$\bar{R}z \vdash \bar{R}z \quad \bar{R}z \vdash \bar{R}z$}
\RightLabel{$\wedge_R$}
\UnaryInfC{$\bar{R}z \vdash \bar{R}z \wedge \bar{R}z$}
\RightLabel{w_R}
\UnaryInfC{$\bar{R}z \vdash \bar{R}z \wedge \bar{R}z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{w_L}
\UnaryInfC{$\bar{R}z, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}z \wedge \bar{R}z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$\bar{R}z, R(w^z)^z \vdash \bar{R}z \wedge \bar{R}z \wedge Rw^z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}z, R(w^z)^z \vdash \exists y.(\bar{R}z \wedge \bar{R}y \wedge Rw^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{$\{z \leftarrow w\}$}
\RightLabel{$\exists_R$}
\UnaryInfC{$\bar{R}w, R(w^w)^w \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\RightLabel{$\{w \leftarrow \sqrt{2}\}$}
\RightLabel{c_R}
\UnaryInfC{$\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)$}
\end{prooftree}
$$

GETTING REAL ANALYTICITY

$$
\begin{array}{l}
\bar{R}\sqrt{2}, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y) \quad \{w \leftarrow \sqrt{2}\} \\
\hline
\bar{R}w, R(w^w)^w \vdash \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y) \quad \{z \leftarrow w\} \\
\hline
\bar{R}z, R(w^z)^z \vdash \exists y.(\bar{R}z \wedge \bar{R}y \wedge Rw^y), \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y) \quad \{z \leftarrow w\} \\
\hline
\bar{R}z, R(w^z)^z \vdash \bar{R}z \wedge \bar{R}z \wedge Rw^z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y) \quad \exists_R \\
\hline
\bar{R}z, R(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}z \wedge \bar{R}z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y) \quad \wedge_R \\
\hline
\bar{R}z \vdash \bar{R}z, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y) \quad w_L \\
\hline
\bar{R}z \vdash \bar{R}z \wedge \bar{R}z \quad w_R \\
\hline
\bar{R}z \vdash \bar{R}z \quad \bar{R}z \vdash \bar{R}z \quad \wedge_R \\
\hline
\bar{R}z, R x^z \vdash Rx, \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y) \quad \{x \leftarrow w^z\} \quad \exists_R \\
\hline
\bar{R}y, R x^y \vdash Rx, \exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y) \quad \{y \leftarrow z\} \quad \exists_R \\
\hline
\bar{R}y, R x^y \vdash Rx, \bar{R}x \wedge \bar{R}y \wedge Rx^y \quad \exists_R \\
\hline
\bar{R}y, R x^y \vdash Rx, \bar{R}x \wedge \bar{R}y \quad \wedge_R \\
\hline
\bar{R}y, R x^y \vdash Rx, \bar{R}x \quad 2w_L \qquad \bar{R}y, R x^y \vdash Rx, \bar{R}y \quad w_L \qquad R x^y \vdash Rx^y, R x^y \quad w_L \\
\hline
\vdash Rx, \bar{R}x \quad \neg_R \qquad \bar{R}y \vdash \bar{R}y \quad w_R \qquad R x^y \vdash Rx^y \quad w_R
\end{array}
$$

We can remove weakenings (for clarity).

GETTING REAL ANALYTICITY



We have a real subformula property
because terms that go up either disappear or eventually come down.

Nonelementary speed-up via quantifier shifts

$$
\begin{array}{rl}
 & \vdash \exists x.\forall y.(\overline{Px} \vee Py) \\
\hline
\exists & \vdash \exists x.\forall y.(\overline{Px} \vee Py),\ \exists x.\forall y.(\overline{Px} \vee Py) \\
\hline
\forall & \vdash \forall y.(\overline{Px} \vee Py),\ \exists x.\forall y.(\overline{Px} \vee Py) \\
\hline
\exists & \vdash \overline{Px} \vee Pz,\ \exists x.\forall y.(\overline{Px} \vee Py) \\
\hline
\forall & \vdash \overline{Px} \vee Pz,\ \forall y.(\overline{Pz} \vee Py) \\
\hline
\vee & \vdash \overline{Px} \vee Pz,\ \overline{Pz} \vee Py \\
\hline
\vee & \vdash \overline{Px},\ Pz,\ \overline{Pz} \vee Py \\
\hline
\vee & \vdash \overline{Px},\ Pz,\ \overline{Pz},\ Py \\
\hline
w & \vdash Pz,\ \overline{Pz},\ Py \\
\hline
w & \vdash Pz,\ \overline{Pz}
\end{array}
$$

Take a proof of the drinker formula.

Nonelementary speed-up via quantifier shifts

$$
\begin{array}{rl}
 & \vdash Pz,\ \overline{Pz} \\
\mathrm{w} & \hline
 & \vdash Pz,\ \overline{Pz},\ Py \\
\mathrm{w} & \hline
 & \vdash \overline{Px},\ Pz,\ \overline{Pz},\ Py \\
\vee & \hline
 & \vdash \overline{Px},\ Pz,\ \overline{Pz} \vee Py \\
\vee & \hline
 & \vdash \overline{Px} \vee Pz,\ \overline{Pz} \vee Py \\
\wedge & \hline
 & \vdash \overline{Px} \vee Pz,\ \forall y.(\overline{Pz} \vee Py) \\
\exists & \hline
 & \vdash \overline{Px} \vee Pz,\ \exists x.\forall y.(\overline{Px} \vee Py) \\
\forall & \hline
 & \vdash \forall y.(\overline{Px} \vee Py),\ \exists x.\forall y.(\overline{Px} \vee Py) \\
\exists & \hline
 & \vdash \exists x.\forall y.(\overline{Px} \vee Py),\ \exists x.\forall y.(\overline{Px} \vee Py) \\
\mathrm{c} & \hline
 & \vdash \exists x.\forall y.(\overline{Px} \vee Py)
\end{array}
$$

Take a proof of the drinker formula.

Gentzen bureaucracy demands contractions (and 'nonanalytic' $\exists$ rules)

Nonelementary speed-up via quantifier shifts

Corollary of Aguilera–Baaz*: Deep inference has a nonelementary speed-up over cut-free Gentzen proofs of the predicate calculus.

Moral reason:

$$
\begin{array}{rl}
 & \vdash Pz,\ \overline{Pz} \\ \hline
\mathsf{w} & \vdash Pz,\ \overline{Pz},\ Py \\ \hline
\mathsf{w} & \vdash \overline{Px},\ Pz,\ \overline{Pz},\ Py \\ \hline
\vee & \vdash \overline{Px},\ Pz,\ \overline{Pz} \vee Py \\ \hline
\vee & \vdash \overline{Px} \vee Pz,\ \overline{Pz} \vee Py \\ \hline
\vee & \vdash \overline{Px} \vee Pz,\ \forall y.(\overline{Pz} \vee Py) \\ \hline
\exists & \vdash \overline{Px} \vee Pz,\ \exists x.\forall y.(\overline{Px} \vee Py) \\ \hline
\forall & \vdash \forall y.(\overline{Px} \vee Py),\ \exists x.\forall y.(\overline{Px} \vee Py) \\ \hline
\exists & \vdash \exists x.\forall y.(\overline{Px} \vee Py),\ \exists x.\forall y.(\overline{Px} \vee Py) \\ \hline
\mathsf{c} & \vdash \exists x.\forall y.(\overline{Px} \vee Py)
\end{array}
$$

Take a proof of the drinker formula.

Gentzen bureaucracy demands contractions (and 'nonanalytic' $\exists$ rules) ...

... deep inference does not.

$$
\begin{array}{rl}
 & t \\ \hline
\mathsf{i} & \exists x.\overline{Px} \vee \forall y.Py \\ \hline
= & \exists x.(\overline{Px} \vee \forall y.Py) \\ \hline
= & \exists x.\forall y.(\overline{Px} \vee Py)
\end{array}
$$

* J.P. Aguilera and M. Baaz. Unsound Inferences Make Proofs Shorter. 2016, t.a. on JSL.

Nonelementary speed-up via quantifier shifts

Corollary of Aguilera–Baaz*: Deep inference has a nonelementary speed-up over cut-free Gentzen proofs of the predicate calculus.

Moral reason:

$$
\begin{array}{rl}
 & \vdash P z \,,\ \overline{P z} \\ \hline
\mathsf{w} & \vdash P z \,,\ \overline{P z} \,,\ P y \\ \hline
\mathsf{w} & \vdash \overline{P x} \,,\ P z \,,\ \overline{P z} \,,\ P y \\ \hline
\vee & \vdash \overline{P x} \,,\ P z \,,\ \overline{P z} \vee P y \\ \hline
\vee & \vdash \overline{P x} \vee P z \,,\ \overline{P z} \vee P y \\ \hline
\wedge & \vdash \overline{P x} \vee P z \,,\ \forall y.(\overline{P z} \vee P y) \\ \hline
\exists & \vdash \overline{P x} \vee P z \,,\ \exists x.\forall y.(\overline{P x} \vee P y) \\ \hline
\forall & \vdash \forall y.(\overline{P x} \vee P y) \,,\ \exists x.\forall y.(\overline{P x} \vee P y) \\ \hline
\exists & \vdash \exists x.\forall y.(\overline{P x} \vee P y) \,,\ \exists x.\forall y.(\overline{P x} \vee P y) \\ \hline
\mathsf{c} & \vdash \exists x.\forall y.(\overline{P x} \vee P y)
\end{array}
$$

Take a proof of the drinker formula.

Gentzen bureaucracy demands contractions (and 'nonanalytic' $\exists$ rules)...

... deep inference does not.

$$
\begin{array}{rl}
 & t \\ \hline
\mathsf{i} & \exists x.\overline{P x} \vee \forall y.P y \\ \hline
= & \exists x.(\overline{P x} \vee \forall y.P y) \\ \hline
= & \exists x.\forall y.(\overline{P x} \vee P y)
\end{array}
$$

Quantifier shift

* J.P. Aguilera and M. Baaz. Unsound Inferences Make Proofs Shorter. 2016, t.a. on JSL.

PROPOSAL FOR A BETTER PROOF SYSTEM WITH QUANTIFICATION

$$\bar{i}\,\frac{A \wedge \bar{A}}{f} \qquad \bar{w}\,\frac{A}{t} \qquad \bar{c}\,\frac{A}{A \wedge A} \qquad s\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$i\,\frac{t}{A \vee \bar{A}} \qquad w\,\frac{f}{A} \qquad c\,\frac{A \vee A}{A}$$

Rules of system SKS

Standard deep inference rules

PROPOSAL FOR A BETTER PROOF SYSTEM WITH QUANTIFICATION

$$\bar{i}\,\frac{A \wedge \bar{A}}{f} \qquad \bar{w}\,\frac{A}{t} \qquad \bar{c}\,\frac{A}{A \wedge A} \qquad s\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$i\,\frac{t}{A \vee \bar{A}} \qquad w\,\frac{f}{A} \qquad c\,\frac{A \vee A}{A}$$

Rules of system SKS

Standard deep inference rules

Connective exchange is analytic

PROPOSAL FOR A BETTER PROOF SYSTEM WITH QUANTIFICATION

Rules

$$\bar{i}\ \frac{A \wedge \bar{A}}{f} \qquad \bar{w}\ \frac{A}{t} \qquad \bar{c}\ \frac{A}{A \wedge A} \qquad s\ \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \qquad \forall\ \frac{\forall x.\, A}{A}$$

$$i\ \frac{t}{A \vee \bar{A}} \qquad w\ \frac{f}{A} \qquad c\ \frac{A \vee A}{A} \qquad \exists\ \frac{A}{\exists x.\, A}$$

New: quantifiers only protect variables

PROPOSAL FOR A BETTER PROOF SYSTEM WITH QUANTIFICATION

Rules

$$\bar{i}\,\frac{A \wedge \bar{A}}{f} \qquad \bar{w}\,\frac{A}{t} \qquad \bar{c}\,\frac{A}{A \wedge A} \qquad s\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \qquad \forall\,\frac{\forall x.\, A}{A}$$

$$i\,\frac{t}{A \vee \bar{A}} \qquad w\,\frac{f}{A} \qquad c\,\frac{A \vee A}{A} \qquad \exists\,\frac{A}{\exists x.\, A}$$

Equations of system SKS

Standard deep inference equations

$$(A \vee B) \vee C = A \vee (B \vee C)$$
$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

$$A \vee B = B \vee A$$
$$A \wedge B = B \wedge A$$

$$f \wedge f = f \qquad A \vee f = A$$
$$t \vee t = t \qquad A \wedge t = A$$

$$\exists x.\, f = \forall x.\, f = f$$
$$\exists x.\, t = \forall x.\, t = t$$

PROPOSAL FOR A BETTER PROOF SYSTEM WITH QUANTIFICATION

Rules

$$\bar{i}\ \frac{A \wedge \bar{A}}{f} \qquad \bar{w}\ \frac{A}{t} \qquad \bar{c}\ \frac{A}{A \wedge A} \qquad s\ \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \qquad \forall\ \frac{\forall x.\, A}{A}$$

$$i\ \frac{t}{A \vee \bar{A}} \qquad w\ \frac{f}{A} \qquad c\ \frac{A \vee A}{A} \qquad \exists\ \frac{A}{\exists x.\, A}$$

Equations

$$(A \vee B) \vee C = A \vee (B \vee C)$$
$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

$$A \vee B = B \vee A$$
$$A \wedge B = B \wedge A$$

$$f \wedge f = f \qquad A \vee f = A$$
$$t \vee t = t \qquad A \wedge t = A$$

$$\exists x.\, f = \forall x.\, f = f$$
$$\exists x.\, t = \forall x.\, t = t$$

Quantifier shifts

$$\left.\begin{aligned} \exists x.\, A \vee B &= \exists x.\, (A \vee B) \\ \exists x.\, A \wedge B &= \exists x.\, (A \wedge B) \\ \forall x.\, A \vee B &= \forall x.\, (A \vee B) \\ \forall x.\, A \wedge B &= \forall x.\, (A \wedge B) \end{aligned}\right\} \quad x \notin fv\, B$$

PROPOSAL FOR A BETTER PROOF SYSTEM WITH QUANTIFICATION

Rules

$$\bar{i}\,\frac{A \wedge \bar{A}}{f} \qquad \bar{w}\,\frac{A}{t} \qquad \bar{c}\,\frac{A}{A \wedge A} \qquad s\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \qquad \forall\,\frac{\forall x.\, A}{A}$$

$$i\,\frac{t}{A \vee \bar{A}} \qquad w\,\frac{f}{A} \qquad c\,\frac{A \vee A}{A} \qquad \exists\,\frac{A}{\exists x.\, A}$$

Equations

$$(A \vee B) \vee C = A \vee (B \vee C)$$
$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

$$A \vee B = B \vee A$$
$$A \wedge B = B \wedge A$$

$$f \wedge f = f \qquad A \vee f = A$$
$$t \vee t = t \qquad A \wedge t = A$$

$$\exists x.\, f = \forall x.\, f = f$$
$$\exists x.\, t = \forall x.\, t = t$$

$$\left.\begin{aligned} \exists x.\, A \vee B &= \exists x.\, (A \vee B) \\ \exists x.\, A \wedge B &= \exists x.\, (A \wedge B) \\ \forall x.\, A \vee B &= \forall x.\, (A \vee B) \\ \forall x.\, A \wedge B &= \forall x.\, (A \wedge B) \end{aligned}\right\} \quad x \notin fv\, B$$

$$\varphi\{x \leftarrow \tau\} = \varphi\{x \Leftarrow \tau\}$$

Now: substitutions

PROPOSAL FOR A BETTER PROOF SYSTEM WITH QUANTIFICATION

Rules

$$\bar{i}\,\frac{A \wedge \bar{A}}{f} \qquad \bar{w}\,\frac{A}{t} \qquad \bar{c}\,\frac{A}{A \wedge A} \qquad s\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \qquad \forall\,\frac{\forall x.\, A}{A}$$

$$i\,\frac{t}{A \vee \bar{A}} \qquad w\,\frac{f}{A} \qquad c\,\frac{A \vee A}{A} \qquad \exists\,\frac{A}{\exists x.\, A}$$

Equations

$$(A \vee B) \vee C = A \vee (B \vee C)$$
$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

$$A \vee B = B \vee A$$
$$A \wedge B = B \wedge A$$

$$f \wedge f = f \qquad A \vee f = A$$
$$t \vee t = t \qquad A \wedge t = A$$

$$\exists x.\, f = \forall x.\, f = f$$
$$\exists x.\, t = \forall x.\, t = t$$

$$\left.\begin{aligned} \exists x.\, A \vee B &= \exists x.\, (A \vee B) \\ \exists x.\, A \wedge B &= \exists x.\, (A \wedge B) \\ \forall x.\, A \vee B &= \forall x.\, (A \vee B) \\ \forall x.\, A \wedge B &= \forall x.\, (A \wedge B) \end{aligned}\right\} \quad x \notin fv\, B$$

$$\varphi\{x \leftarrow \tau\} = \varphi\{x \Leftarrow \tau\}$$

indicated actual

PROPOSAL FOR A BETTER PROOF SYSTEM WITH QUANTIFICATION

$$\bar{i}\,\frac{A \wedge \bar{A}}{f} \qquad \bar{w}\,\frac{A}{t} \qquad \bar{c}\,\frac{A}{A \wedge A} \qquad s\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \qquad \forall\,\frac{\forall x.\, A}{A}$$

$$i\,\frac{t}{A \vee \bar{A}} \qquad w\,\frac{f}{A} \qquad c\,\frac{A \vee A}{A} \qquad \exists\,\frac{A}{\exists x.\, A}$$

$$(A \vee B) \vee C = A \vee (B \vee C)$$
$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

$$A \vee B = B \vee A$$
$$A \wedge B = B \wedge A$$

$$f \wedge f = f \qquad A \vee f = A$$
$$t \vee t = t \qquad A \wedge t = A$$

$$\exists x.\, f = \forall x.\, f = f$$
$$\exists x.\, t = \forall x.\, t = t$$

$$\left.\begin{aligned} \exists x.\, A \vee B &= \exists x.\, (A \vee B) \\ \exists x.\, A \wedge B &= \exists x.\, (A \wedge B) \\ \forall x.\, A \vee B &= \forall x.\, (A \vee B) \\ \forall x.\, A \wedge B &= \forall x.\, (A \wedge B) \end{aligned}\right\} \quad x \notin fv\, B$$

$$\varphi\{x \leftarrow \tau\} = \varphi\{x \Leftarrow \tau\}$$

Soundness

Simple inspection

PROPOSAL FOR A BETTER PROOF SYSTEM WITH QUANTIFICATION

$$\bar{i}\,\frac{A \wedge \bar{A}}{f} \qquad \bar{w}\,\frac{A}{t} \qquad \bar{c}\,\frac{A}{A \wedge A} \qquad s\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \qquad \forall\,\frac{\forall x.\, A}{A}$$

$$i\,\frac{t}{A \vee \bar{A}} \qquad w\,\frac{f}{A} \qquad c\,\frac{A \vee A}{A} \qquad \exists\,\frac{A}{\exists x.\, A}$$

$$(A \vee B) \vee C = A \vee (B \vee C)$$
$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

$$A \vee B = B \vee A$$
$$A \wedge B = B \wedge A$$

$$f \wedge f = f \qquad A \vee f = A$$
$$t \vee t = t \qquad A \wedge t = A$$

$$\exists x.\, f = \forall x.\, f = f$$
$$\exists x.\, t = \forall x.\, t = t$$

$$\left.\begin{aligned} \exists x.\, A \vee B &= \exists x.\, (A \vee B) \\ \exists x.\, A \wedge B &= \exists x.\, (A \wedge B) \\ \forall x.\, A \vee B &= \forall x.\, (A \vee B) \\ \forall x.\, A \wedge B &= \forall x.\, (A \wedge B) \end{aligned}\right\} \quad x \notin fv\, B$$

$$\varphi\{x \leftarrow \tau\} = \varphi\{x \leftarrow \tau\}$$

Soundness

Simple inspection

Completeness

Completess of deep inference system SKS

$$+$$

$$\frac{A\{x \Leftarrow \tau\}}{\exists x.\, A} = \frac{A\{x \Leftarrow \tau\}}{\boxed{\dfrac{A}{\exists x.\, A}}\,\{x \leftarrow \tau\}}$$

(same for $\forall$)

DEEP INFERENCE = freely compose proofs via any given connectives — and substitution (w.i.p.)

DEEP INFERENCE

freely compose proofs via any given connectives — and substitution (w.i.p.)

$$R\sqrt{2} \qquad \vee\ \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}\ \vee \qquad \exists x.\,\exists y.\,(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})$$

DEEP INFERENCE

freely compose proofs via any given connectives — and substitution (w.i.p.)

$$2c\,\frac{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vee c\,\frac{\exists x.\,\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y}) \vee \exists x.\,\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\exists x.\,\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}$$

DEEP INFERENCE

freely compose proofs via any given connectives — and substitution (w.i.p.)

$$\mathsf{i}\,\frac{\mathsf{t}}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}}$$

$$\mathsf{s}\,\frac{\mathsf{i}\,\frac{\mathsf{t}}{Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y)} \wedge Rx^{y}}{Rx \vee Ry \vee 2\exists\,\frac{\bar{R}x \wedge \bar{R}y \wedge Rx^{y}}{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}}\ \{x \leftarrow \sqrt{2}\} \vee \mathsf{s}\,\frac{\bar{R}x \wedge \mathsf{i}\,\frac{\mathsf{t}}{Ry \vee \bar{R}x^{y} \vee (\bar{R}y \wedge Rx^{y})}}{Ry \vee \bar{R}x^{y} \vee 2\exists\,\frac{\bar{R}x \wedge \bar{R}y \wedge Rx^{y}}{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}}\ \{x \leftarrow \sqrt{2}^{\sqrt{2}}\}\ \{y \leftarrow \sqrt{2}\}$$

$$\mathsf{2c}\,\frac{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vee \mathsf{c}\,\frac{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y}) \vee \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}$$

DEEP INFERENCE

freely compose proofs via any given connectives — and substitution (w.i.p.)

$$\mathsf{i}\,\frac{\mathsf{t}}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}}$$

$$\mathsf{s}\,\frac{\mathsf{i}\,\dfrac{\mathsf{t}}{Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y)} \wedge Rx^{y}}{Rx \vee Ry \vee \mathsf{2}\exists\,\dfrac{\bar{R}x \wedge \bar{R}y \wedge Rx^{y}}{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}}\ \{x \leftarrow \sqrt{2}\} \vee \mathsf{s}\,\frac{\bar{R}x \wedge \mathsf{i}\,\dfrac{\mathsf{t}}{Ry \vee \bar{R}x^{y} \vee (\bar{R}y \wedge Rx^{y})}}{Ry \vee \bar{R}x^{y} \vee \mathsf{2}\exists\,\dfrac{\bar{R}x \wedge \bar{R}y \wedge Rx^{y}}{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}}\ \{x \leftarrow \sqrt{2}^{\sqrt{2}}\}\ \{y \leftarrow \sqrt{2}\}$$

$$\mathsf{2c}\,\frac{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vee \mathsf{c}\,\frac{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y}) \vee \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}$$

The two possibilities are clearly represented — nice semantics of proofs.

REAL ANALYTICITY IN DEEP INFERENCE

freely compose proofs via any given connectives — and substitution (w.i.p.)

$$
=\dfrac{
=\dfrac{\mathsf{i}\,\dfrac{t}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}}}{
\boxed{
\boxed{2s\,\dfrac{\mathsf{i}\,\dfrac{t}{Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y)} \wedge Rx^{y}}{Rx \vee Ry \vee 2\exists\,\dfrac{\bar{R}x \wedge \bar{R}y \wedge Rx^{y}}{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}}}\{x \leftarrow \sqrt{2}\} \vee
\boxed{2s\,\dfrac{\bar{R}x \wedge \mathsf{i}\,\dfrac{t}{Ry \vee \bar{R}x^{y} \vee (\bar{R}y \wedge Rx^{y})}}{Ry \vee \bar{R}x^{y} \vee 2\exists\,\dfrac{\bar{R}x \wedge \bar{R}y \wedge Rx^{y}}{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}}}\{x \leftarrow \sqrt{2}^{\sqrt{2}}\}
}\{y \leftarrow \sqrt{2}\}
}
}{
2c\,\dfrac{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vee c\,\dfrac{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y}) \vee \exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}{\exists x.\exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^{y})}
}
$$

Real subformula property!

Conjecture $= \dfrac{A\{x \Leftarrow \tau\}}{A\{x \leftarrow \tau\}}$ is admissible.

PERSPECTIVES

Generalised quantification:

$$\{x \leftarrow T\} = \forall x$$

$$\{x \leftarrow 3 \vee 4\} = \{x \leftarrow 3\} \vee \{x \leftarrow 4\}$$

$$\{x \leftarrow \varepsilon \ldots\}$$

$$\ldots$$

In any case, substitution shift:

$$A\sigma * B = (A * B)\sigma \qquad \text{if } B\sigma = B$$